Monotonicity Formulas and Boundary Obstacle Problems

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Thank you for the invitation!

Obstacle problems play a pervasive role in the applied sciences, from temperature control to linear elasticity, from fluid dynamics to financial mathematics. In this talk I will describe how seemingly different phenomena can be expressed in terms of the same mathematical model of obstacle type, as well as some recent developments in the regularity theory for solutions and their free boundaries.

In particular, I would like to focus on one of the central tools in the regularity theory, namely families of monotonicity formulas.

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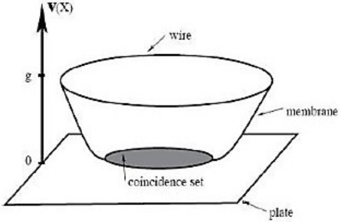

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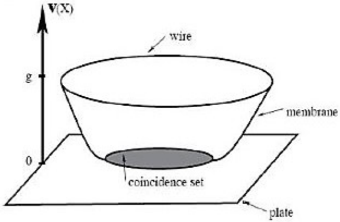

Applications include the study of fluid filtration in porous media, constrained heating, elasto-plasticity, control theory, and financial mathematics.

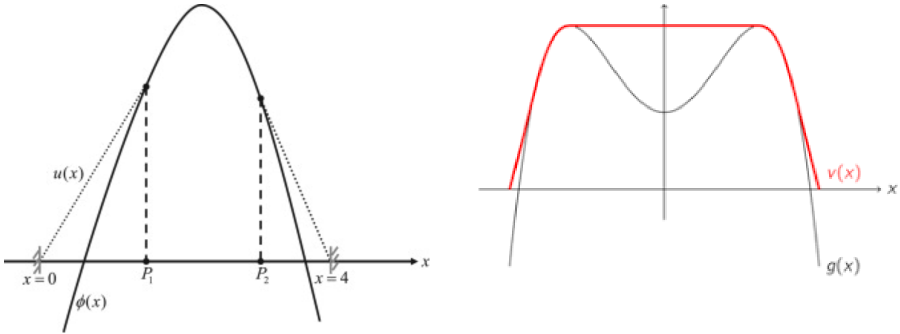

Figure: Examples of 1-dimensional obstacle problems

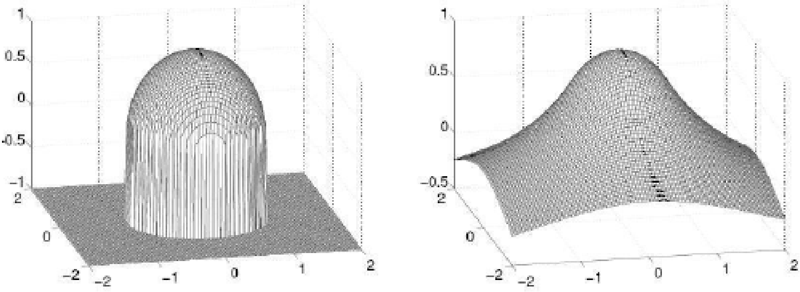

Figure: A two-dimensional obstacle problem: the obstacle on the left, the solution on the right

Here $\Omega = [-2,2] \times [-2,2]$, and the obstacle

$$\psi(x,y) = \begin{cases} \sqrt{1 - x^2 - y^2}, & \text{if } x^2 + y^2 \leq 1 \\ -1, & \text{elsewhere} \end{cases}$$

Mathematically, the obstacle problem consists of studying the properties of minimizers of the Dirichlet integral

$$J(u) = \int_D |\nabla u|^2 \, dx$$

in a domain $D \subset \mathbb{R}^n$, among all configurations $u(x)$ (representing the vertical displacement of the membrane) with prescribed boundary values $u|_{\partial D} = f(x)$, and constrained to remain above the obstacle $\varphi(x)$.

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The solution breaks down into a region where the solution is equal to the obstacle function, known as the *coincidence set*, and a region where the solution is above the obstacle. The interface between the two regions $\partial\{u > \varphi\}$ is the so-called *free boundary*.

Variational inequalities

The theory of Variational Inequalities (V.I.) was born in the early sixties. Its “founding fathers” were G. Stampacchia and G. Fichera.
Stampacchia was motivated by a problem in potential theory, whereas Fichera was motivated by a question in mechanics (more on this later...)

