Conformal geometry on four manifolds

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Conformal Geometry

- On (M^n, g), compact Riemannian manifold

A metric $\hat{g}$ is conformal to g, if $\hat{g} = \rho g$ for some $\rho > 0$. Denote $\rho = e^{2w}$, and $g_w = e^{2w} g$. Conformal means "angle preserving".

- Geometric Analysis: Using methods in analysis (e.g. PDE method) to study problems in geometry:

The sign of the curvature,
The size of the curvature,
The sign of some integral of the curvature polynomials.

- Conformal Geometry: Study of conformal invariants, conformal invariant operators.

In this talk, we restrict our attention to integral conformal invariants on four manifolds, and some geometric applications.

Outline of talk

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1. Introduction: Gaussian curvature on compact surfaces, Yamabe problem.

2. Conformal invariants on compact closed 4-manifolds, σ_2 and Q curvature, geometric application.

3. Conformal invariants on compact 4-manifolds with boundary, (Q, T) curvature.

4. Conformally compact Einstein manifolds, renormalized volume.

5. Compactness results for conformally compact Einstein manifolds of dimension 3+1.

§1. Gaussian curvature on compact surfaces

- On a compact surface (M^2, g), K_g the Gaussian curvature of g.
  Gauss-Bonnet formula:

$$2\pi\chi(M) = \int_M K_g dv_g,$$

where $\chi(M)$ is the Euler characteristic of M.
• Uniformization Theorem:
Classify (orientable) (M^2, g) according to sign of $\int_M K_g dv_g$.
Under conformal change of metric $g_w = e^{2w} g$, solve $K_{g_w} \equiv -1, 0, 1$
according to the sign of $\int_M K_g dv_g$.
When $K_{g_w} \equiv -1$; (M^2, g) is isometric to $(H^2/\Gamma, h_c)$.
When $K_{g_w} \equiv 0$; (M^2, g) is isometric to $(\mathbb{R}^2/\Gamma, |dx|^2)$.
When $K_{g_w} \equiv 1$; (M^2, g) is isometric to (S^2, g_c).

§1. Gaussian curvature on compact surfaces

- One can solve $K_{g_w} \equiv c$ by

$$-\Delta_g w + K_g = K_{g_w} e^{2w} \ on \ M.$$

Variational Approach: Moser's functional J_g

$$J_g[w] = \int_M |\nabla_g w|^2 dv_g + 2\int_M K_g w dv_g - (\int_M K_g dv_g) \log \frac{\int_M dv_{g_w}}{\int_M dv_g}.$$

- Ray-Singer-Polyakov formula: assume $vol(g_w) = vol(g)$,

$$J_g[w] = 12\pi \ log \ (\frac{det \ (-\Delta)_g}{det \ (-\Delta)_{g_w}}).$$

Works of Moser-Trudinger, Onofri, Osgood-Phillips-Sarnak, Nirenberg's problem.......

Second order operator on (M^n, g), Yamabe problem

- On (M^n, g), $n \geqslant 3$, the conformal Laplace operator L_g

$L_g = -\Delta_g + c_n R_g$ where $c_n = \frac{n-2}{4(n-1)}$, and R_g denotes the scalar curvature of the metric g.

- Under conformal change of metrics $\hat{g} = u^{\frac{4}{n-2}} g$, $u > 0$.

$$L_g u \ = c_n \ \hat{R} \ u^{\frac{n+2}{n-2}}.$$

The famous Yamabe problem is to solve above equation for $\hat{R}$ a constant c; settled by Yamabe, Trudinger, Aubin and Schoen, '60-'84.

- The problem is variational.

$$\mathcal{F}_g[u] = \int_{M^n} R_{\hat{g}} dv_{\hat{g}}.$$

The sign of c agrees with the sign of the second order Yamabe invariant:

$$Y(M, g) := \inf_{\hat{g} \in [g]} \frac{\int_M R_{\hat{g}} dv_{\hat{g}}}{(vol\ \hat{g})^{\frac{n-2}{n}}}.$$

§2. σ_2 curvature on four manifolds

• On (M^4, g) a closed, compact 4-manifold,
Gauss-Bonnet-Chern formula:

$$8\pi^2\chi(M) = \int_M \frac{1}{4}|W_g|^2 dv_g + \int_M \frac{1}{6}(R_g^2 - 3|Ric_g|^2) dv_g,$$

where $\chi(M)$ is the Euler characteristic of M, W_g the Weyl curvature, R_g the scalar curvature and Ric_g the Ricci curvature of g.

