# Isometric embedding via strongly symmetric positive systems

Jeanne N. Clelland, University of Colorado, Boulder

Joint with Gui-Qiang Chen, Marshall Slemrod, Dehua Wang, and Deane Yang

Midwest Geometry Conference
Iowa State University

September 7, 2019

# The isometric embedding problem

## The isometric embedding problem

Let $(M, g)$ be an $n$-dimensional Riemannian manifold. A map $\mathbf{y} : M \to \mathbb{R}^N$ is called an *isometric embedding* if $\mathbf{y}$ is injective and, in local coordinates $\mathbf{x} = (x^1, \ldots, x^n)$ on $M$,

$$\partial_i \mathbf{y} \cdot \partial_j \mathbf{y} = g_{ij}, \qquad 1 \leq i, j \leq n, \tag{1}$$

where $g = g_{ij} dx^i dx^j$ and $\partial_i$ denotes $\frac{\partial}{\partial x^i}$.

## The isometric embedding problem

Let $(M, g)$ be an $n$-dimensional Riemannian manifold. A map $\mathbf{y} : M \to \mathbb{R}^N$ is called an *isometric embedding* if $\mathbf{y}$ is injective and, in local coordinates $\mathbf{x} = (x^1, \ldots, x^n)$ on $M$,

$$\partial_i \mathbf{y} \cdot \partial_j \mathbf{y} = g_{ij}, \qquad 1 \le i, j \le n, \tag{1}$$

where $g = g_{ij} dx^i dx^j$ and $\partial_i$ denotes $\frac{\partial}{\partial x^i}$.

The *local isometric embedding problem* asks whether, given $(M, g)$ and $\mathbf{x}_0 \in M$, there exists an isometric embedding of some neighborhood of $\mathbf{x}_0$ into $\mathbb{R}^N$—i.e., whether the PDE system (1) has local solutions in some neighborhood of $\mathbf{x}_0$.

## The isometric embedding problem

Let $(M, g)$ be an $n$-dimensional Riemannian manifold. A map $\mathbf{y} : M \to \mathbb{R}^N$ is called an *isometric embedding* if $\mathbf{y}$ is injective and, in local coordinates $\mathbf{x} = (x^1, \ldots, x^n)$ on $M$,

$$\partial_i \mathbf{y} \cdot \partial_j \mathbf{y} = g_{ij}, \qquad 1 \leq i, j \leq n, \tag{1}$$

where $g = g_{ij} dx^i dx^j$ and $\partial_i$ denotes $\frac{\partial}{\partial x^i}$.

The *local isometric embedding problem* asks whether, given $(M, g)$ and $\mathbf{x}_0 \in M$, there exists an isometric embedding of some neighborhood of $\mathbf{x}_0$ into $\mathbb{R}^N$—i.e., whether the PDE system (1) has local solutions in some neighborhood of $\mathbf{x}_0$.

This problem is overdetermined when $N < \frac{1}{2}n(n+1)$, underdetermined when $N > \frac{1}{2}n(n+1)$, and determined when $N = \frac{1}{2}n(n+1)$.

**The underdetermined case:**

**The underdetermined case:**

**Nash (1956):** If $(M^n, g)$ is $C^k$ with $3 \leq k \leq \infty$, then there exists a global $C^k$ isometric embedding of $M$ into some $\mathbb{R}^N$ with $N \leq \frac{1}{2}n(n+1)(3n+11)$.

**The underdetermined case:**

**Nash (1956):** If $(M^n, g)$ is $C^k$ with $3 \leq k \leq \infty$, then there exists a global $C^k$ isometric embedding of $M$ into some $\mathbb{R}^N$ with $N \leq \frac{1}{2}n(n+1)(3n+11)$.

**Greene (1970):** If $(M^n, g)$ is $C^\infty$, then every $\mathbf{x}_0 \in M$ has a neighborhood which has a $C^\infty$ isometric embedding into some $\mathbb{R}^N$ with $N \leq \frac{1}{2}n(n+1) + n$.

**The determined case:**

**The determined case:**

**Cartan-Janet (1927):** If $(M^n, g)$ is real analytic and $N = \frac{1}{2}n(n+1)$, then every $\mathbf{x}_0 \in M$ has a neighborhood which has a real analytic isometric embedding into $\mathbb{R}^N$.

**The case $n = 2, N = 3$:**

**The case $n = 2, N = 3$:**

If $(M^2, g)$ is $C^\infty$, then local isometric embeddings of varying regularity exist in a neighborhood of any point where:

**The case $n = 2, N = 3$:**

If $(M^2, g)$ is $C^\infty$, then local isometric embeddings of varying regularity exist in a neighborhood of any point where:

- $K(\mathbf{x}_0) \neq 0$;

**The case $n = 2, N = 3$:**

If $(M^2, g)$ is $C^\infty$, then local isometric embeddings of varying regularity exist in a neighborhood of any point where:

- $K(\mathbf{x}_0) \neq 0$;
- $K(\mathbf{x}_0) = 0$ and $\nabla K(\mathbf{x}_0) \neq 0$;

**The case $n = 2, N = 3$:**

If $(M^2, g)$ is $C^\infty$, then local isometric embeddings of varying regularity exist in a neighborhood of any point where:

- $K(\mathbf{x}_0) \neq 0$;
- $K(\mathbf{x}_0) = 0$ and $\nabla K(\mathbf{x}_0) \neq 0$;
- $K(\mathbf{x}_0)$ vanishes to finite order in certain precise ways.

(C.-S. Lin, Q. Han, J.-X. Hong, M. Khuri)

**The case $n = 3, N = 6$:**

**The case $n = 3, N = 6$:**

**Bryant-Griffiths-Yang (1983):** If $(M^3, g)$ is $C^\infty$, then $C^\infty$ local isometric embeddings exist in a neighborhood of any point where the rank of the Einstein tensor is at least 2.

**The case $n = 3, N = 6$:**

**Bryant-Griffiths-Yang (1983):** If $(M^3, g)$ is $C^\infty$, then $C^\infty$ local isometric embeddings exist in a neighborhood of any point where the rank of the Einstein tensor is at least 2.

**Nakamura-Maeda (1989):** If $(M^3, g)$ is $C^\infty$, then $C^\infty$ local isometric embeddings exist in a neighborhood of any point where the Riemann curvature tensor is nonzero.

**The case $n = 4, N = 10$:**

**The case $n = 4, N = 10$:**

**Bryant-Griffiths-Yang (1983), Goodman-Yang (1988):**
There exists a finite set of algebraic relations among the Riemann curvature tensor and its covariant derivatives, with the property that a local isometric embedding exists in a neighborhood of any point where these relations do not all hold.

For $n \geq 3$, these results are all based on the Nash-Moser implicit function theorem. In order to apply Nash-Moser, one must show that:

For $n \geq 3$, these results are all based on the Nash-Moser implicit function theorem. In order to apply Nash-Moser, one must show that:

- There exist "approximate solutions," i.e., local embeddings $\mathbf{y}_0 : M \to \mathbb{R}^N$ so that the induced metric

$$\bar{g}_{ij} = \partial_i \mathbf{y}_0 \cdot \partial_j \mathbf{y}_0$$

  is "close" to $g_{ij}$.

For $n \geq 3$, these results are all based on the Nash-Moser implicit function theorem. In order to apply Nash-Moser, one must show that:

- There exist "approximate solutions," i.e., local embeddings $\mathbf{y}_0 : M \to \mathbb{R}^N$ so that the induced metric

$$\bar{g}_{ij} = \partial_i \mathbf{y}_0 \cdot \partial_j \mathbf{y}_0$$

  is "close" to $g_{ij}$.
- For any such $\mathbf{y}_0$, the linear PDE system obtained by linearizing the system (1) at $\mathbf{y}_0$ has a local $C^\infty$ solution $\mathbf{v}(\mathbf{x})$.

For $n \geq 3$, these results are all based on the Nash-Moser implicit function theorem. In order to apply Nash-Moser, one must show that:

- There exist "approximate solutions," i.e., local embeddings $\mathbf{y}_0 : M \to \mathbb{R}^N$ so that the induced metric

$$\bar{g}_{ij} = \partial_i \mathbf{y}_0 \cdot \partial_j \mathbf{y}_0$$

  is "close" to $g_{ij}$.
- For any such $\mathbf{y}_0$, the linear PDE system obtained by linearizing the system (1) at $\mathbf{y}_0$ has a local $C^\infty$ solution $\mathbf{v}(\mathbf{x})$.
- The solution $\mathbf{v}(\mathbf{x})$ to the linearized system satisfies "smooth tame estimates."

**The easy part:**

**The easy part:**

Approximate solutions are provided by the Cartan-Janet theorem:

**The easy part:**

Approximate solutions are provided by the Cartan-Janet theorem:

Let $\bar{g}$ be a real analytic metric that agrees with $g$ to high order at $\mathbf{x}_0$; then there exists a neighborhood $\Omega \subset M$ of $\mathbf{x}_0$ and a real analytic isometric embedding $\mathbf{y}_0 : \Omega \to \mathbb{R}^N$ for $\bar{g}$.

**The easy part:**

Approximate solutions are provided by the Cartan-Janet theorem:

Let $\bar{g}$ be a real analytic metric that agrees with $g$ to high order at $\mathbf{x}_0$; then there exists a neighborhood $\Omega \subset M$ of $\mathbf{x}_0$ and a real analytic isometric embedding $\mathbf{y}_0 : \Omega \to \mathbb{R}^N$ for $\bar{g}$.

By shrinking $\Omega$ if necessary, we can ensure that $\bar{g}$ is sufficiently close to $g$.

**The hard part:**

**The hard part:**

The linearized system is generally not "nice," and showing that it has solutions that satisfy the necessary estimates can be very challenging.

**The hard part:**

The linearized system is generally not "nice," and showing that it has solutions that satisfy the necessary estimates can be very challenging.

In particular, for $n \geq 3$, the linearized system is never elliptic, so standard estimation techniques for elliptic systems don't work.

Bryant-Griffiths-Yang show that:

Bryant-Griffiths-Yang show that:

- For $n = 3$ and the Einstein tensor having rank at least 2, the approximate embedding $\mathbf{y}_0$ can be chosen so that the linearized system becomes either *symmetric hyperbolic* or *strictly hyperbolic*. They then show that any such system has a solution that satisfies smooth tame estimates.

Bryant-Griffiths-Yang show that:

- For $n = 3$ and the Einstein tensor having rank at least 2, the approximate embedding $\mathbf{y}_0$ can be chosen so that the linearized system becomes either *symmetric hyperbolic* or *strictly hyperbolic*. They then show that any such system has a solution that satisfies smooth tame estimates.
- For $n = 3$ and $R(\mathbf{0}) \neq 0$, or $n = 4$ and $(R(\mathbf{0}), \nabla R(\mathbf{0}))$ in some dense open set, the approximate embedding $\mathbf{y}_0$ can be chosen so that this system has *real principal type*.

Bryant-Griffiths-Yang show that:

- For $n = 3$ and the Einstein tensor having rank at least 2, the approximate embedding $\mathbf{y}_0$ can be chosen so that the linearized system becomes either *symmetric hyperbolic* or *strictly hyperbolic*. They then show that any such system has a solution that satisfies smooth tame estimates.
- For $n = 3$ and $R(\mathbf{0}) \neq 0$, or $n = 4$ and $(R(\mathbf{0}), \nabla R(\mathbf{0}))$ in some dense open set, the approximate embedding $\mathbf{y}_0$ can be chosen so that this system has *real principal type*.