V.I. have stimulated new and deep results in PDEs (regularity theory for non linear equations, free boundary problems, etc.), and have found applications in a wide range of fields (engineering, optimization and control, physics, etc.)

The Obstacle Problem as a V.I.

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The obstacle problem can be reformulated as a variational inequality on a Hilbert space. In fact, solving the obstacle problem is equivalent to seeking a function $u \in K = \{v \in W^{1,2}(D) \mid v|_{\partial D} = f(x),\ v \geq \varphi\}$ such that

$$\int_D \nabla u \cdot \nabla (v - u)\, dx \ \geq \ 0 \qquad \text{for all } v \in K.$$

Variational arguments show that the solution to the obstacle problem is harmonic away from the contact set

$$\Delta u = 0 \qquad \text{in } \{u > \varphi\},$$

and that it is superharmonic on the contact set

$$\Delta u \leq 0 \qquad \text{in } \{u = \varphi\}.$$

Hence, the solution is a superharmonic function.

Questions:

1. How regular is the function u?
2. What are the geometric properties of the coincidence set? Is the free boundary a regular surface?

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The study of the classical obstacle problem, initiated in the 60's with the pioneering works of G. Stampacchia, H. Lewy, J. L. Lions, has led to beautiful and deep developments in calculus of variations and geometric partial differential equations. The crowning achievement has been the development, due to L. Caffarelli, of the theory of free boundaries.

Optimal regularity of the solution The solution to the obstacle problem is $C^{1,1}$ (i.e. it has bounded second derivatives) when the obstacle itself has such regularity (Frehse, 1972). In general, the second derivatives of the solutions are discontinuous across the free boundary.

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Coupling these results, we have that the free boundary is smooth at the so-called *regular* point.

The Signorini Problem

A problem in linear elasticity, first proposed by Signorini in 1959, was one of the driving forces in the study of V.I. In its original formulation, it consists of finding the elastic equilibrium configuration of an anisotropic non-homogeneous elastic body, resting on a rigid frictionless surface and subject only to its mass forces.

The existence and uniqueness of solutions was proved by Fichera in 1963.

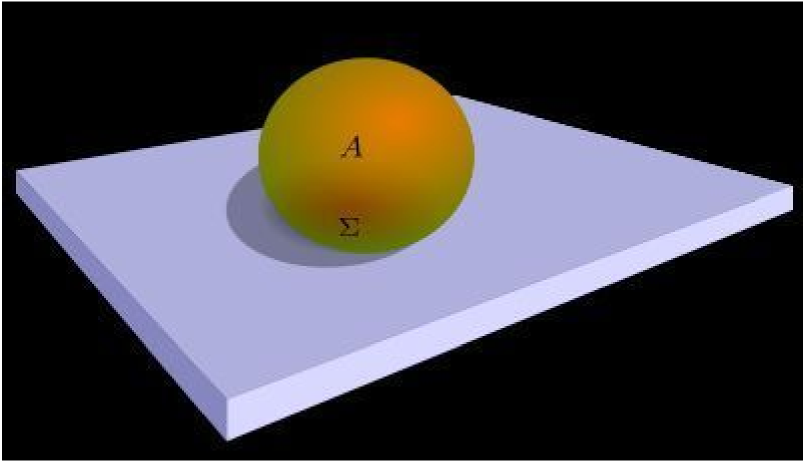
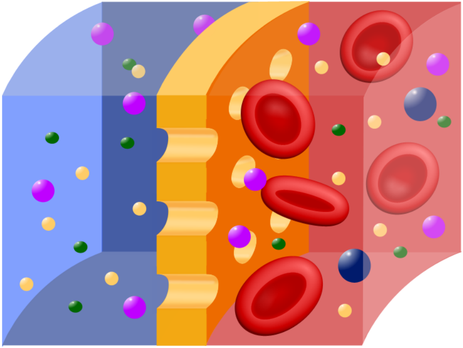


Figure: What will be the equilibrium configuration of an elastic body resting on a rigid frictionless plane?

Other applications include optimal control of temperature across a surface, in the modeling of semipermeable membranes where some saline concentration can flow through the membrane only in one direction, and financial math (when the random variation of underlying asset changes in a discontinuous fashion, as a Levi process).