• Weyl curvature measures the obstruction to being conformally flat. On (M^n, g), $n \geqslant 4$. $W_g \equiv 0$ in a neighborhood of a point if an only if $g = e^{2w}|dx|^2$ for some function w. Thus for example, (S^n, g_c) has $W_{g_c} \equiv 0$.

• $g_w = e^{2w}g$, $|W_{g_w}| = e^{-2w}|W_g|$, thus on 4-manifolds $|W_{g_w}|^2 dv_{g_w} = |W_g|^2 dv_g$ a pointwise conformal invariant; thus

$$g \to \int_M |W|_g^2 dv_g$$

is an integral conformal invariant.

§2. σ_2 curvature on four manifolds

- Denote

$$\sigma_2(g) = \frac{1}{6}(R_g^2 - 3|Ric_g|^2)$$

and conclude

$$g \to \int_M \sigma_2(g) dv_g$$

is also an integral conformal invariant.

- We now justify the name of σ_2.

From the perspective of conformal geometry, a natural basis of the full curvature tensor R_m are Weyl tensor W, Schouten tensor A.

$$A_g = Ric_g - \frac{R}{2(n-1)} g.$$

Decomposition of R_m:

$$(R_m)_g = W_g \oplus \frac{1}{n-2} A_g \owedge g.$$

§2. σ_2 curvature on four manifolds

- When $k = 1$, $\sigma_1(A_g) = \sum_i \lambda_i = Tr_g\ A_g = \frac{n-2}{2(n-1)} R_g$.

- When $k = 2$, $\sigma_2(A_g) = \sum_{i<j} \lambda_i \lambda_j = \frac{1}{2}(|Tr_g\ A_g|^2 - |A_g|^2)$, where λs are the eigenvalues of the tensor A_g.

- On (M^4, g),

$$\sigma_2(g) = \sigma_2(A_g) = \frac{1}{6}(R_g^2 - 3|Ric_g|^2).$$

- When $k = n$, $\sigma_n(A_g) =$ determinant *of* A_g, an equation of Monge-Ampère type.

§2. σ_2 curvature on four manifolds

- To solve the "generalized Yamabe" problem:

$$\sigma_2(A_{g_w}) = \text{constant}. \qquad (1)$$

- To do so, we have

$$A_{g_w} = (n-2)\{-\nabla_g^2 w + dw \otimes dw - \frac{|\nabla_g w|^2}{2}\} + A_g.$$

- To illustrate that (1) is a fully non-linear equation, we have when $n = 4$,

$$\begin{aligned} \sigma_2(A_{g_w})e^{4w} &= \sigma_2(A_g) + 2((\Delta_g w)^2 - |\nabla_g^2 w|^2)) \\ &+ \Delta_g w |\nabla_g w|^2 + (\nabla_g w, \nabla_g |\nabla_g w|^2) \\ &+ \text{lower order terms.} \end{aligned}$$

- Compared to

$$\sigma_2(\nabla^2 u) = ((\Delta u)^2 - |\nabla^2 u|^2)).$$

§2. Variational functional to study σ_2

• Recall on (M^n, g) when $n \geqslant 3$, $\hat{g} = u^{\frac{4}{n-2}} g$, $\mathcal{F}_g(u) := \int_{M^n} R_{\hat{g}} dv_{\hat{g}}$ is the variational functional for the Yamabe problem.
When $n = 2$, $\mathcal{F}_g$ is replaced by the Moser's functional J_g to study the Gaussian curvature equation.

• When $n > 2$ and $n \neq 4$, denote $\hat{g} = e^{2w} g$, the functional $(\mathcal{F}_2)_g(w) := \int_{M^n} \sigma_2(\hat{g}) dv_{\hat{g}}$ is variational for σ_2.

• We now describe a variational approach to study σ_2 curvature in dimension 4 and the corresponding Moser's functional.