Nakamura-Maeda and Goodman-Yang then showed that any system of real principal type has a solution that satisfies smooth tame estimates. Proving these estimates requires the use of sophisticated microlocal analysis and Fourier integral operators.

**Our approach to a more straightforward proof:**

## Our approach to a more straightforward proof:

1. Show that, for $n = 3$ and $R(\mathbf{0}) \neq 0$, the approximate embedding $\mathbf{y}_0$ can be chosen so that the linearized system becomes *strongly symmetric positive* after a carefully chosen change of variables.

**Our approach to a more straightforward proof:**

1. Show that, for $n = 3$ and $R(\mathbf{0}) \neq 0$, the approximate embedding $\mathbf{y}_0$ can be chosen so that the linearized system becomes *strongly symmetric positive* after a carefully chosen change of variables.
2. Show that any such system has a solution that satisfies smooth tame estimates.

**Our approach to a more straightforward proof:**

1. Show that, for $n = 3$ and $R(\mathbf{0}) \neq 0$, the approximate embedding $\mathbf{y}_0$ can be chosen so that the linearized system becomes *strongly symmetric positive* after a carefully chosen change of variables.
2. Show that any such system has a solution that satisfies smooth tame estimates.

**Advantages:**

**Our approach to a more straightforward proof:**

1. Show that, for $n = 3$ and $R(\mathbf{0}) \neq 0$, the approximate embedding $\mathbf{y}_0$ can be chosen so that the linearized system becomes *strongly symmetric positive* after a carefully chosen change of variables.
2. Show that any such system has a solution that satisfies smooth tame estimates.

**Advantages:**

- Step (2) is fairly straightforward, requiring none of the sophisticated analysis needed for prior proofs.

**Our approach to a more straightforward proof:**

1. Show that, for $n = 3$ and $R(\mathbf{0}) \neq 0$, the approximate embedding $\mathbf{y}_0$ can be chosen so that the linearized system becomes *strongly symmetric positive* after a carefully chosen change of variables.
2. Show that any such system has a solution that satisfies smooth tame estimates.

**Advantages:**

- Step (2) is fairly straightforward, requiring none of the sophisticated analysis needed for prior proofs.
- Step (1) requires only linear algebra.

## Symmetric positive linear systems

## Symmetric positive linear systems

Friedrichs (1958) introduced the notion of a *symmetric positive* linear system of $s$ first order PDEs

$$A^i \partial_i \mathbf{v} + B\mathbf{v} = \mathbf{h} \tag{2}$$

for a function $\mathbf{v} : \mathbb{R}^n \to \mathbb{R}^s$, in order to handle some cases where the system does not fall into one of the standard types (elliptic, hyperbolic, parabolic).

## Symmetric positive linear systems

Friedrichs (1958) introduced the notion of a *symmetric positive* linear system of $s$ first order PDEs

$$A^i \partial_i \mathbf{v} + B\mathbf{v} = \mathbf{h} \tag{2}$$

for a function $\mathbf{v} : \mathbb{R}^n \to \mathbb{R}^s$, in order to handle some cases where the system does not fall into one of the standard types (elliptic, hyperbolic, parabolic).

The system (2) is called *symmetric* if the coefficient matrices $A^1, \ldots, A^n$ are symmetric $s \times s$ matrices.

Suppose that a symmetric system (2) is given on the closure of a bounded, open domain $\Omega \subset \mathbb{R}^n$ with piecewise smooth boundary $\partial\Omega$.

Suppose that a symmetric system (2) is given on the closure of a bounded, open domain $\Omega \subset \mathbb{R}^n$ with piecewise smooth boundary $\partial\Omega$.

In order to compute an $L^2(\bar{\Omega}, \mathbb{R}^s)$ estimate for a possible solution $\mathbf{v}(\mathbf{x})$, we might try the following:

Suppose that a symmetric system (2) is given on the closure of a bounded, open domain $\Omega \subset \mathbb{R}^n$ with piecewise smooth boundary $\partial\Omega$.

In order to compute an $L^2(\bar{\Omega}, \mathbb{R}^s)$ estimate for a possible solution $\mathbf{v}(\mathbf{x})$, we might try the following:

Multiply the system (2) on the left by $\mathbf{v}^\mathsf{T}$ to obtain the scalar equation

$$\mathbf{v}^\mathsf{T} A^i \, \partial_i \mathbf{v} + \mathbf{v}^\mathsf{T} B \, \mathbf{v} = \mathbf{v}^\mathsf{T} \mathbf{h}. \tag{3}$$

Suppose that a symmetric system (2) is given on the closure of a bounded, open domain $\Omega \subset \mathbb{R}^n$ with piecewise smooth boundary $\partial\Omega$.

In order to compute an $L^2(\bar{\Omega}, \mathbb{R}^s)$ estimate for a possible solution $\mathbf{v}(\mathbf{x})$, we might try the following:

Multiply the system (2) on the left by $\mathbf{v}^\mathsf{T}$ to obtain the scalar equation

$$\mathbf{v}^\mathsf{T} A^i \, \partial_i \mathbf{v} + \mathbf{v}^\mathsf{T} B \, \mathbf{v} = \mathbf{v}^\mathsf{T} \mathbf{h}. \tag{3}$$

After some straightforward manipulations using the product rule and taking into account the fact that the matrices $A^i$ are symmetric, this can be written as

$$\mathbf{v}^\mathsf{T} \left( B + B^\mathsf{T} - \partial_i A^i \right) \mathbf{v} = 2\mathbf{v}^\mathsf{T} \mathbf{h} - \partial_i \left( \mathbf{v}^\mathsf{T} A^i \mathbf{v} \right). \tag{4}$$

**Definition** (Friedrichs): The symmetric system (2) is called *symmetric positive* if the quadratic form $Q_0(\mathbf{x}) : \mathbb{R}^s \to \mathbb{R}$ defined by

$$Q_0(\xi) = \xi^{\mathsf{T}} \left( B + B^{\mathsf{T}} - \partial_i A^i \right) \xi$$

is positive definite for all $\mathbf{x} \in \bar{\Omega}$.

**Definition** (Friedrichs): The symmetric system (2) is called *symmetric positive* if the quadratic form $Q_0(\mathbf{x}) : \mathbb{R}^s \to \mathbb{R}$ defined by

$$Q_0(\xi) = \xi^{\mathsf{T}} \left( B + B^{\mathsf{T}} - \partial_i A^i \right) \xi$$

is positive definite for all $\mathbf{x} \in \bar{\Omega}$.

For a symmetric positive system, we have

$$\mathbf{v}^{\mathsf{T}} \left( B + B^{\mathsf{T}} - \partial_i A^i \right) \mathbf{v} \geq \lambda_0 |\mathbf{v}|^2$$

for some $\lambda_0 > 0$.

Therefore, it follows from (4) that

Therefore, it follows from (4) that

$$\lambda_0|\mathbf{v}|^2 \leq \mathbf{v}^{\mathsf{T}}\left(B + B^{\mathsf{T}} - \partial_i A^i\right)\mathbf{v}$$

Therefore, it follows from (4) that

$$\begin{aligned}\lambda_0|\mathbf{v}|^2 &\leq \mathbf{v}^\mathsf{T}\left(B + B^\mathsf{T} - \partial_i A^i\right)\mathbf{v}\\ &\leq 2\mathbf{v}^\mathsf{T}\mathbf{h} - \partial_i\left(\mathbf{v}^\mathsf{T} A^i \mathbf{v}\right)\end{aligned}$$

Therefore, it follows from (4) that

$$\begin{aligned}\lambda_0|\mathbf{v}|^2 &\leq \mathbf{v}^\mathsf{T}\Big(B + B^\mathsf{T} - \partial_i A^i\Big)\mathbf{v} \\ &\leq 2\mathbf{v}^\mathsf{T}\mathbf{h} - \partial_i\left(\mathbf{v}^\mathsf{T} A^i \mathbf{v}\right) \\ &\leq \frac{\lambda_0}{2}|\mathbf{v}|^2 + \frac{2}{\lambda_0}|\mathbf{h}|^2 - \partial_i\left(\mathbf{v}^\mathsf{T} A^i \mathbf{v}\right).\end{aligned}$$

Therefore, it follows from (4) that

$$\lambda_0|\mathbf{v}|^2 \le \mathbf{v}^\mathsf{T}\left(B + B^\mathsf{T} - \partial_i A^i\right)\mathbf{v}$$
$$\le 2\mathbf{v}^\mathsf{T}\mathbf{h} - \partial_i\left(\mathbf{v}^\mathsf{T}A^i\mathbf{v}\right)$$
$$\le \frac{\lambda_0}{2}|\mathbf{v}|^2 + \frac{2}{\lambda_0}|\mathbf{h}|^2 - \partial_i\left(\mathbf{v}^\mathsf{T}A^i\mathbf{v}\right).$$

Integrating over $\Omega$ and applying Stokes' theorem yields

$$\|\mathbf{v}\|_0^2 \le C_0\|\mathbf{h}\|_0^2 - \frac{2}{\lambda_0}\int_{\partial\Omega}(\mathbf{v}^\mathsf{T}\beta(\mathbf{x})\mathbf{v})\,dS,$$

where, for $\mathbf{x} \in \partial\Omega$, $\beta(\mathbf{x})$ is the *characteristic matrix*

$$\beta(\mathbf{x}) = \nu_i(\mathbf{x})A^i(\mathbf{x}).$$

**Definition:** Given a symmetric positive linear operator

$$P = A^i \partial_i + B$$

on the closure of a domain $\Omega \subset \mathbb{R}^n$, we call the domain *$P$-convex* for the system (2) if the characteristic matrix

$$\beta(\mathbf{x}) = \sum_{i=1}^{n} \nu_i(\mathbf{x}) A^i(\mathbf{x}),$$

where $\nu(\mathbf{x}) = (\nu_1(\mathbf{x}), \ldots, \nu_n(\mathbf{x}))$ denotes the outer unit normal vector to $\partial\Omega$ at $\mathbf{x} \in \partial\Omega$, is positive definite at each point $\mathbf{x} \in \partial\Omega$.

**Theorem** (Friedrichs, 1958): Suppose that the system (2) is symmetric positive on $\bar{\Omega}$ and that $\Omega$ is $P$-convex. Then the system (2) has a unique solution $\mathbf{v} \in L^2(\bar{\Omega}, \mathbb{R}^s)$. Moreover, we have a smooth tame estimate of the form

$$\|\mathbf{v}\|_0 \leq C_0 \|\mathbf{h}\|_0,$$

where the constant $C_0$ depends only on the minimum eigenvalue $\lambda_0$ of the quadratic form $Q_0$ on $\bar{\Omega}$.

**Example:** Consider the following ODE:

$$(x - x_0)u' + bu = h(x). \tag{5}$$

**Example:** Consider the following ODE:

$$(x - x_0)u' + bu = h(x). \tag{5}$$

It is straightforward to verify that (5) is symmetric positive if $b > \frac{1}{2}$, and an interval $\Omega = (x_1, x_2)$ is $P$-convex if and only if $x_0 \in (x_1, x_2)$, i.e., if and only if the regular singular point of this ODE lies in the domain.