Semipermeable Membranes and Osmosis



Picture Source: Wikipedia

- **Semipermeable membrane** is a membrane that is permeable only for a certain type of molecules (*solvents*) and blocks other molecules (*solutes*).

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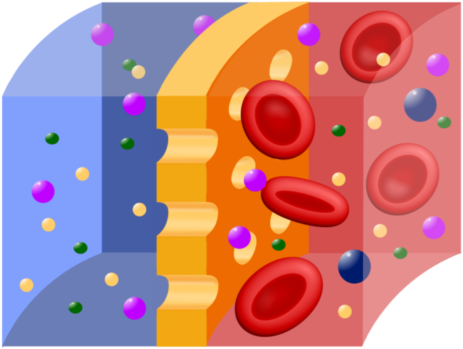

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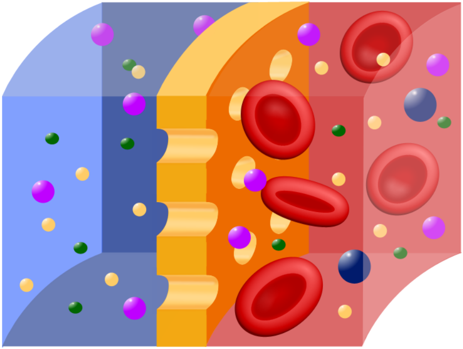

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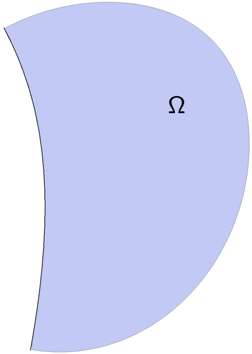
- The flow occurs in one direction. The flow continues until a sufficient pressure builds up on the other side of the membrane (to compensate for osmotic pressure), which then shuts the flow. This process is known as **osmosis**.

Mathematical Formulation

- Given open $\Omega \subset \mathbb{R}^n$



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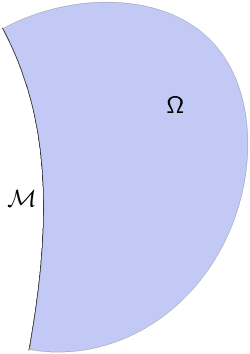

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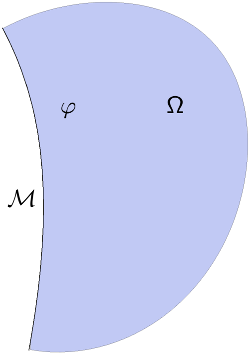

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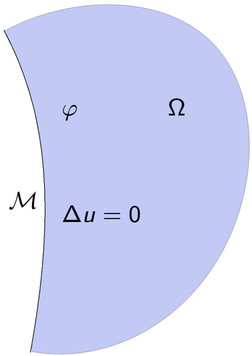
- $u : \Omega = \to \mathbb{R}$ *pressure* of the chemical solution, that satisfies the equation

$$\Delta u = 0 \quad \text{in } \Omega$$



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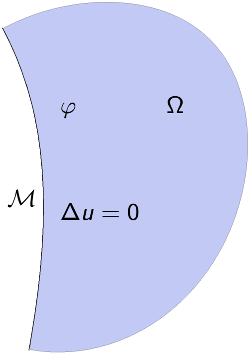

- On $\mathcal{M}$ we have the following boundary conditions (*finite permeability*)

$$\begin{aligned} u > \varphi \quad &\Rightarrow \quad \partial_\nu u = 0 && \textbf{(no flow)} \\ u \leq \varphi \quad &\Rightarrow \quad \partial_\nu u = \lambda(u - \varphi) && \textbf{(flow)} \end{aligned}$$

- Letting $\lambda \to \infty$ we obtain the following conditions on $\mathcal{M}$ (*infinite permeability*)

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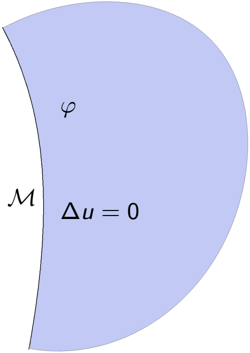

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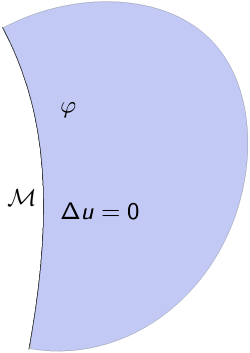

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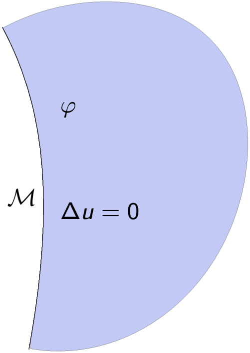
- Since u should stay above φ on $\mathcal{M}$, φ is known as the thin obstacle. The problem is also known as the Thin Obstacle Problem.