§2. Link between σ_2 to Paneitz operator and Q-curvature

- Recall on (M^n, g), $n \geqslant 3$, the second order conformal Laplacian operator $L = -\Delta + \frac{n-2}{4(n-1)} R$, we have,

$$L_{\hat{g}}(\varphi) = u^{-\frac{n+2}{n-2}} L_g(u\,\varphi\,) \text{ for all } \varphi \in C^\infty(M^n), \text{ where } \hat{g} = u^{\frac{4}{n-2}} g.$$

- Paneitz operator in 1983 on (M^n, g), $n \geqslant 5$.

$$P_4^n = (-\Delta)^2 + \delta\,(a_n R\,g + b_n \mathrm{Ric})\,d + \frac{n-4}{2} Q_4^n.$$

$$(P_4^n)_{\hat{g}}(\varphi) \;=\; u^{-\frac{n+4}{n-4}} (P_4^n)_g(u\,\varphi\,) \text{ for all } \varphi \in C^\infty(M^n), \text{ where } \hat{g} = u^{\frac{4}{n-4}} g.$$

- Notice that $P_4^n(1) = \frac{n-4}{2} Q_4^n$, so we can read Q_4^n from P_4^n when $n \neq 4$.

§2. Branson's Q-curvature

- Branson pointed out that $P := P_4^4$ and $Q := Q_4^4$ are well defined.

(which we named as Branson's Q-curvature.)

$$P_g\varphi = (-\Delta)^2\varphi + \delta\left(\frac{2}{3}Rg - 2\mathrm{Ric}\right)d\varphi,$$

$$2Q_g = -\frac{1}{6}\Delta R_g + \frac{1}{6}(R_g^2 - 3|Ric_g|^2).$$

- $$P_g w + 2Q_g = 2Q_{g_w}e^{4w} \text{ on } M^4, \text{ where } g_w = e^{2w}g.$$

Compared to

$$-\Delta_g w + K_g = K_{g_w}e^{2w} \text{ on } M^2 .$$

- For examples:

On $(R^4, |dx|^2)$, $P = \Delta^2$,
On (S^4, g_c), $P = \Delta^2 - 2\Delta$,
On (M^4, g), g Einstein, $P = (-\Delta) \circ (L)$.

§2. Link between σ_2 to Q-curvature

Thus we have

$$2Q_g = -\frac{1}{6}\Delta R_g + \sigma_2(A_g). \tag{2}$$

- Following Moser, the functional to study constant Q_{g_w} curvature:

$$II[w] = \langle Pw, w\rangle + 4\int Qwdv - \left(\int Qdv\right) \log \frac{\int e^{4w}dv}{\int dv}.$$

- Consider the functional III with Euler equation $\Delta R = \ constant$,

$$III[w] = \frac{1}{3}\left(\int R_{g_w}^2 dv_{g_w} - \int R^2 dv\right),$$

- $$\mathcal{F}[w] = II[w] - \frac{1}{12} III[w].$$

Proposition (Chang -Yang '02, Brendle-Viaclovsky '04)
$\mathcal{F}$ is the Lagrangian functional for σ_2 curvature.

§2. Link between σ_2 and Q curvature

- Strategy to solve $\sigma_2(A_{g_w}) \equiv 1$. Solve for critical point of

$$\mathcal{F}_\delta := II - (\frac{1}{12} - \delta) III$$

starting at $\delta = \frac{1}{12}$, then let $\delta \to 0$.

Remark: PDE analogue of solving $1 = \delta(\Delta^2 u) + \sigma_2(\nabla^2 u)$, let $\delta \to 0$.

- On M^4, denote

$$\mathcal{A} := \{g| \ Y(M,g) > 0, \int_M \sigma_2(A_g) dv_g > 0\}.$$

- Theorem (Chang-Gursky-Yang '01-'03)
  $g \in \mathcal{A}$ if and only if there is some $g_w \in [g]$ with $R_{g_w} > 0$ and $\sigma_2(A_{g_w}) > 0$, i.e. $g_w \in \Gamma_2^+$.

§2. A uniqueness result

- We then apply elliptic PDE method to show if $g \in \mathcal{A}$, then there exists some $g_w \in [g]$ with $\sigma_2(A_{g_w}) = 1$ and $R_{g_w} > 0$.

- An uniqueness result.

Theorem (Gursky-Steets '16)
If $g \in \mathcal{A}$ and (M^4, g) is not conformal to (S^4, g_c), then $g_w \in [g]$ with $g_w \in \Gamma_2^+$ with $\sigma_2(A_{g_w}) = 1$ is unique.

The result was established by constructing some norm for metrics in Γ_2^+, with respect to which the functional $\mathcal{F}$ is convex.

The result is **surprising** in contrast with the famous example of Schoen'87 that on $(S^1 \times S^n, g_{prod})$, where $n \geqslant 2$, the metric with constant scalar curvature (and the same volume) is not unique.