**Example:** Consider the following ODE:

$$(x - x_0)u' + bu = h(x). \tag{5}$$

It is straightforward to verify that (5) is symmetric positive if $b > \frac{1}{2}$, and an interval $\Omega = (x_1, x_2)$ is $P$-convex if and only if $x_0 \in (x_1, x_2)$, i.e., if and only if the regular singular point of this ODE lies in the domain.

The general solution of (5) is

$$u(x) = \frac{1}{(x - x_0)^b} \int_{x_0}^{x} (y - x_0)^{b-1} h(y)\, dy + \frac{C}{(x - x_0)^b},$$

which is continuous at $x = x_0$ and satisfies the desired estimate if and only if $C = 0$.

Thus we see that:

Thus we see that:

- The $P$-convexity condition forces the uniqueness of a continuous solution of (5) on $\Omega$, *without* specifying any initial or boundary data for $u$.

Thus we see that:

- The $P$-convexity condition forces the uniqueness of a continuous solution of (5) on $\Omega$, *without* specifying any initial or boundary data for $u$.
- Symmetric positivity on a domain $\Omega$ does not necessarily guarantee the existence of a $P$-convex neighborhood of $\mathbf{x}_0 \in \Omega$.

For our purposes, Friedrich's theorem has two important shortcomings:

For our purposes, Friedrich's theorem has two important shortcomings:

1. Recall that our starting point will be an approximate local embedding $\mathbf{y}_0 : M \to \mathbb{R}^N$ that may be defined on an arbitrarily small neighborhood of a given point $\mathbf{x}_0 \in M$. So we have no way to guarantee that we have a $P$-convex domain for the linearized system.

For our purposes, Friedrich's theorem has two important shortcomings:

1. Recall that our starting point will be an approximate local embedding $\mathbf{y}_0 : M \to \mathbb{R}^N$ that may be defined on an arbitrarily small neighborhood of a given point $\mathbf{x}_0 \in M$. So we have no way to guarantee that we have a $P$-convex domain for the linearized system.
2. We need estimates for $\|\mathbf{v}\|_k$ for all $k \geq 0$, but even if the coefficients $A^i, B$ and the inhomogeneous term $\mathbf{h}$ are all $C^\infty$, Friedrichs's theorem does not guarantee any higher order regularity for the solution $\mathbf{v}$.

What happens if we try to compute a first-order estimate for the solution $\mathbf{v}$?

What happens if we try to compute a first-order estimate for the solution $\mathbf{v}$?

If we differentiate the system (2) and perform manipulations similar to those above, we obtain

$$\partial_j \mathbf{v}^\mathsf{T} \left( B + B^\mathsf{T} - \partial_i A^i \right) \partial_j \mathbf{v} + (\partial_j \mathbf{v}^\mathsf{T})(\partial_j A^i + \partial_i A^j)(\partial_i \mathbf{v}) \\ = 2\partial_j \mathbf{v}^\mathsf{T} \left( \partial_j \mathbf{h} - (\partial_j B)\mathbf{v} \right) - \partial_i \left( \partial_j \mathbf{v}^\mathsf{T} A^i \partial_j \mathbf{v} \right).$$

**Definition:** The symmetric system (2) is called *strongly symmetric positive* if the quadratic forms $Q_0(\mathbf{x}) : \mathbb{R}^s \to \mathbb{R}$ and $Q_1(\mathbf{x}) : \mathbb{R}^{ns} \to \mathbb{R}$ defined by

$$Q_0(\mathbf{x})(\xi) = \xi^\mathsf{T} \left( B + B^\mathsf{T} - \partial_i A^i \right) \xi,$$
$$Q_1(\mathbf{x})(\xi_1, \ldots, \xi_n) = \xi_j^\mathsf{T} \left( \partial_j A^i + \partial_i A^j \right) \xi_i$$

are positive definite for all $\mathbf{x} \in \bar{\Omega}$.

For a strongly symmetric positive system on the closure of a $P$-convex domain $\Omega$, a similar argument to that above yields a smooth tame first-order estimate of the form

$$\|\mathbf{v}\|_1 \leq C_1 \left( \|\mathbf{h}\|_1 + \|\mathbf{h}\|_0 \|B\|_{2+[\frac{n}{2}]} \right),$$

where the constant $C_1$ depends only on the minimum eigenvalues $\lambda_0, \lambda_1$ of the quadratic forms $Q_0, Q_1$ on $\bar{\Omega}$.

Perhaps surprisingly, it turns out that higher-order estimates require no further assumptions.

Perhaps surprisingly, it turns out that higher-order estimates require no further assumptions.

Successive differentiations of the system (2) lead to expressions of the form

$$\sum_{j_1,\dots,j_k=1}^{n} Q_0(\partial^k_{j_1,\dots,j_k}\mathbf{v}) \\ + k \sum_{j_1,\dots,j_{k-1}=1}^{n} Q_1(\partial^k_{j_1,\dots,j_{k-1},1}\mathbf{v},\dots,\partial^k_{j_1,\dots,j_{k-1},n}\mathbf{v}).$$

Perhaps surprisingly, it turns out that higher-order estimates require no further assumptions.

Successive differentiations of the system (2) lead to expressions of the form

$$\sum_{j_1,\ldots,j_k=1}^{n} Q_0(\partial^k_{j_1,\ldots,j_k}\mathbf{v}) \\ + k \sum_{j_1,\ldots,j_{k-1}=1}^{n} Q_1(\partial^k_{j_1,\ldots,j_{k-1},1}\mathbf{v},\ldots,\partial^k_{j_1,\ldots,j_{k-1},n}\mathbf{v}).$$

If the system (2) is strongly symmetric positive on the closure of a $P$-convex domain $\Omega$, then we can obtain smooth tame estimates for $\|\mathbf{v}\|_k$ for all $k \geq 0$. In particular, the solution $\mathbf{v}$ promised by Friedrichs's theorem is $C^\infty$.

By applying Nash-Moser, this leads to the following theorem for *nonlinear* systems, proven for real analytic systems by Moser (1966) and for $C^\infty$ systems by K. Tso (1992):

By applying Nash-Moser, this leads to the following theorem for *nonlinear* systems, proven for real analytic systems by Moser (1966) and for $C^\infty$ systems by K. Tso (1992):

**Theorem** (Tso): Let $\Phi : C^\infty(\bar{\Omega}, \mathbb{R}^s) \to C^\infty(\bar{\Omega}, \mathbb{R}^s)$ be a $C^\infty$, nonlinear first-order partial differential operator. Given a smooth function $\mathbf{f} : \bar{\Omega} \to \mathbb{R}^s$, consider the PDE system

$$\Phi(\mathbf{u}) = \mathbf{f}(\mathbf{x}). \tag{6}$$

Suppose that the linearization of $\Phi$ at any function in some $C^1$-neighborhood of $\mathbf{u}_0$ is strongly symmetric positive and that $\Omega$ is $P$-convex for the associated linear operators. Then there exist an integer $\beta$ and $\epsilon > 0$ such that, for any $C^\infty$ function $\mathbf{f} : \bar{\Omega} \to \mathbb{R}^s$ with $\|\Phi(\mathbf{u}_0) - \mathbf{f}\|_\beta < \epsilon$, there exists a $C^\infty$ solution $\mathbf{u} : \bar{\Omega} \to \mathbb{R}^s$ to the nonlinear system (6).

Unfortunately, Tso's theorem isn't quite enough for us; we need a *local* version that can be applied to an arbitrarily small neighborhood of a point $\mathbf{x}_0$, *without* the requirement of $P$-convexity.

**Theorem 1** (Chen, C—, Slemrod, Wang, Yang): Let $\Phi : C^\infty(\Omega, \mathbb{R}^s) \to C^\infty(\Omega, \mathbb{R}^s)$ be a $C^\infty$, nonlinear first-order partial differential operator. Given a smooth function $\mathbf{f} : \Omega \to \mathbb{R}^s$, consider the PDE system

$$\Phi(\mathbf{u}) = \mathbf{f}(\mathbf{x}). \tag{7}$$

Suppose that the linearization of $\Phi$ at any function in some $C^1$-neighborhood of $\mathbf{u}_0$ is strongly symmetric positive at some point $\mathbf{x}_0 \in \Omega$. Then there exist a neighborhood $\Omega_0 \subset \Omega$ of $\mathbf{x}_0$, an integer $\beta$ and $\epsilon > 0$ such that, for any $C^\infty$ function $\mathbf{f} : \bar{\Omega}_0 \to \mathbb{R}^s$ with $\|\Phi(\mathbf{u}_0) - \mathbf{f}\|_\beta < \epsilon$, there exists a $C^\infty$ solution $\mathbf{u} : \bar{\Omega}_0 \to \mathbb{R}^s$ to the restriction of the nonlinear system (7) to $\bar{\Omega}_0$.

## Outline of proof:

## Outline of proof:

WLOG, assume that $\mathbf{x}_0 = \mathbf{0}$. Write the Taylor expansions for the coefficients of the linearized system at $\mathbf{u}_0$:

$$B(\mathbf{x}) = \bar{B} + \hat{B}(\mathbf{x}), \qquad A^i(\mathbf{x}) = \bar{A}^i + \sum_{j=1}^{n} x^j \bar{A}^i_j + \hat{A}^i(\mathbf{x}).$$

## Outline of proof:

WLOG, assume that $\mathbf{x}_0 = \mathbf{0}$. Write the Taylor expansions for the coefficients of the linearized system at $\mathbf{u}_0$:

$$B(\mathbf{x}) = \bar{B} + \hat{B}(\mathbf{x}), \qquad A^i(\mathbf{x}) = \bar{A}^i + \sum_{j=1}^{n} x^j \bar{A}^i_j + \hat{A}^i(\mathbf{x}).$$

Strong symmetric positivity at $\mathbf{x} = \mathbf{0}$ is equivalent to the assumption that the quadratic forms $\bar{Q}_0 : \mathbb{R}^s \to \mathbb{R}$, $\bar{Q}_1 : \mathbb{R}^{ns} \to \mathbb{R}$ defined by

$$\bar{Q}_0(\xi) = \xi^\mathsf{T}\Big(\bar{B} + \bar{B}^\mathsf{T} - \sum_{i=1}^{n} \bar{A}^i_i\Big)\xi,$$

$$\bar{Q}_1(\xi_1, \ldots, \xi_n) = \sum_{i,j=1}^{n} \xi_j^\mathsf{T}\left(\bar{A}^i_j + \bar{A}^j_i\right)\xi_i$$

are positive definite.

**Step 1: Restriction to a small ball**

**Step 1: Restriction to a small ball**

Choose $r > 0$ so that $\bar{B}_r \subset \Omega$ and the restrictions of the remainder terms $\hat{B}(\mathbf{x})$ and $\hat{A}^i(\mathbf{x})$ to $\bar{B}_r$ are sufficiently small.

Restrict the system (7) to the closure of the domain $\Omega_0 = B_r$.