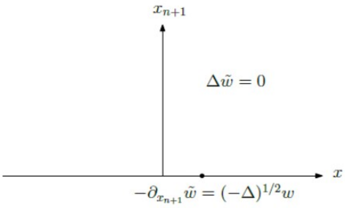
Connection with the Fractional Laplacian

An alternative interpretation of the thin obstacle problem is given as a "standard" obstacle problem for the *fractional Laplacian*

$$(-\Delta)^s u(x) = C(n,s) \lim_{\varepsilon \to 0} \int_{\mathbb{R}^n \setminus B_\varepsilon(x)} \frac{u(x) - u(y)}{|x-y|^{n+2s}}\, dy,$$

where $C(n,s)$ is a dimensional constant. The connection in the case $s = 1/2$ comes through the Dirichlet-to-Neumann map.

If u_0 is a solution of the obstacle problem for $(-\Delta)^{1/2}$, then its harmonic extension to $\mathbb{R}^n \times (0, +\infty)$ solves the corresponding Signorini problem, and viceversa. Therefore, the two problems are equivalent and any regularity result for one of them can be carried to the other one.



Caffarelli and Silvestre have extended this characterization, showing that there exists a PDE realization of $(-\Delta)^s$ for every $s \in (0,1)$, $s \neq \frac{1}{2}$. (*more on this later...*)

Variational Formulation

Let Ω be a domain in $\mathbb{R}^n$ and $\mathcal{M}$ a smooth $(n-1)$-dimensional manifold in $\mathbb{R}^n$ that divides Ω into two parts: Ω_+ and Ω_-. For given functions $\varphi : \mathcal{M} \to \mathbb{R}$ and $g : \partial\Omega \to \mathbb{R}$ satisfying $g > \varphi$ on $\mathcal{M} \cap \partial\Omega$, the thin obstacle problem consists of minimizing the Dirichlet integral

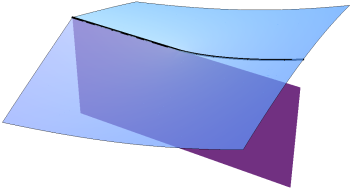
$$D_\Omega(u) = \int_\Omega |\nabla u|^2 dx$$

on the closed convex set

$$\mathfrak{K} = \{u \in W^{1,2}(\Omega) : u = g \text{ on } \partial\Omega,\ u \geq \varphi \text{ on } \mathcal{M} \cap \Omega\}.$$

This is equivalent to solving in $\mathfrak{K}$ the V.I.

$$\int_\Omega \nabla u \nabla (v-u) \geq 0 \quad \text{for every } v \in \mathfrak{K}.$$



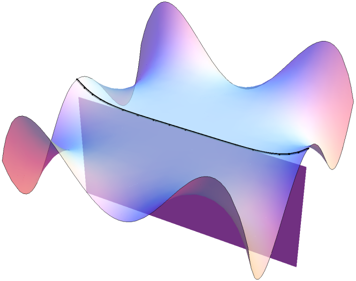


Figure: Graphs of $\mathrm{Re}(x_1 + i\,x_2)^{3/2}$ and $\mathrm{Re}(x_1 + i\,x_2)^6$

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The main goals are still to understand the properties of the coincidence set $\Lambda(u) := \{x \in \mathcal{M} : u = \varphi\}$ and its boundary (in the relative topology of $\mathcal{M}$) $\Gamma(u) := \partial_{\mathcal{M}}\Lambda(u)$, i.e., the free boundary. In order to do so, one needs to establish the optimal regularity of the solution across the free boundary.

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When $\mathcal{M}$ and φ are smooth, Caffarelli proved in 1979 that the minimizer u in the thin obstacle problem is of class $C^{1,\alpha}_{\mathrm{loc}}(\Omega_\pm \cup \mathcal{M})$.