§2. Diffeomorphism theorem

- Theorem (Chang-Gursky-Yang '03)

Suppose (M, g) is a closed 4-manifold with $g \in \mathcal{A}$.
(a). If $\int_M ||W||_g^2 dv_g < \int_M \sigma_2(A_g) dv_g$ then M is diffeomorphic to either $S^4 or\ \mathbb{RP}^4$.
(b). If $\int_M ||W||_g^2 dv_g = \int_M \sigma_2(A_g) dv_g$ and M is not diffeomorphic to S^4 or $\mathbb{RP}^4$, then (M, g) is conformally equivalent to $(\mathbb{CP}^2, g_{FS})$.

Here $||W||^2 := \frac{1}{4}|W|^2$.

- Part (a) of the theorem above is an L^2 version of an earlier result of Margerin '98; applying Ricci flow method pioneered by Hamilton '86.

- In part (b), the assumption $g \in \mathcal{A}$ excludes out the case when $(M, g) = (S^1 \times S^3, g_{prod})$, where $||W||_g = \sigma_2(A_g) \equiv 0$.

§2. Diffeomorphism Theorem

- For a metric $g \in \mathcal{A}$, we define the conformally invariant constant $\beta = \beta([g])$:

$$\int ||W||_g^2 dv_g = \beta \int_M \sigma_2(A_g) dv_g.$$

- Previous Theorem says when $0 < \beta < 1$, (M^4, g) is diffeomorphic to the standard S^4 or $\mathbb{RP}^4$.

- Lemma : Given $g \in \mathcal{A}$, if $1 < \beta < 2$, then

Either M^4 is homemorphic to S^4 or $\mathbb{RP}^4$ (when $b_2^+ = b_2^- = 0$)
or M^4 is homeomorphic to $\mathbb{CP}^2$ (when $b_2^+ = 1, b_2^- = 0$).
We remark that $\beta = 2$ for the product metric on $S^2 \times S^2$.

- Proof relies on the Signature formula:

$$12\pi^2 \tau = \int_{M^4} (||W^+||^2 - ||W^-||^2) dv,$$

where $\tau = b_2^+ - b_2^-$,

§2. Perturbation Result on $\mathbb{CP}^2$

- Theorem (Chang-Gursky-S. Zhang '18) There exists $\epsilon > 0$ such that if (M, g) is a four manifold with $b_2^+ > 0$ and with a metric of positive Yamabe type satisfying with $1 < \beta < 1 + \epsilon$, then (M, g) is diffeomorphic to $(\mathbb{CP}^2, g_{FS})$.

Remark: Proof again uses method of Ricci flow.

- What is the class of 4-manifolds which allows a metric in the class $\mathcal{A}$? By the work of Donaldson '83, Freedman '82, the homeomorphism type of the class of simply-connected 4-manifolds (M^4, g) with $R_g > 0$ consists of S^4 together with $k\mathbb{CP}^2 \# l\overline{\mathbb{CP}^2}$ and $k(S^2 \times S^2)$.
- $g \in \mathcal{A}$ implies $4 + 5l > k$. There are many examples in this class beyond S^4, $\mathbb{CP}^2$ and $S^2 \times S^2$ constructed by different authors.

- It would be an ambitious program to find out the entire class of 4-manifolds with metrics in $\mathcal{A}$, and to classify their diffeomorphism types by the (relative) size of the integral conformal invariants discussed here.

§3. (Q, T) curvature on 4-manifold with boundary

- We recall compact surface with boundary (X^2, M^1, g) where the metric g is defined on $X^2 \cup M^1$; the Gauss-Bonnet formula

$$2\pi\chi(X) = \int_X K\ dv + \oint_M k\ d\sigma,$$

where k is the geodesic curvature on M.

- Under conformal change of metric g_w on X, we have

$$\frac{\partial}{\partial n} w + k \ = \ k_{g_w}\, e^w \ \text{ on M.}$$

- Chang-Qing, '85-'87

Replace $(-\Delta, \frac{\partial}{\partial n})$ on (X^2, M^1, g) by (P_4, P_3) on (X^4, M^3, g).
Replace (K, k) on (X^2, M^1, g) by (Q, T) on (X^4, M^3, g).
Where $(P_4 := P, Q)$ are as before.