## Step 2: Extension to $\mathbb{R}^n$

**Step 2: Extension to $\mathbb{R}^n$**

We will apply the following variant of the Stein extension theorem:

**Theorem** (Stein): For any $r > 0$, there exists an extension operator $\mathcal{E}_r : L^1(\bar{B}_r) \to L^1(\mathbb{R}^n)$ and constants $M_{k,p}$, $1 \leq p \leq \infty,\ 0 \leq k < \infty$, such that, for all $f \in W^{k,p}(\bar{B}_r)$,

$$\|\mathcal{E}_r f\|_{k,p} \leq M_{k,p} \|f\|_{k,p}.$$

Moreover, the constants $M_{k,p}$ are independent of $r$.

Apply this theorem to the remainder terms $\hat{B}(\mathbf{x})$, $\hat{A}^i(\mathbf{x})$, and $\mathbf{h}(\mathbf{x})$ on $\bar{B}_r$.

Apply this theorem to the remainder terms $\hat{B}(\mathbf{x})$, $\hat{A}^i(\mathbf{x})$, and $\mathbf{h}(\mathbf{x})$ on $\bar{B}_r$.

This allows us to extend the system (7) on $\bar{B}_r$ to a new system

$$\tilde{A}^i\,\partial_i \mathbf{v} + \tilde{B}\,\mathbf{v} = \tilde{\mathbf{h}} \tag{8}$$

on all of $\mathbb{R}^n$, where

$$\begin{aligned}\tilde{B}(\mathbf{x}) &= \bar{B} + (\mathcal{E}_r \hat{B})(\mathbf{x}),\\ \tilde{A}^i(\mathbf{x}) &= \bar{A}^i + x^j\,\bar{A}^i_j + (\mathcal{E}_r \hat{A}^i)(\mathbf{x}),\\ \tilde{\mathbf{h}}(\mathbf{x}) &= (\mathcal{E}_r \mathbf{h})(\mathbf{x}).\end{aligned}$$

## Step 3: Restriction to a large ball

## Step 3: Restriction to a large ball

Consider the restriction of the extended system (8) to a large ball $\bar{B}_R$.

## Step 3: Restriction to a large ball

Consider the restriction of the extended system (8) to a large ball $\bar{B}_R$.

The normal vector to $\partial B_R$ is $\nu(\mathbf{x}) = \frac{1}{R}\mathbf{x}$.

## Step 3: Restriction to a large ball

Consider the restriction of the extended system (8) to a large ball $\bar{B}_R$.

The normal vector to $\partial B_R$ is $\nu(\mathbf{x}) = \frac{1}{R}\mathbf{x}$.

Therefore, the characteristic matrix for $\mathbf{x} \in \partial B_R$ is

$$\begin{aligned} \beta(\mathbf{x}) &= \nu_i(\mathbf{x})\tilde{A}^i(\mathbf{x}) = \tfrac{1}{R}x^i\tilde{A}^i(\mathbf{x}) \\ &= \tfrac{1}{R}\left(x^i\bar{A}^i + x^i x^j \bar{A}^i_j + x^i(\mathcal{E}_r\hat{A}^i)(\mathbf{x})\right) \\ &\approx \tfrac{1}{R}x^i x^j \bar{A}^i_j \end{aligned}$$

for large $R$.

**Proposition:** As $R \to \infty$, the quadratic form

$$Q_\beta(\mathbf{x})(\xi) = \xi^{\mathsf{T}} \beta(\mathbf{x}) \xi$$

defined by $\beta(\mathbf{x})$ is asymptotic to

$$\frac{1}{2R} \tilde{Q}_1(\mathbf{x})(x^1 \xi, \ldots, x^n \xi) \geq \frac{1}{2} \lambda_0 R |\xi|^2,$$

where $\lambda_0 > 0$ is the minimum eigenvalue of $\tilde{Q}_1(\mathbf{x})$ for $\mathbf{x} \in \mathbb{R}^n$.

**Proposition:** As $R \to \infty$, the quadratic form

$$Q_\beta(\mathbf{x})(\xi) = \xi^\mathsf{T}\beta(\mathbf{x})\xi$$

defined by $\beta(\mathbf{x})$ is asymptotic to

$$\frac{1}{2R}\tilde{Q}_1(\mathbf{x})(x^1\xi, \ldots, x^n\xi) \geq \frac{1}{2}\lambda_0 R|\xi|^2,$$

where $\lambda_0 > 0$ is the minimum eigenvalue of $\tilde{Q}_1(\mathbf{x})$ for $\mathbf{x} \in \mathbb{R}^n$.

It follows that, for sufficiently large $R$, the characteristic matrix $\beta(\mathbf{x})$ is positive definite for all $\mathbf{x} \in \partial B_R$. Therefore, $B_R$ is a $P$-convex domain for the extended system (8).

## Step 4: Smooth tame estimates

## Step 4: Smooth tame estimates

Tso's theorem guarantees the existence of smooth tame estimates for the extended function $\tilde{\mathbf{v}} : \bar{B}_R \to \mathbb{R}^s$, in terms of the Sobolev norms of the function $\tilde{A}^i, \tilde{B}$, and $\tilde{\mathbf{h}}$ on $\bar{B}_R$.

## Step 4: Smooth tame estimates

Tso's theorem guarantees the existence of smooth tame estimates for the extended function $\tilde{\mathbf{v}} : \bar{B}_R \to \mathbb{R}^s$, in terms of the Sobolev norms of the function $\tilde{A}^i, \tilde{B}$, and $\tilde{\mathbf{h}}$ on $\bar{B}_R$.

These, in turn, are bounded in terms of the Sobolev norms of the functions $A^i, B$, and $\mathbf{h}$ on $\bar{B}_r$.

## Step 4: Smooth tame estimates

Tso's theorem guarantees the existence of smooth tame estimates for the extended function $\tilde{\mathbf{v}} : \bar{B}_R \to \mathbb{R}^s$, in terms of the Sobolev norms of the function $\tilde{A}^i, \tilde{B}$, and $\tilde{\mathbf{h}}$ on $\bar{B}_R$.

These, in turn, are bounded in terms of the Sobolev norms of the functions $A^i, B$, and $\mathbf{h}$ on $\bar{B}_r$.

Thus we obtain estimates for the solution $\mathbf{v} : \bar{B}_r \to \mathbb{R}^s$ as follows:

$$\begin{aligned}\|\mathbf{v}\|_k \leq \|\tilde{\mathbf{v}}\|_k &\leq C_k \left( \|\tilde{\mathbf{h}}\|_k + \|\tilde{\mathbf{h}}\|_0 \|\mathbf{u}_0\|_{k+3+[\frac{n}{2}]} \right) \\ &\leq \tilde{C}_k M_{k,2} \left( \|\mathbf{h}\|_k + \|\mathbf{h}\|_0 \|\mathbf{u}_0\|_{k+3+[\frac{n}{2}]} \right).\end{aligned}$$

## Step 4: Smooth tame estimates

Tso's theorem guarantees the existence of smooth tame estimates for the extended function $\tilde{\mathbf{v}} : \bar{B}_R \to \mathbb{R}^s$, in terms of the Sobolev norms of the function $\tilde{A}^i, \tilde{B}$, and $\tilde{\mathbf{h}}$ on $\bar{B}_R$.

These, in turn, are bounded in terms of the Sobolev norms of the functions $A^i, B$, and $\mathbf{h}$ on $\bar{B}_r$.

Thus we obtain estimates for the solution $\mathbf{v} : \bar{B}_r \to \mathbb{R}^s$ as follows:

$$\begin{aligned} \|\mathbf{v}\|_k \le \|\tilde{\mathbf{v}}\|_k &\le C_k \left( \|\tilde{\mathbf{h}}\|_k + \|\tilde{\mathbf{h}}\|_0 \|\mathbf{u}_0\|_{k+3+[\frac{n}{2}]} \right) \\ &\le \tilde{C}_k M_{k,2} \left( \|\mathbf{h}\|_k + \|\mathbf{h}\|_0 \|\mathbf{u}_0\|_{k+3+[\frac{n}{2}]} \right). \end{aligned}$$

Theorem 1 then follows from Nash-Moser.

**And now, back to isometric embedding!**

**And now, back to isometric embedding!**

**Theorem 2** (Chen, C—, Slemrod, Wang, Yang): Let $(M, g)$ be a $C^\infty$ Riemannian manifold of dimension $n = 2$ or $n = 3$; let $N = \frac{1}{2}n(n+1)$; let $\mathbf{x}_0 \in M$, and suppose that the Riemann curvature tensor $R(\mathbf{x}_0)$ is nonzero. Then there exists a neighborhood $\Omega \subset M$ of $\mathbf{x}_0$ for which there is a $C^\infty$ isometric embedding $\mathbf{y} : \Omega \to \mathbb{R}^N$.

**Strategy for the proof:**

**Strategy for the proof:**

- Choose local coordinates $\mathbf{x} = (x^1, \ldots, x^n)$ on $M$ so that $\mathbf{x}_0 = \mathbf{0}$. Given a $C^\infty$ metric $g$ on a neighborhood $\Omega$ of $\mathbf{x} = \mathbf{0}$, choose a real analytic metric $\bar{g}$ on $\Omega$ that agrees with $g$ to sufficiently high order at $\mathbf{x} = \mathbf{0}$. By the Cartan-Janet theorem, there exists a real analytic isometric embedding (possibly on a smaller neighborhood) $\mathbf{y}_0 : \Omega \subset M \to \mathbb{R}^N$ of $(\Omega, \bar{g})$ into $\mathbb{R}^N$.

**Strategy for the proof:**

- Choose local coordinates $\mathbf{x} = (x^1, \ldots, x^n)$ on $M$ so that $\mathbf{x}_0 = \mathbf{0}$. Given a $C^\infty$ metric $g$ on a neighborhood $\Omega$ of $\mathbf{x} = \mathbf{0}$, choose a real analytic metric $\bar{g}$ on $\Omega$ that agrees with $g$ to sufficiently high order at $\mathbf{x} = \mathbf{0}$. By the Cartan-Janet theorem, there exists a real analytic isometric embedding (possibly on a smaller neighborhood) $\mathbf{y}_0 : \Omega \subset M \to \mathbb{R}^N$ of $(\Omega, \bar{g})$ into $\mathbb{R}^N$.
- The linearization of the isometric embedding system at $\mathbf{y}_0$ is a first-order PDE system of $N$ equations for the unknown function $\mathbf{v} : \Omega \to \mathbb{R}^N$. This system decomposes into a system of $n$ first-order PDEs for the tangential components of $\mathbf{v}$, together with $(N - n)$ equations that determine the normal components of $\mathbf{v}$ algebraically in terms of the tangential components.

- We show that, under the hypotheses of Theorem 2, the embedding $\mathbf{y}_0$ can be chosen so that the tangential subsystem becomes strongly symmetric positive after a fairly simple, but carefully chosen, change of variables. Consequently, it follows from the argument given in the proof of Theorem 1 that the tangential components of $\mathbf{v}$ satisfy the smooth tame estimates required for Nash-Moser.

- We show that, under the hypotheses of Theorem 2, the embedding $\mathbf{y}_0$ can be chosen so that the tangential subsystem becomes strongly symmetric positive after a fairly simple, but carefully chosen, change of variables. Consequently, it follows from the argument given in the proof of Theorem 1 that the tangential components of $\mathbf{v}$ satisfy the smooth tame estimates required for Nash-Moser.
- The remaining algebraic equations will imply the necessary estimates for the normal components of $\mathbf{v}$. Theorem 2 then follows directly from the Nash-Moser implicit function theorem .