Normalization

Simplifying assumptions:

1. Vanishing thin obstacle φ.
2. The manifold $\mathcal{M}$ is a flat portion of the boundary of the relevant domain: $\mathcal{M}=\mathbb{R}^{n-1} \times \{0\}$. In this case the thin obstacle problem is known as the Signorini problem.

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Since we are interested in properties of minimizers near free boundary points, after translation, rotation and scaling arguments we may consider a function u defined in the upper half-ball $B_1^+ := B_1 \cap \mathbb{R}^n_+$ satisfying

$$\Delta u = 0 \quad \text{in } B_1^+ \tag{0.1}$$

$$u \geq 0, \quad -\partial_{x_n} u \geq 0, \quad u\,\partial_{x_n} u = 0 \quad \text{on } B_1' \tag{0.2}$$

$$0 \in \Gamma(u) = \partial\Lambda(u) := \partial\{(x',0) \in B_1' \mid u(x',0) = 0\}, \tag{0.3}$$

where $\Lambda(u)$ is the coincidence set and the boundary is in the relative topology of B_1'. Here $B_1' := B_1 \cap (\mathbb{R}^{n-1}\times\{0\})$.
We denote by $\mathfrak{S}$ the class of solutions of the normalized Signorini problem (0.1)–(0.3).

Recent Developments

- Athanasopoulos-Caffarelli (2006): Optimal $C^{1,1/2}$ interior regularity with flat $\mathcal{M}$ and $\varphi = 0$.
- Athanasopoulos-Caffarelli-Salsa (2008): Fine regularity properties of the free boundary. Namely, the set of regular free boundary points is locally a C^1-manifold of dimension $n-2$.

- In the particular case $\Omega = \mathbb{R}^{n-1} \times (0, \infty)$ and $\mathcal{M} = \mathbb{R}^{n-1} \times \{0\}$, the Signorini problem can be interpreted as an obstacle problem for the fractional Laplacian on $\mathbb{R}^{n-1}$:

$$u - \varphi \geq 0, \quad (-\Delta_{x'})^s u \geq 0, \quad (u - \varphi)(-\Delta_{x'})^s u = 0,$$

  with $s = 1/2$.
  - Silvestre (2007): Almost optimal regularity of solutions, namely $u \in C^{1,\alpha}(\mathbb{R}^{n-1})$ for any $\alpha < s$, $0 < s < 1$.
  - Caffarelli-Salsa-Silvestre (2008): Optimal regularity $C^{1,s}(\mathbb{R}^{n-1})$, free boundary regularity.

Interesting aspect: In the above results, the thin obstacle φ is allowed to be nonzero.

- Garofalo-Petrosyan (2009): Structure of the singular set of solutions to the thin obstacle problem by construction of two one-parameter families of monotonicity formulas (of Weiss and Monneau type).

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- Higher regularity of the free boundary around regular points:
  - De Silva-Savin (2014)
    C^∞ regularity (based on boundary Harnack estimates in slit domains)
  - Koch-Petrosyan-Shi (2014)
    Analiticity (based on a partial hodograph-Legendre transformation)

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$$u_r(x) := \frac{u(rx)}{\left(\frac{1}{r^{n-1}} \int_{\partial B_r} u^2\right)^{1/2}},$$

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and studied the limits as $r \to 0+$, known as the blowups.

Generally the blowups might be different over different subsequences $r = r_j \to 0+$.

One needs a tool to control the rescalings.

Almgren's Frequency Function

The crucial tool is Almgren's Frequency Function

$$N(r,u) = \frac{r\int_{B_r} |\nabla u|^2}{\int_{\partial B_r} u^2}$$

The name comes from fact that if u is a harmonic function in B_1, homogeneous of degree κ, then $N(r,u) = \kappa$.

Monotonicity of the Frequency

Theorem 1 (Athanasopolous-Caffarelli-Salsa, 2007)

Let $u \in \mathfrak{S}$, then the function

$$N(r,u) := \frac{r\int_{B_r} |\nabla u|^2}{\int_{\partial B_r} u^2}$$

is monotone increasing in r for $0 < r < 1$. Moreover, $N(r,u) \equiv \kappa$ for $0 < r < 1$ iff u is homogeneous of order κ in B_1, i.e.

$$x \cdot \nabla u - \kappa u = 0 \quad \text{in } B_1.$$

When u is a harmonic function this is a classical result of Almgren (1979), extended by Garofalo-Lin in 1986 to solutions of divergence form elliptic PDEs.