§3. (Q, T) curvature on 4-manifold with boundary

- Construct $(P_3,\ T)$, where P_3 bidegree $(0,3)$ with

$$(P_3)_g w + T_g \ = \ T_{g_w} e^{3w} \ \text{on} \ M^3 \ .$$

- $(B^4, S^3, |dx|^2)$,

$$P_4 = (-\Delta)^2, \ P_3 = -\left(\frac{1}{2} \ \frac{\partial}{\partial n} \ \Delta + \tilde{\Delta} \frac{\partial}{\partial n} + \tilde{\Delta} \right), \ T = 2,$$

where $\tilde{\Delta}$ the intrinsic boundary Laplacian.

- Gauss-Bonnet-Chern formula:

$$8\pi^2 \chi(X^4, M^3) = \int_{X^4} (||W||^2 \ + \ 2Q) \ dv \ + \oint_{M^3} (\mathcal{L} + 2T) \ d\sigma.$$

Where $\mathcal{L} d\sigma$ is a pointwise conformal invariant.

§3. (Q, T) curvature on 4-manifold with boundary

- $$\int_X Qdv + \oint_M Td\sigma$$

is an integral conformal invariant.

- In general, the formula for T is lengthy; but when (X^4, g) is with **totally geodesic boundary**, that is, its second fundamental form vanishes. we have

$$T = \frac{1}{12}\frac{\partial}{\partial n}R.$$

- Recall

$$2Q_g = -\frac{1}{6}\Delta R_g + \sigma_2(A_g).$$

- Thus in this case, we have

$$2(\int_X Qdv + \oint_M Td\sigma) = \int_X \sigma_2 dv.$$

We now will present some geometric content of this formula.

§4. Conformally compact Einstein manifolds

Given a compact manifold (M^n, h), when is it the boundary of a conformally compact Einstein manifold (X^{n+1}, g^+) with $r^2 g^+|_M = h$? This problem of finding "conformal filling in" is motivated by:

- The AdS/CFT correspondence in quantum gravity (proposed by Maldacena around 1998)

- Geometric considerations to study the structure of non-compact asymptotically hyperbolic manifolds.

§4. Conformally compact Einstein manifolds, Definition

• On a manifold X with boundary M, we call r a defining function on X, if $r > 0$ *on* X, $r = 0$ *on* M and $dr \neq 0$ on M.

(X^{n+1}, g^+) is conformally compact if $(\bar{X}^{n+1}, r^2 g^+)$ is compact. Denote $h = r^2 g^+|_M$, we call $(M^n, [h])$ the conformal infinity of (X^{n+1}, g^+).

If $\mathrm{Ric}[g^+] = -n\, g^+$, we call (X^{n+1}, M^n, g^+) a conformally compact (Poincaré) Einstein (CCE) manifold.

• We remark on a CCE manifold, special r (called geodesic defining function) can be chosen so that $r^2 g^+$ is with totally geodesic boundary.

§4. Examples of CCE manifold

- Example 1.

On $(\mathbb{R}_+^{n+1}, \mathbb{R}^n, g_{\mathbb{H}})$, where $g_{\mathbb{H}} = \frac{dx^2+dy^2}{y^2}$, $x \in \mathbb{R}^n$, $\quad y > 0$. Choose $r = y$, then $(\mathbb{R}_+^{n+1}, dx^2 + dy^2)$ is not compact, but conformal to $g_{\mathbb{H}}$, with conformal infinity $(\mathbb{R}^n, [dx^2])$.

- Example 2.

On $(B^{n+1}, S^n, g_{\mathbb{H}})$, where $(B^{n+1},\ g_{\mathbb{H}} = (\frac{2}{1-|y|^2})^2|dy|^2)$. Choose

$$r = 2\,\frac{1-|y|}{1+|y|},$$

$$g_{\mathbb{H}} = g^+ = r^{-2}\left(dr^2 + (1-\frac{r^2}{4})^2\, g_c\right).$$

with $(S^n, [g_c])$ as conformal infinity.

- Example 3. AdS-Schwarzchild space

§4. Existence and Uniqueness results on CCE manifolds

Some existence and non-existence results.

• "Ambient Metric" of Fefferman-Graham '85. On any compact manifold (M^n, h), h analytic, there is CCE metric on some $M^n \times (0, \epsilon)$ of M. Gursky-Székelyhidi '17, extend to smooth h .

• Graham-Lee: Any h in a small smooth neighborhood of g_c on S^n.