## The linearized isometric embedding equations

## The linearized isometric embedding equations

Set $\mathbf{y}(\mathbf{x}) = \mathbf{y}_0(\mathbf{x}) + \mathbf{v}(\mathbf{x})$, where $\mathbf{v}(\mathbf{x})$ is assumed to be small, and substitute into the isometric embedding system to obtain:

$$\partial_i \mathbf{y}_0 \cdot \partial_j \mathbf{y}_0 + (\partial_i \mathbf{y}_0 \cdot \partial_j \mathbf{v} + \partial_j \mathbf{y}_0 \cdot \partial_i \mathbf{v}) + \partial_i \mathbf{v} \cdot \partial_j \mathbf{v} = g_{ij}.$$

## The linearized isometric embedding equations

Set $\mathbf{y}(\mathbf{x}) = \mathbf{y}_0(\mathbf{x}) + \mathbf{v}(\mathbf{x})$, where $\mathbf{v}(\mathbf{x})$ is assumed to be small, and substitute into the isometric embedding system to obtain:

$$\partial_i \mathbf{y}_0 \cdot \partial_j \mathbf{y}_0 + (\partial_i \mathbf{y}_0 \cdot \partial_j \mathbf{v} + \partial_j \mathbf{y}_0 \cdot \partial_i \mathbf{v}) + \partial_i \mathbf{v} \cdot \partial_j \mathbf{v} = g_{ij}.$$

The linearization of the system at $\mathbf{y}_0$ is obtained by eliminating the terms that are quadratic in $\mathbf{v}$:

$$\partial_i \mathbf{y}_0 \cdot \partial_j \mathbf{v} + \partial_j \mathbf{y}_0 \cdot \partial_i \mathbf{v} = h_{ij}, \qquad 1 \leq i, j \leq n, \tag{9}$$

where $h_{ij} = g_{ij} - \bar{g}_{ij}$.

The linearized system

$$\partial_i \mathbf{y}_0 \cdot \partial_j \mathbf{v} + \partial_j \mathbf{y}_0 \cdot \partial_i \mathbf{v} = h_{ij}, \qquad 1 \leq i, j \leq n \tag{9}$$

can be reformulated as a system of $n$ linear PDEs for the $n$ tangential components of $\mathbf{v}$, together with a system of $(N - n)$ algebraic equations for the normal components:

The linearized system

$$\partial_i \mathbf{y}_0 \cdot \partial_j \mathbf{v} + \partial_j \mathbf{y}_0 \cdot \partial_i \mathbf{v} = h_{ij}, \qquad 1 \leq i, j \leq n \tag{9}$$

can be reformulated as a system of $n$ linear PDEs for the $n$ tangential components of $\mathbf{v}$, together with a system of $(N - n)$ algebraic equations for the normal components:

For $i = 1, \ldots, n$, let $\bar{v}_i(\mathbf{x})$ be the function

$$\bar{v}_i(\mathbf{x}) = \partial_i \mathbf{y}_0(\mathbf{x}) \cdot \mathbf{v}(\mathbf{x}).$$

The linearized system

$$\partial_i \mathbf{y}_0 \cdot \partial_j \mathbf{v} + \partial_j \mathbf{y}_0 \cdot \partial_i \mathbf{v} = h_{ij}, \qquad 1 \leq i, j \leq n \tag{9}$$

can be reformulated as a system of $n$ linear PDEs for the $n$ tangential components of $\mathbf{v}$, together with a system of $(N-n)$ algebraic equations for the normal components:

For $i = 1, \ldots, n$, let $\bar{v}_i(\mathbf{x})$ be the function

$$\bar{v}_i(\mathbf{x}) = \partial_i \mathbf{y}_0(\mathbf{x}) \cdot \mathbf{v}(\mathbf{x}).$$

Then the system (9) can be written as

$$\partial_i \bar{v}_j + \partial_j \bar{v}_i - 2\partial^2_{ij} \mathbf{y}_0 \cdot \mathbf{v} = h_{ij}, \qquad 1 \leq i, j \leq n. \tag{10}$$

## Brief digression: The second fundamental form

## Brief digression: The second fundamental form

Since $\mathbf{y}_0$ is an embedding, the tangent vectors $\{\partial_1 \mathbf{y}_0, \ldots, \partial_n \mathbf{y}_0\}$ are linearly independent and span an $n$-dimensional subspace $T_{\mathbf{x}} \subset \mathbb{R}^N$.

## Brief digression: The second fundamental form

Since $\mathbf{y}_0$ is an embedding, the tangent vectors $\{\partial_1 \mathbf{y}_0, \ldots, \partial_n \mathbf{y}_0\}$ are linearly independent and span an $n$-dimensional subspace $T_{\mathbf{x}} \subset \mathbb{R}^N$.

We can therefore decompose the second derivatives $\partial^2_{ij}\mathbf{y}_0$ as follows:

$$\partial^2_{ij}\mathbf{y}_0 = \Gamma^k_{ij}\partial_k \mathbf{y}_0 + H_{ij},$$

where, for each $1 \le i, j \le n$, the vector-valued function $H_{ij} = H_{ji} : \Omega \to \mathbb{R}^N$ satisfies $H_{ij} \cdot \partial_k \mathbf{y}_0 = 0$ for $1 \le k \le n$.

## Brief digression: The second fundamental form

Since $\mathbf{y}_0$ is an embedding, the tangent vectors $\{\partial_1 \mathbf{y}_0, \ldots, \partial_n \mathbf{y}_0\}$ are linearly independent and span an $n$-dimensional subspace $T_{\mathbf{x}} \subset \mathbb{R}^N$.

We can therefore decompose the second derivatives $\partial^2_{ij} \mathbf{y}_0$ as follows:

$$\partial^2_{ij} \mathbf{y}_0 = \Gamma^k_{ij} \partial_k \mathbf{y}_0 + H_{ij},$$

where, for each $1 \leq i, j \leq n$, the vector-valued function $H_{ij} = H_{ji} : \Omega \to \mathbb{R}^N$ satisfies $H_{ij} \cdot \partial_k \mathbf{y}_0 = 0$ for $1 \leq k \leq n$.

The quadratic form $H_{ij} dx^i dx^j$ is the *second fundamental form* of the embedding $\mathbf{y}_0$.

Let $(\mathbf{e}_{n+1}, \ldots, \mathbf{e}_N)$ be a smoothly varying orthonormal basis for the normal bundle of the embedded submanifold $\mathbf{y}_0(\Omega) \subset \mathbb{R}^N$. Then we can write the second fundamental form of $\mathbf{y}_0$ as

$$H_{ij} dx^i \circ dx^j = \mathbf{e}_\alpha \otimes H^\alpha_{ij} dx^i \circ dx^j$$

for scalar-valued functions $H^\alpha_{ij}$ on $\Omega$.

Let $(\mathbf{e}_{n+1}, \ldots, \mathbf{e}_N)$ be a smoothly varying orthonormal basis for the normal bundle of the embedded submanifold $\mathbf{y}_0(\Omega) \subset \mathbb{R}^N$. Then we can write the second fundamental form of $\mathbf{y}_0$ as

$$H_{ij} dx^i \circ dx^j = \mathbf{e}_\alpha \otimes H^\alpha_{ij} dx^i \circ dx^j$$

for scalar-valued functions $H^\alpha_{ij}$ on $\Omega$.

**Definition:** The embedding $\mathbf{y}_0 : \Omega \to \mathbb{R}^N$ is *nondegenerate* if, for each $\mathbf{x} \in \Omega$, the $\frac{1}{2}n(n-1)$ matrices

$$H^\alpha(\mathbf{x}) = [H^\alpha_{ij}(\mathbf{x})]$$

are linearly independent, or equivalently, if the vectors $H_{ij}(\mathbf{x})$ span the normal space $T^\perp_{\mathbf{x}} \subset \mathbb{R}^N$.

Let $(\mathbf{e}_{n+1}, \ldots, \mathbf{e}_N)$ be a smoothly varying orthonormal basis for the normal bundle of the embedded submanifold $\mathbf{y}_0(\Omega) \subset \mathbb{R}^N$. Then we can write the second fundamental form of $\mathbf{y}_0$ as

$$H_{ij} dx^i \circ dx^j = \mathbf{e}_\alpha \otimes H^\alpha_{ij} dx^i \circ dx^j$$

for scalar-valued functions $H^\alpha_{ij}$ on $\Omega$.

**Definition:** The embedding $\mathbf{y}_0 : \Omega \to \mathbb{R}^N$ is *nondegenerate* if, for each $\mathbf{x} \in \Omega$, the $\frac{1}{2}n(n-1)$ matrices

$$H^\alpha(\mathbf{x}) = [H^\alpha_{ij}(\mathbf{x})]$$

are linearly independent, or equivalently, if the vectors $H_{ij}(\mathbf{x})$ span the normal space $T^\perp_{\mathbf{x}} \subset \mathbb{R}^N$.

We will assume henceforth that $\mathbf{y}_0$ is nondegenerate.

Let $\mathrm{II}_{\mathbf{x}}$ denote the span of the matrices $H^{\alpha}$.

Let $\mathrm{II}_{\mathbf{x}}$ denote the span of the matrices $H^{\alpha}$.

**Definition:** The *annihilator* $\mathrm{II}_{\mathbf{x}}^{\perp}$ of $\mathrm{II}_{\mathbf{x}}$ is the subspace of the space $\mathcal{S}_n$ of symmetric $n \times n$ matrices defined by

$$\mathrm{II}_{\mathbf{x}}^{\perp} = \{A \in \mathcal{S}_n \ : \ \langle A, H^{\alpha} \rangle = 0, \quad n+1 \leq \alpha \leq N\},$$

where

$$\langle A, H^{\alpha} \rangle = A^{ij} H^{\alpha}_{ij}.$$

The fact that $\mathbf{y}_0$ is nondegenerate implies that:

The fact that $\mathbf{y}_0$ is nondegenerate implies that:

- $\dim(\mathrm{II}_{\mathbf{x}}) = \frac{1}{2}n(n-1)$ for all $\mathbf{x} \in \Omega$.

The fact that $\mathbf{y}_0$ is nondegenerate implies that:

- $\dim(\mathrm{II}_{\mathbf{x}}) = \frac{1}{2}n(n-1)$ for all $\mathbf{x} \in \Omega$.
- $\dim(\mathrm{II}_{\mathbf{x}}^{\perp}) = n$ for all $\mathbf{x} \in \Omega$.

The fact that $\mathbf{y}_0$ is nondegenerate implies that:

- $\dim(\mathrm{II}_{\mathbf{x}}) = \frac{1}{2}n(n-1)$ for all $\mathbf{x} \in \Omega$.
- $\dim(\mathrm{II}_{\mathbf{x}}^{\perp}) = n$ for all $\mathbf{x} \in \Omega$.

Let $A^1, \ldots, A^n : \Omega \to \mathcal{S}_n$ be chosen so that for each $\mathbf{x} \in \Omega$, the matrices $A^1(\mathbf{x}), \ldots, A^n(\mathbf{x})$ comprise a basis of $\mathrm{II}_{\mathbf{x}}^{\perp}$.

Write $A^k = [A^{kij}]$, where $A^{kij} = A^{kji}$.

The fact that $\mathbf{y}_0$ is nondegenerate implies that:

- $\dim(\mathrm{II}_{\mathbf{x}}) = \frac{1}{2}n(n-1)$ for all $\mathbf{x} \in \Omega$.
- $\dim(\mathrm{II}_{\mathbf{x}}^{\perp}) = n$ for all $\mathbf{x} \in \Omega$.

Let $A^1, \dots, A^n : \Omega \to \mathcal{S}_n$ be chosen so that for each $\mathbf{x} \in \Omega$, the matrices $A^1(\mathbf{x}), \dots, A^n(\mathbf{x})$ comprise a basis of $\mathrm{II}_{\mathbf{x}}^{\perp}$.

Write $A^k = [A^{kij}]$, where $A^{kij} = A^{kji}$.

**And now, back to the linearized isometric embedding system...**

The decomposition

$$\partial^2_{ij}\mathbf{y}_0 = \Gamma^k_{ij}\partial_k\mathbf{y}_0 + H_{ij}$$

allows us to write the linearized system

$$\partial_i\bar{v}_j + \partial_j\bar{v}_i - 2\partial^2_{ij}\mathbf{y}_0\cdot\mathbf{v} = h_{ij}, \qquad 1\le i,j\le n \tag{10}$$

as

$$\partial_i\bar{v}_j + \partial_j\bar{v}_i - 2(\Gamma^k_{ij}\bar{v}_k + H_{ij}\cdot\mathbf{v}) = h_{ij}, \qquad 1\le i,j\le n. \tag{11}$$

The decomposition

$$\partial^2_{ij}\mathbf{y}_0 = \Gamma^k_{ij}\partial_k\mathbf{y}_0 + H_{ij}$$

allows us to write the linearized system

$$\partial_i\bar{v}_j + \partial_j\bar{v}_i - 2\partial^2_{ij}\mathbf{y}_0\cdot\mathbf{v} = h_{ij}, \qquad 1\le i,j\le n \tag{10}$$

as

$$\partial_i\bar{v}_j + \partial_j\bar{v}_i - 2(\Gamma^k_{ij}\bar{v}_k + H_{ij}\cdot\mathbf{v}) = h_{ij}, \qquad 1\le i,j\le n. \tag{11}$$

By pairing each of the (symmetric!) matrices $A^k$ with the system (11), we obtain a system of $n$ equations for the functions $\bar{\mathbf{v}} = (\bar{v}_1,\dots,\bar{v}_n)$:

$$A^{kij}(\partial_i\bar{v}_j - \Gamma^\ell_{ij}\bar{v}_\ell) = \tfrac{1}{2}A^{kij}h_{ij}, \qquad 1\le k\le n. \tag{12}$$

Now, suppose that $\bar{\mathbf{v}}(\mathbf{x})$ is any solution of the reduced linear system (12). The nondegeneracy assumption guarantees that the algebraic equations

$$\begin{aligned} \mathbf{v} \cdot \partial_i \mathbf{y}_0 &= \bar{v}_i, && 1 \le i \le n, \\ -2\mathbf{v} \cdot H_{ij} &= h_{ij} - \partial_i \bar{v}_j - \partial_j \bar{v}_i + 2\Gamma_{ij}^k \bar{v}_k, && 1 \le i, j \le n \end{aligned}$$

can be solved uniquely for $\mathbf{v}(\mathbf{x})$.

Now, suppose that $\bar{\mathbf{v}}(\mathbf{x})$ is any solution of the reduced linear system (12). The nondegeneracy assumption guarantees that the algebraic equations

$$\begin{aligned} \mathbf{v} \cdot \partial_i \mathbf{y}_0 &= \bar{v}_i, & 1 \leq i \leq n, \\ -2\mathbf{v} \cdot H_{ij} &= h_{ij} - \partial_i \bar{v}_j - \partial_j \bar{v}_i + 2\Gamma_{ij}^k \bar{v}_k, & 1 \leq i, j \leq n \end{aligned}$$

can be solved uniquely for $\mathbf{v}(\mathbf{x})$.

So it suffices to show that we can arrange for the reduced system (12) to be strongly symmetric positive; this will imply all the necessary estimates required for the Nash-Moser Theorem.

Now, suppose that $\bar{\mathbf{v}}(\mathbf{x})$ is any solution of the reduced linear system (12). The nondegeneracy assumption guarantees that the algebraic equations

$$\begin{aligned} \mathbf{v} \cdot \partial_i \mathbf{y}_0 &= \bar{v}_i, && 1 \le i \le n, \\ -2\mathbf{v} \cdot H_{ij} &= h_{ij} - \partial_i \bar{v}_j - \partial_j \bar{v}_i + 2\Gamma^k_{ij} \bar{v}_k, && 1 \le i, j \le n \end{aligned}$$

can be solved uniquely for $\mathbf{v}(\mathbf{x})$.

So it suffices to show that we can arrange for the reduced system (12) to be strongly symmetric positive; this will imply all the necessary estimates required for the Nash-Moser Theorem.

**This is the hard part!**

We can write the reduced system (12) in the form

$$\bar{A}^i \partial_i \bar{\mathbf{v}} + B\bar{\mathbf{v}} = \mathbf{h},$$

where

$$\bar{A}^i = [A^{kij}] = \begin{bmatrix} A^{1i1} & \cdots & A^{1in} \\ \vdots & & \vdots \\ A^{ni1} & \cdots & A^{nin} \end{bmatrix},$$

$$B = [B^{kj}] = [-A^{k\ell m}\Gamma^j_{\ell m}], \qquad \mathbf{h} = [\tfrac{1}{2}A^{k\ell m}h_{\ell m}],$$

$$A^{kij}H^\alpha_{ij} = 0, \quad 1 \le k \le n, \quad n+1 \le \alpha \le N.$$

We can write the reduced system (12) in the form

$$\bar{A}^i \partial_i \bar{\mathbf{v}} + B\bar{\mathbf{v}} = \mathbf{h},$$

where

$$\bar{A}^i = [A^{kij}] = \begin{bmatrix} A^{1i1} & \cdots & A^{1in} \\ \vdots & & \vdots \\ A^{ni1} & \cdots & A^{nin} \end{bmatrix},$$

$$B = [B^{kj}] = [-A^{k\ell m}\Gamma^j_{\ell m}], \qquad \mathbf{h} = [\tfrac{1}{2}A^{k\ell m}h_{\ell m}],$$

$$A^{kij}H^\alpha_{ij} = 0, \quad 1 \leq k \leq n, \quad n+1 \leq \alpha \leq N.$$

**GOAL:** Show that we can choose the approximate embedding $\mathbf{y}_0 : \Omega \to \mathbb{R}^N$ so that this system becomes strongly symmetric positive at $\mathbf{x} = \mathbf{0}$. Then the local isometric embedding theorem follows from Theorem 1.

First, let $\bar{g}$ be a real analytic metric that agrees with $g$ to high order at $\mathbf{x}_0$. Cartan-Janet guarantees existence of an approximate local nondegenerate isometric embedding $\mathbf{y}_0 : \Omega \to \mathbb{R}^N$ for $\bar{g}$.

First, let $\bar{g}$ be a real analytic metric that agrees with $g$ to high order at $\mathbf{x}_0$. Cartan-Janet guarantees existence of an approximate local nondegenerate isometric embedding $\mathbf{y}_0 : \Omega \to \mathbb{R}^N$ for $\bar{g}$.

The Riemann curvature tensors for $g$ and $\bar{g}$ and their first derivatives agree at $\mathbf{x} = \mathbf{0}$, so we need not distinguish between them.

First, let $\bar{g}$ be a real analytic metric that agrees with $g$ to high order at $\mathbf{x}_0$. Cartan-Janet guarantees existence of an approximate local nondegenerate isometric embedding $\mathbf{y}_0 : \Omega \to \mathbb{R}^N$ for $\bar{g}$.

The Riemann curvature tensors for $g$ and $\bar{g}$ and their first derivatives agree at $\mathbf{x} = \mathbf{0}$, so we need not distinguish between them.

Choose a local coordinate system $\mathbf{x} = (x^1, \ldots, x^n)$ based at $\mathbf{x} = \mathbf{0}$ that is *normal* with respect to the metric $g$, i.e., $\Gamma^k_{ij}(\mathbf{0}) = 0$ for $1 \leq i, j, k \leq n$.

First, let $\bar{g}$ be a real analytic metric that agrees with $g$ to high order at $\mathbf{x}_0$. Cartan-Janet guarantees existence of an approximate local nondegenerate isometric embedding $\mathbf{y}_0 : \Omega \to \mathbb{R}^N$ for $\bar{g}$.

The Riemann curvature tensors for $g$ and $\bar{g}$ and their first derivatives agree at $\mathbf{x} = \mathbf{0}$, so we need not distinguish between them.

Choose a local coordinate system $\mathbf{x} = (x^1, \ldots, x^n)$ based at $\mathbf{x} = \mathbf{0}$ that is *normal* with respect to the metric $g$, i.e., $\Gamma^k_{ij}(\mathbf{0}) = 0$ for $1 \leq i, j, k \leq n$.

Choose the basis $(\mathbf{e}_{n+1}, \ldots, \mathbf{e}_N)$ for the normal bundle so that

$$\nabla^{\perp}_{\mathbf{w}} \mathbf{e}_{\alpha}(\mathbf{0}) = \mathbf{0}$$

for $n + 1 \leq \alpha \leq N$ and all $\mathbf{w} \in T_{\mathbf{0}} M$.

## First issue: Symmetrization

## First issue: Symmetrization

Recall that we have $A^{kij} = A^{kji}$, but no other symmetry assumptions among the $A^{kij}$. Thus the coefficient matrices $\bar{A}^i = [A^{kij}]$ are *not* necessarily symmetric.

## First issue: Symmetrization

Recall that we have $A^{kij} = A^{kji}$, but no other symmetry assumptions among the $A^{kij}$. Thus the coefficient matrices $\bar{A}^i = [A^{kij}]$ are *not* necessarily symmetric.

**Question:** Can we always choose a basis $A^1, \ldots, A^n$ for $\mathrm{II}_{\mathbf{x}}^{\perp}$ for which the $A^{kij}$ are symmetric in all their indices, and hence the matrices $\bar{A}^i$ are symmetric?

Equivalently, can we always find a fully symmetric solution $A^{kij}$ to the annihilator equations

$$A^{kij} H^{\alpha}_{ij} = 0?$$

**First issue: Symmetrization**

Recall that we have $A^{kij} = A^{kji}$, but no other symmetry assumptions among the $A^{kij}$. Thus the coefficient matrices $\bar{A}^i = [A^{kij}]$ are *not* necessarily symmetric.

**Question:** Can we always choose a basis $A^1, \ldots, A^n$ for $\mathrm{II}_{\mathbf{x}}^{\perp}$ for which the $A^{kij}$ are symmetric in all their indices, and hence the matrices $\bar{A}^i$ are symmetric?