• Gursky-Han '17 and Gursky-Han-Stolz '18 construct many examples of boundary conformal classes that have no Poincaré-Einstein extensions. For example, S^{4k-1} for $k \geqslant 2$ admits infinitely many conformal classes (with positive Yamabe invariant) which cannot be extended to Poincaré-Einstein metrics in B^{4k}.

§4. CCE existence and uniqueness

Some uniqueness and non-uniqueness results.

• Qing '03, and many others later have established (B^{n+1}, g_H) as the unique CCE manifold with (S^n, g_c) as its conformal infinity.

• Chang-Ge-Qing '18 have extended the uniqueness of CCE extensions constructed by Graham-Lee '91 for metrics $\{h\}$ on S^3 in a neighbor of g_c. I will soon present the proof.

• Hawking-Page '83 non-uniqueness result for Ads-Schwarzchild space with conformal infinity $(S^1 \times S^2, [g_{prod}])$.

§4. Renormalized volume

- "Renormalized volume" in the CCE setting, introduced by Maldacena in 1998. (Witten '98, Henningson-Skenderis '98 and Graham '00).

- On CCE manifolds (X^{n+1}, M^n, g^+) with geodesic defining function r, For n even,

$$\mathrm{Vol}_{g^+}(\{r > \epsilon\}) = c_0\epsilon^{-n} + c_2\epsilon^{-n+2} + \cdots + c_{n-2}\epsilon^{-2} + L\log\frac{1}{\epsilon} + V + o(1).$$

V is called the renormalized volume, L is independent of $h \in [h]$ where $h = r^2 g^+|_M$,

- Theorem (Graham-Zworski, Fefferman-Graham '02) For n even,

$$L = c_n \oint_{M^n} Q_h \, dv_h.$$

§4. Renormalized volume

- On (X^{n+1}, M^n, g^+), for n odd,

$$\begin{aligned} \mathrm{Vol}_{g^+}(\{r > \epsilon\}) = c_0\epsilon^{-n} + c_2\epsilon^{-n+2} + \cdots\cdot\cdot \\ + c_{n-1}\epsilon^{-1} + V + o(1). \end{aligned}$$

V is called the renormalized volume. V is independent of $g \in [g]$, and hence a conformal invariant.

- Theorem (M. Anderson '01, Chang-Qing-Yang '06)

On (X^4, M^3, g^+) conformal compact Einstein manifold, we have

$$V = \frac{1}{6}\int_{X^4} \sigma_2(A_g)dv_g$$

for any compactified metric g with totally geodesic boundary.

§4. Renormalized volume

- Proof of Theorem

Lemma 1 (Fefferman-Graham '03)
On (X^4, M^3, g^+) CCE, (M, h) conformal infinity.

$$- \Delta_{g^+} w \ = \ 3 \ on \ \ X^4, \tag{3}$$

then w has the asymptotic behavior $w = log\, r + A + Br^3$ near M, where A, B are functions even in r, $A|_M = 0$, and $V = \int_M B|_M$.

- Lemma 2 (Chang-Qing-Yang '06)

Consider the metric $g^* = g_w = e^{2w} g^+$, with w as in (3), then g^* is totally geodesic on boundary with
(a) $Q_{g*} \equiv 0$, and (b) $B|_M = \frac{1}{36} \frac{\partial}{\partial n} R_{g*} = \frac{1}{6} T_{g*}$.

- To see (a), recall we have g^+ is Einstein with $Ric_{g^+} = -3g^+$,

$$P_{g^+} = (-\Delta_{g^+}) \circ (-\Delta_{g^+} - 2); \ while \ 2Q_{g^+} = 6.$$

Thus

$$P_{g^+} w + 2Q_{g^+} = 0 = 2e^{2w} Q_{g*}.$$

§4. Renormalized volume

- Recall the statement of the theorem.

Theorem On (X^4, M^3, g^+) CCE manifold, we have

$$V = \frac{1}{6}\int_{X^4} \sigma_2(A_g) dv_g$$

for any compactified metric g with totally geodesic boundary.

- To prove the theorem we apply Lemmas 1 and 2 and get

$$\begin{aligned} 6V &= \oint_M B|_M d\sigma_{g*} = \frac{1}{6}\oint_M \frac{\partial}{\partial n} R_{g*} \\ &= 2(\int_X Q_{g*} + \oint_M T_{g*}) = \int_{X^4} \sigma_2(A_{g*}) dv_{g*}. \end{aligned}$$

§5. Compactness of CCE manifolds

• An open question: Does the entire class of metrics (S^3, h) with positive scalar curvature allow CCE filling in B^4?