Equivalently, can we always find a fully symmetric solution $A^{kij}$ to the annihilator equations

$$A^{kij} H^{\alpha}_{ij} = 0?$$

**Answer:** Yes if $n = 2$ or $n = 3$; No if $n \geq 4$.

## Proof:

**Proof:**

- When $n = 2, N = 3$, the annihilator equations are a system of 2 homogeneous linear equations for the 4 components of a symmetric tensor $A^{kij}\mathbf{e}_i \circ \mathbf{e}_j \circ \mathbf{e}_k$, so there is a 2-dimensional solution space.

**Proof:**

- When $n = 2, N = 3$, the annihilator equations are a system of 2 homogeneous linear equations for the 4 components of a symmetric tensor $A^{kij}\mathbf{e}_i \circ \mathbf{e}_j \circ \mathbf{e}_k$, so there is a 2-dimensional solution space.
- When $n = 3, N = 6$, the annihilator equations are a system of 9 homogeneous linear equations for the 10 components of a symmetric tensor $A^{kij}\mathbf{e}_i \circ \mathbf{e}_j \circ \mathbf{e}_k$, so there is a 1-dimensional solution space.

**Proof:**

- When $n = 2, N = 3$, the annihilator equations are a system of 2 homogeneous linear equations for the 4 components of a symmetric tensor $A^{kij}\mathbf{e}_i \circ \mathbf{e}_j \circ \mathbf{e}_k$, so there is a 2-dimensional solution space.
- When $n = 3, N = 6$, the annihilator equations are a system of 9 homogeneous linear equations for the 10 components of a symmetric tensor $A^{kij}\mathbf{e}_i \circ \mathbf{e}_j \circ \mathbf{e}_k$, so there is a 1-dimensional solution space.
- When $n \geq 4$, there are more equations than components of a symmetric tensor, and so generically there are no solutions. (e.g., for $n = 4, N = 10$, there are 24 equations for 20 components.)

**Proof:**

- When $n = 2, N = 3$, the annihilator equations are a system of 2 homogeneous linear equations for the 4 components of a symmetric tensor $A^{kij}\mathbf{e}_i \circ \mathbf{e}_j \circ \mathbf{e}_k$, so there is a 2-dimensional solution space.
- When $n = 3, N = 6$, the annihilator equations are a system of 9 homogeneous linear equations for the 10 components of a symmetric tensor $A^{kij}\mathbf{e}_i \circ \mathbf{e}_j \circ \mathbf{e}_k$, so there is a 1-dimensional solution space.
- When $n \geq 4$, there are more equations than components of a symmetric tensor, and so generically there are no solutions. (e.g., for $n = 4, N = 10$, there are 24 equations for 20 components.)

Henceforth, we will assume that $n \leq 3$ and the $A^{kij}$ are fully symmetric.

## Main issue: Strong symmetric positivity

## Main issue: Strong symmetric positivity

Henceforth, we will only be concerned with the values of $H_{ij}^{\alpha}, A^{kij}$, and their derivatives at $\mathbf{x} = \mathbf{0}$. We will denote the derivatives by

$$h_{ijk}^{\alpha} = \partial_k H_{ij}^{\alpha}, \qquad a_{\ell}^{kij} = \partial_{\ell} A^{kij},$$

and we will write

$$h_k^{\alpha} = \left[h_{ijk}^{\alpha}\right], \qquad a_{\ell}^{k} = \left[a_{\ell}^{kij}\right].$$

## Main issue: Strong symmetric positivity

Henceforth, we will only be concerned with the values of $H^{\alpha}_{ij}, A^{kij}$, and their derivatives at $\mathbf{x} = \mathbf{0}$. We will denote the derivatives by

$$h^{\alpha}_{ijk} = \partial_k H^{\alpha}_{ij}, \qquad a^{kij}_{\ell} = \partial_{\ell} A^{kij},$$

and we will write

$$h^{\alpha}_{k} = \left[h^{\alpha}_{ijk}\right], \qquad a^{k}_{\ell} = \left[a^{kij}_{\ell}\right].$$

The Cartan-Janet theorem implies that these values may be chosen arbitrarily, subject only to the nondegeneracy condition on the $H^{\alpha}_{ij}$ and the following constraints:

## Gauss equations and their first derivatives:

$$\sum_{\alpha=n+1}^{N} (H_{ik}^{\alpha} H_{j\ell}^{\alpha} - H_{i\ell}^{\alpha} H_{jk}^{\alpha}) = R_{ijk\ell};$$

$$\sum_{\alpha=n+1}^{N} (H_{ik}^{\alpha} h_{j\ell m}^{\alpha} + H_{j\ell}^{\alpha} h_{ikm}^{\alpha} - H_{i\ell}^{\alpha} h_{jkm}^{\alpha} - H_{jk}^{\alpha} h_{i\ell m}^{\alpha}) = \partial_m R_{ijk\ell};$$

**Gauss equations and their first derivatives:**

$$\sum_{\alpha=n+1}^{N} (H_{ik}^{\alpha} H_{j\ell}^{\alpha} - H_{i\ell}^{\alpha} H_{jk}^{\alpha}) = R_{ijk\ell};$$

$$\sum_{\alpha=n+1}^{N} (H_{ik}^{\alpha} h_{j\ell m}^{\alpha} + H_{j\ell}^{\alpha} h_{ikm}^{\alpha} - H_{i\ell}^{\alpha} h_{jkm}^{\alpha} - H_{jk}^{\alpha} h_{i\ell m}^{\alpha}) = \partial_m R_{ijk\ell};$$

**Codazzi equations:**

$$h_{ijk}^{\alpha} = h_{ikj}^{\alpha} = h_{jik}^{\alpha};$$

**Gauss equations and their first derivatives:**

$$\sum_{\alpha=n+1}^{N} (H^\alpha_{ik} H^\alpha_{j\ell} - H^\alpha_{i\ell} H^\alpha_{jk}) = R_{ijk\ell};$$

$$\sum_{\alpha=n+1}^{N} (H^\alpha_{ik} h^\alpha_{j\ell m} + H^\alpha_{j\ell} h^\alpha_{ikm} - H^\alpha_{i\ell} h^\alpha_{jkm} - H^\alpha_{jk} h^\alpha_{i\ell m}) = \partial_m R_{ijk\ell};$$

**Codazzi equations:**

$$h^\alpha_{ijk} = h^\alpha_{ikj} = h^\alpha_{jik};$$

**Annihilator equations and their first derivatives:**

$$A^{kij} H^\alpha_{ij} = 0;$$

$$A^{kij} h^\alpha_{ij\ell} + H^\alpha_{ij} a^{kij}_\ell = 0.$$

From the normal coordinates condition, we have

$$B(\mathbf{0}) = \mathbf{0}.$$

From the normal coordinates condition, we have

$$B(\mathbf{0}) = \mathbf{0}.$$

Thus, the quadratic forms $\bar{Q}_0 : \mathbb{R}^n \to \mathbb{R}$ and $\bar{Q}_1 : \mathbb{R}^{n^2} \to \mathbb{R}$ are given by

$$\bar{Q}_0 = -a_i^i, \qquad \bar{Q}_1 = \begin{bmatrix} 2a_1^1 & \cdots & a_n^1 + a_1^n \\ \vdots & \cdots & \vdots \\ a_n^1 + a_1^n & \cdots & 2a_n^n \end{bmatrix}.$$

From the normal coordinates condition, we have

$$B(\mathbf{0}) = \mathbf{0}.$$

Thus, the quadratic forms $\bar{Q}_0 : \mathbb{R}^n \to \mathbb{R}$ and $\bar{Q}_1 : \mathbb{R}^{n^2} \to \mathbb{R}$ are given by

$$\bar{Q}_0 = -a^i_i, \qquad \bar{Q}_1 = \begin{bmatrix} 2a^1_1 & \cdots & a^1_n + a^n_1 \\ \vdots & \cdots & \vdots \\ a^1_n + a^n_1 & \cdots & 2a^n_n \end{bmatrix}.$$

From this, strong symmetric positivity appears impossible.

From the normal coordinates condition, we have

$$B(\mathbf{0}) = \mathbf{0}.$$

Thus, the quadratic forms $\bar{Q}_0 : \mathbb{R}^n \to \mathbb{R}$ and $\bar{Q}_1 : \mathbb{R}^{n^2} \to \mathbb{R}$ are given by

$$\bar{Q}_0 = -a_i^i, \qquad \bar{Q}_1 = \begin{bmatrix} 2a_1^1 & \cdots & a_n^1 + a_1^n \\ \vdots & \cdots & \vdots \\ a_n^1 + a_1^n & \cdots & 2a_n^n \end{bmatrix}.$$

From this, strong symmetric positivity appears impossible.

Amazingly, a change of variables may save the day!

**Lemma:** Under a change of variables of the form

$$x^i = \bar{x}^i + \tfrac{1}{2} c^i_{jk} \bar{x}^j \bar{x}^k, \qquad \bar{\mathbf{v}} = (I + \bar{x}^i S_i)\bar{\mathbf{w}}, \tag{13}$$

where $c^i_{jk} = c^i_{kj} \in \mathbb{R}$ and $S_1, \ldots, S_n$ are constant $n \times n$ matrices, the symmetric linear system (12) is transformed to a symmetric system

$$\tilde{A}^i \partial_i \bar{\mathbf{w}} + \tilde{B} \bar{\mathbf{w}} = \tilde{\mathbf{h}},$$

with

$$\begin{aligned} \tilde{A}^i &= A^i + \bar{x}^k \left( S_k^\mathsf{T} A^i + A^i S_k - c^i_{jk} A^j \right) + O(\bar{x}^2), \\ \tilde{B} &= B + A^i S_i + O(\bar{x}). \end{aligned}$$

**Lemma:** Under a change of variables of the form

$$x^i = \bar{x}^i + \tfrac{1}{2} c^i_{jk} \bar{x}^j \bar{x}^k, \qquad \bar{\mathbf{v}} = (I + \bar{x}^i S_i)\bar{\mathbf{w}}, \tag{13}$$

where $c^i_{jk} = c^i_{kj} \in \mathbb{R}$ and $S_1, \ldots, S_n$ are constant $n \times n$ matrices, the symmetric linear system (12) is transformed to a symmetric system

$$\tilde{A}^i \partial_i \bar{\mathbf{w}} + \tilde{B} \bar{\mathbf{w}} = \tilde{\mathbf{h}},$$

with

$$\tilde{A}^i = A^i + \bar{x}^k \left( S_k^\mathsf{T} A^i + A^i S_k - c^i_{jk} A^j \right) + O(\bar{x}^2),$$
$$\tilde{B} = B + A^i S_i + O(\bar{x}).$$

So even if $B(\mathbf{0}) = \mathbf{0}$—which makes strong symmetric positivity impossible—the same may not be true of $\tilde{B}(\mathbf{0})$ if the matrices $S_i$ are chosen carefully.

**Proof:** Straightforward chain rule slog.

**Proof:** Straightforward chain rule slog.

The associated quadratic form $\tilde{\bar{Q}}_0$ for the transformed system is given by

$$\begin{aligned}\tilde{\bar{Q}}_0 &= -a^i_i + c^i_{ij} A^j \\ &= \bar{Q}_0 + c^i_{ij} A^j,\end{aligned} \tag{14}$$

and the $(i, j)$th block of $\tilde{\bar{Q}}_1$ is given by

$$\begin{aligned}(\tilde{\bar{Q}}_1)_{ij} &= (a^i_j + a^j_i) - (c^i_{jk} + c^j_{ik})A^k + (S_i^\mathsf{T} A^j + A^j S_i) + (S_j^\mathsf{T} A^i + A^i S_j) \\ &= (\bar{Q}_1)_{ij} - (c^i_{jk} + c^j_{ik})A^k + (S_i^\mathsf{T} A^j + A^j S_i) + (S_j^\mathsf{T} A^i + A^i S_j).\end{aligned} \tag{15}$$

**Theorem** $2'$ (Chen, C—, Slemrod, Wang, Yang): Suppose that either $n = 2$ and $K(\mathbf{0}) \neq 0$, or $n = 3$ and $R(\mathbf{0}) \neq 0$. Then there exists a neighborhood $\Omega \subset M$ of $\mathbf{x} = \mathbf{0}$ and an approximate embedding $\mathbf{y}_0 : \Omega \to \mathbb{R}^N$ such that the linearized isometric embedding system can be transformed to a strongly symmetric positive system in a neighborhood of $\mathbf{x} = \mathbf{0}$ via a change of variables of the form (13).

**Theorem** $2'$ (Chen, C—, Slemrod, Wang, Yang): Suppose that either $n = 2$ and $K(\mathbf{0}) \neq 0$, or $n = 3$ and $R(\mathbf{0}) \neq 0$. Then there exists a neighborhood $\Omega \subset M$ of $\mathbf{x} = \mathbf{0}$ and an approximate embedding $\mathbf{y}_0 : \Omega \to \mathbb{R}^N$ such that the linearized isometric embedding system can be transformed to a strongly symmetric positive system in a neighborhood of $\mathbf{x} = \mathbf{0}$ via a change of variables of the form (13).

The existence of local isometric embeddings then follows from Theorem 1.

**Outline of Proof:**

**Outline of Proof:**

**Step 1:** Given any nonzero $R$, choose nondegenerate $H_{ij}^{\alpha}$ subject to the Gauss equations

$$\sum_{\alpha=n+1}^{N} (H_{ik}^{\alpha} H_{j\ell}^{\alpha} - H_{i\ell}^{\alpha} H_{jk}^{\alpha}) = R_{ijk\ell},$$

and fully symmetric $A^{kij}$ subject to the annihilator equations

$$A^{kij} H_{ij}^{\alpha} = 0.$$

**Step 2:** Choose $\lambda, \mu > 0$, set

$$\tilde{\bar{Q}}_0 = \lambda I_{n\times n}, \qquad \tilde{\bar{Q}}_1 = \mu I_{n^2\times n^2},$$

and solve as many of the equations

$$\tilde{\bar{Q}}_0 = -a_i^i + c_{ij}^i A^j, \tag{14}$$

$$(\tilde{\bar{Q}}_1)_{ij} = (a_j^i + a_i^j) - (c_{jk}^i + c_{ik}^j)A^k + (S_i^\top A^j + A^j S_i) + (S_j^\top A^i + A^i S_j). \tag{15}$$

as possible for a subset of the $c_{jk}^i$ and the entries of $S_i$.

**Step 2:** Choose $\lambda, \mu > 0$, set

$$\tilde{\bar{Q}}_0 = \lambda I_{n\times n}, \qquad \tilde{\bar{Q}}_1 = \mu I_{n^2\times n^2},$$

and solve as many of the equations

$$\tilde{\bar{Q}}_0 = -a^i_i + c^i_{ij} A^j, \tag{14}$$

$$(\tilde{\bar{Q}}_1)_{ij} = (a^i_j + a^j_i) - (c^i_{jk} + c^j_{ik})A^k + (S_i^\top A^j + A^j S_i) + (S_j^\top A^i + A^i S_j). \tag{15}$$

as possible for a subset of the $c^i_{jk}$ and the entries of $S_i$.

The remaining equations determine an affine subspace $\mathcal{A}$ of "admissible" values for $(a^{kij}_\ell)$.

**Step 3:** Find the values of $(h^\alpha_{ijk})$ that satisfy the derivatives of the annihilator equations

$$A^{kij} h^\alpha_{ij\ell} + H^\alpha_{ij} a^{kij}_\ell = 0$$

for some $(a^{kij}_\ell) \in \mathcal{A}$.

**Step 3:** Find the values of $(h^\alpha_{ijk})$ that satisfy the derivatives of the annihilator equations

$$A^{kij} h^\alpha_{ij\ell} + H^\alpha_{ij} a^{kij}_\ell = 0$$

for some $(a^{kij}_\ell) \in \mathcal{A}$.

These values determine an affine subspace $\mathcal{H}$ of "admissible" values for $(h^\alpha_{ijk})$.

**Step 4:** Show that all possible values of $(\partial_m R_{ijk\ell})$ may be obtained as the right-hand sides of the derivatives of the Gauss equations

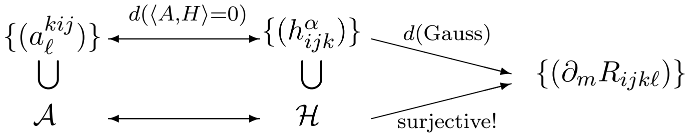
$$\sum_{\alpha=n+1}^{N} (H^{\alpha}_{ik} h^{\alpha}_{j\ell m} + H^{\alpha}_{j\ell} h^{\alpha}_{ikm} - H^{\alpha}_{i\ell} h^{\alpha}_{jkm} - H^{\alpha}_{jk} h^{\alpha}_{i\ell m}) = \partial_m R_{ijk\ell}$$

for some $(h^{\alpha}_{ijk}) \in \mathcal{H}$.

**Step 4:** Show that all possible values of $(\partial_m R_{ijk\ell})$ may be obtained as the right-hand sides of the derivatives of the Gauss equations

$$\sum_{\alpha=n+1}^{N} (H^\alpha_{ik} h^\alpha_{j\ell m} + H^\alpha_{j\ell} h^\alpha_{ikm} - H^\alpha_{i\ell} h^\alpha_{jkm} - H^\alpha_{jk} h^\alpha_{i\ell m}) = \partial_m R_{ijk\ell}$$

for some $(h^\alpha_{ijk}) \in \mathcal{H}$.



**Conclusion:** for any nonzero $R$ and any values of $\partial_m R$, there exist values of $H^\alpha_{ij}, A^{kij}, h^\alpha_{ijk}, a^{kij}_\ell$ that satisfy all necessary constraints, and for which there exists a change of variables of the form (13) that renders the linearized isometric embedding system strongly symmetric positive.

**Conclusion:** for any nonzero $R$ and any values of $\partial_m R$, there exist values of $H_{ij}^{\alpha}, A^{kij}, h_{ijk}^{\alpha}, a_{\ell}^{kij}$ that satisfy all necessary constraints, and for which there exists a change of variables of the form (13) that renders the linearized isometric embedding system strongly symmetric positive.

This completes the proof of Theorem $2'$.

**Details for $n = 2$**

## Details for $n = 2$

When $n = 2$, there is only one second fundamental form matrix $H^3$. According to the Gauss equations, we may choose

$$H^3 = \begin{bmatrix} K & 0 \\ 0 & 1 \end{bmatrix}.$$

## Details for $n = 2$

When $n = 2$, there is only one second fundamental form matrix $H^3$. According to the Gauss equations, we may choose

$$H^3 = \begin{bmatrix} K & 0 \\ 0 & 1 \end{bmatrix}.$$

Then, according to the annihilator equations, we may choose

$$A^1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad A^2 = \begin{bmatrix} 1 & 0 \\ 0 & -K \end{bmatrix}.$$

For any fixed $\lambda, \mu > 0$, the equations

$$\tilde{\tilde{Q}}_0 = \lambda I_{2\times 2}, \qquad \tilde{\tilde{Q}}_1 = \mu I_{4\times 4}$$

can be solved for $c^i_{jk}$ and $S_i = [s^{jk}_i]$ if and only if

$$(a^{122}_1 + a^{222}_2 + \lambda) + K(a^{111}_1 + a^{112}_2 + \lambda) = 0. \tag{16}$$

(This solution makes use of the assumption that $K \neq 0$.)

For any fixed $\lambda, \mu > 0$, the equations

$$\tilde{\tilde{Q}}_0 = \lambda I_{2\times 2}, \qquad \tilde{\tilde{Q}}_1 = \mu I_{4\times 4}$$

can be solved for $c^i_{jk}$ and $S_i = [s_i^{jk}]$ if and only if

$$(a_1^{122} + a_2^{222} + \lambda) + K(a_1^{111} + a_2^{112} + \lambda) = 0. \tag{16}$$

(This solution makes use of the assumption that $K \neq 0$.)

Thus, $\mathcal{A}$ is the 7-dimensional affine subspace of the 8-dimensional space of $(a_\ell^{kij})$ values defined by equation (16).

Now consider the derivatives of the annihilator equations, which may be written in matrix form as

$$\langle A^k, h^3_\ell \rangle + \langle H^3, a^k_\ell \rangle = 0.$$

Now consider the derivatives of the annihilator equations, which may be written in matrix form as

$$\langle A^k, h^3_\ell \rangle + \langle H^3, a^k_\ell \rangle = 0.$$

The defining equation (16) for $\mathcal{A}$ is equivalent to

$$\langle H^3, a^1_1 + a^2_2 \rangle = -(K+1)\lambda,$$

which holds if and only if

$$\langle A^1, h^3_1 \rangle + \langle A^2, h^3_2 \rangle = -\langle H^3, a^1_1 + a^2_2 \rangle = (K+1)\lambda,$$

or, equivalently,

$$3h^3_{112} - Kh^3_{222} = (K+1)\lambda. \tag{17}$$

Now consider the derivatives of the annihilator equations, which may be written in matrix form as

$$\langle A^k, h^3_\ell \rangle + \langle H^3, a^k_\ell \rangle = 0.$$

The defining equation (16) for $\mathcal{A}$ is equivalent to

$$\langle H^3, a^1_1 + a^2_2 \rangle = -(K+1)\lambda,$$

which holds if and only if

$$\langle A^1, h^3_1 \rangle + \langle A^2, h^3_2 \rangle = -\langle H^3, a^1_1 + a^2_2 \rangle = (K+1)\lambda,$$

or, equivalently,

$$3h^3_{112} - Kh^3_{222} = (K+1)\lambda. \tag{17}$$

Thus, $\mathcal{H}$ is the 3-dimensional affine subspace of the 4-dimensional space of $(h^3_{ijk})$ values defined by equation (17).

Finally, consider the derivatives of the Gauss equations, which can be written as

$$Kh^3_{122} + h^3_{111} = k_1,$$
$$Kh^3_{222} + h^3_{112} = k_2.$$

The values of $h^3_{ijk}$ may be chosen arbitrarily, subject only to the condition

$$3h^3_{112} - Kh^3_{222} = (K+1)\lambda; \quad (17)$$

therefore, any given values of $k_1$ and $k_2$ may be realized by an appropriate choice of $h^3_{ijk} \in \mathcal{H}$.

The reasoning in the $n = 3$ case is exactly the same—but the linear algebra is a lot messier!