• The class is path-connected by a result of F. Marques '12.
The index argument for non-existence of Gursky-Han, Gursky-Han-Stolz does not apply.

• We propose to study the "compactness" problem, which hopefully will lead to degree theory argument for the positive answer to the question above. More precisely, we ask the question:

Given a sequence of $(S^3, [h_i])$ metrics with positive Yamabe constants, which are conformal infinity of CCE (B^4, g_i^+); when would

$$\{[h_i]\} \text{ forms a compact family on } S^3$$

$$\Longrightarrow \{[g_i]\} \text{ forms a compact family on } B^4?$$

where g_i is some compactification of $\{g_i^+\}$ with $g_i|_M = h_i$.

§5. Compactness of CCE manifolds

- Report on works of Chang-Yuxin Ge '16 and Chang-Ge-Jie Qing '17.

The difficulty lies in the existence of an "non-local" term.
To see this on (X^4, M^3, g^+) CCE with (M^3, h) conformal infinity, recall the asymptotic behavior

$$g := r^2 g^+ = h + g^{(2)} r^2 + g^{(3)} r^3 + g^{(4)} r^4 + \cdots,$$

where $g^{(2)} = -\frac{1}{2} A_h$ determined by h (a local terms), $Tr_h g^{(3)} = 0$, while

$$g^{(3)}_{\alpha,\beta} = -\frac{1}{3} \frac{\partial}{\partial n} (Ric_g)_{\alpha,\beta}$$

is a non-local term not determined by h.

We remark that h together with $g^{(3)}$ determines the asymptotic behavior of g.

§5. Compactness of CCE manifolds

- For convenience, we choose $h = h^Y \in [h]$, the Yamabe metric on M. But what is a good choice of g? A first attempt is to choose $g := g^Y$, a Yamabe metric among compactified metrics of $[g^+]$, the difficulty of this choice is one does not know how to control the behavior of $g^Y|_M$ in terms of h^Y.
- Instead on (X^4, M^3, g^+) with conformal infinity (M^3, h), we choose the "Fefferman-Graham" compactification $g = g^* = e^{2w} g^+$ where

$$-\Delta_{g^+} w = 3 \ \ on\ X, \text{with } g^*|_M = h$$

- We recall that $Q_{g*} \equiv 0$, hence

$$\int_X \sigma_2(A_{g*}) dv_{g*} = 2 \oint_M T_{g*} d\sigma_h = \frac{1}{3} \oint_M \frac{\partial}{\partial n} R_{g*} d\sigma_h.$$

A model case. On $(B^4, S^3, g_{\mathbb{H}})$,

$$g^* = e^{(1-|x|^2)}|dx|^2 \; on \; B^4.$$

$$Q_{g*} \equiv 0, \quad T_{g*} \equiv 2 \; on \; S^3.$$

$$(g^*)^{(3)} \equiv 0.$$

and

$$\int_{B^4} \sigma_2(A_{g*}) dv_{g*} = 8\pi^2.$$

§5. A perturbation compactness result

- **Theorem 1**. Let $\{(B^4, S^3, g_i^+)\}$ be sequence of CCE manifolds with conformal infinity $(S^3, [h_i])$, assume

1. The boundary Yamabe metrics $\{h_i\}$ form a compact family in C^{k+3} norm with $k \geqslant 2$; with

$$Y(M, [h_i]) \geqslant c_1 > 0;$$

2. There exists some small positive constant $\varepsilon > 0$ such that for all i

$$\int_{B^4} \sigma_2(A_{g_i^*}) dv_{g_i^*} \geqslant 8\pi^2 - \varepsilon.$$

Then the family of the g_i^* is compact in $C^{k+2,\alpha}$ norm for any $\alpha \in (0, 1)$ up to a diffeomorphism fixing the boundary.

- Main step in the proof is to show the curvature of $\{g_i^*\}$ is bounded, which is achieved via some "blow-up" analysis. It is easier to see the argument via some equivalent conditions of (2).

§5. A perturbation compactness result

- On (B^4, S^3), recall Gauss-Bonnet formula

$$8\pi^2\chi(B^4, S^3) = 8\pi^2 = \int_{B^4} (||W||_g^2 + \sigma_2(A_g))dv_g.$$

- It turns out in Theorem 1, the following statements are equivalent:

1. $$\int_X \sigma(A_{g*})dv_{g*} \geqslant 8\pi^2 - \varepsilon.$$
2. $$\int_X ||W||_{g^+}^2 dv_{g^+} \leqslant \varepsilon.$$
3. $$Y(S^3, [g_c]) \geqslant Y(S^3, [h]) > Y(S^3, [g_c]) - \varepsilon_1.$$
4. $$T(g^*) \geqslant 2 - \varepsilon_2, \text{ when } vol(h) = vol(g_c).$$
5. $$|(g^*)^{(3)}| \leqslant \varepsilon_3.$$

§5. A Perturbation compactness theorem

- Corollary There exists some $\varepsilon > 0$ such that on (B^4, S^3, h), if $||h - g_c||_{C^\infty} < \varepsilon$, the CCE filling (B^4, S^3, g^+) of h is unique.

- Sketch proof of the perturbation result.

The major step is to show the curvature of g_i^* remains bounded. Assume not, we will do the "blow up" analysis. That is, we re-scale the metrics $\bar{g}_i = K_i^2 g_i$, $\bar{h}_i = \bar{g}_i|_{S^3}$, where

$$K_i^2 = \max\{\sup_{B^4} |Rm_{g_i}|\} = |Rm_{g_i}|(p_i).$$

We mark the accumulation point of p_i as $0 \in B^4$, for simplicity, we assume $0 \in S^3$. Note that we have $(*)$ $|Rm_{\bar{g}_i}|(0) = 1$.

§5. Proof of the perturbation compactness result

- Step 1: We have $\bar{g}_i = e^{2\bar{w}_i} g_I^+$, $\quad \bar{h}_i = K_i^2 h_i$,

$$(S^3, \bar{h}_i) \longrightarrow (\mathbb{R}^3, dx^2)$$

$$(B^4, \bar{g}_i) \longrightarrow (X_\infty, g_\infty) \text{ in Gromov-Hausdroff sense.}$$

- Step 2:

$$\bar{w}_i \longrightarrow \bar{w}_\infty, \; uniformly\ on\ compact,$$

$$g_\infty = e^{2\bar{w}_\infty} g_\infty^+, \text{ with } Ric_{g_\infty}^+ = -3g_\infty^+ \text{ and } ||W_{g_\infty^+}|| \equiv 0.$$

- Step 3: We then claim up to an isometry

$$(X_\infty, g_\infty^+) \; = \; (\mathbb{R}_+^4, \, g_H := \frac{|dx|^2 + |dy|^2}{y^2}),$$

and apply a Liouville type PDE argument to conclude $\bar{w}_\infty = \log y$.

$$\text{Thus } g_\infty = |dx|^2 + |dy|^2,$$

which contradicts our marking condition $(*)$ $|Rm_{g_\infty}|(0) = 1$.

§5. Compactness of CCE manifolds

- Theorem 2

Let $\{(B^4, S^3, g_i{}^+)\}$ be sequence of CCE on B^4 with boundary S^3. Assume

1. The boundary Yamabe metrics h_i form a compact family in C^{k+3} norm with $k \geqslant 2$; with

$$Y(M, [h_i]) \geqslant c_1 > 0;$$

2. $$\liminf_{r \to 0} \inf_i \inf_{x \in S^3} \oint_{B(x,r)} T_{g_i^*} \geqslant 0.$$

Then the family of metrics g_i^* is compact in $C^{k+2,\alpha}$ norm for any $\alpha \in (0,1)$ up to a diffeomorphism fixing the boundary, provided $k \geqslant 2$.

- It remains to see if in both theorem 1 and 2 above condition (2) can be replaced by the renormalized volume term $\int_{B^4} \sigma_2(A_{g_i^*}) dv_{g_i^*}$ being positive.

§5. Open questions and some future directions

• In this talk, we have only addressed the case of 4-manifolds with $Y(M,g) > 0$ and $\int_M \sigma_2(g) dv_g > 0$, i.e $[g] \in \Gamma_2^+$. It remain to study manifolds with metrics $g \in \Gamma_2^-$?, i.e. $Y(M,g) < 0$, while $\sigma_2(g) > 0$.

• On (M^n, g), $n \geqslant 5$ extensive works have been done to study $\sigma_k(A_g)$, mainly restricted to locally conformal flat manifolds. It remains to locate suitable conformal invariants with geometric connections to study.

THANK YOU FOR YOUR ATTENTION!