The Blowups

It follows easily from the monotonicity formula that, for $r \leq 1$

$$\int_{B_1} |\nabla u_r|^2 = N(1, u_r) = N(r, u) \leq N(1, u),$$

where in the last inequality we have used the monotonicity of the frequency $N(r, u)$ claimed in the previous theorem. The above inequality, and the $C^{1,\alpha}_{loc}$ estimates of Caffarelli, imply that there exists a nonzero function $u_0 \in W^{1,2}(B_1)$, called a blowup of u at the origin, such that for a subsequence $r = r_j \to 0+$

$$\begin{aligned} u_{r_j} &\to u_0 \quad \text{in } W^{1,2}(B_1) \\ u_{r_j} &\to u_0 \quad \text{in } L^2(\partial B_1) \\ u_{r_j} &\to u_0 \quad \text{in } C^1_{\mathrm{loc}}(B_1' \cup B_1^{\pm}) \end{aligned}$$

The monotonicity of the frequency easily implies the following

Proposition 2 (Homogeneity of blowups)

Let $u \in \mathfrak{S}$ and denote by u_0 any blowup of u as described above. Then, $u_0 \in \mathfrak{S}$ and it is a homogeneous function of degree $\kappa = N(0+, u)$.

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The following result was proved in part by Luis Silvestre in his Ph.D. Dissertation, and in part by Caffarelli, Salsa and Silvestre

Lemma 1 (Minimal homogeneity)

Let $u \in \mathfrak{S}$. Then

$$N(0+, u) = \frac{3}{2} \quad or \quad N(0+, u) \geq 2.$$

Combining these results, we obtain that, in a suitable coordinate system,

$$u(x) = C\,\mathrm{Re}(x' \cdot e + i x_n)_+^{3/2} \quad \text{in}$$

for some tangential direction $e \in \partial B_1'$.

This immediately yields the optimal regularity of solutions.

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The regularity of the free boundary follows from a delicate analysis of the blow-ups.

The study of the singular set hinges on the uniqueness of a different type of blow-ups, which is established by means of monotonicity formulas of Weiss-and Monneau type (*more details on this later...*)

Statement of the Parabolic Signorini Problem

Given a domain Ω in $\mathbb{R}x^n$ with a sufficiently regular boundary $\partial\Omega$, a relatively open subset $\mathcal{M} \subset \partial\Omega$, $\mathcal{S} = \partial\Omega \setminus \mathcal{M}$, consider the problem

$$\Delta v - \partial_t v = 0 \quad \text{in } \Omega_T := \Omega \times (0, T] \tag{0.4}$$

$$v \geq \varphi, \quad \partial_\nu v \geq 0, \quad (v - \varphi)\partial_\nu v = 0 \quad \text{on } \mathcal{M}_T := \mathcal{M} \times (0, T], \tag{0.5}$$

$$v = g \quad \text{on } \mathcal{S}_T := \mathcal{S} \times (0, T] \tag{0.6}$$

$$v(\cdot, 0) = \varphi_0 \quad \text{on } \Omega_0 := \Omega \times \{0\} \tag{0.7}$$

where ∂_ν is the outer normal derivative on $\partial\Omega$ and $\varphi : \mathcal{M}_T \to \mathbb{R}$, $\varphi_0 : \Omega_0 \to \mathbb{R}$, $g : \mathcal{S}_T \to \mathbb{R}$ are prescribed functions satisfying the compatibility conditions $\varphi_0 \geq \varphi$ on $\mathcal{M} \times \{0\}$, $g \geq \varphi$ on $\partial\mathcal{S} \times (0, T]$, $g = \varphi$ on $\mathcal{S} \times \{0\}$. The condition (0.5) is known as the *Signorini boundary condition* and the problem (0.4)–(0.7) as the *Signorini problem* for the heat equation.

- The function $u(x,t)$ solves the following variational inequality:

$$\int_\Omega \nabla u \cdot \nabla(u-v) + \partial_t u(u-v) \geq 0$$

$$u \in \mathfrak{K}, \quad \partial_t u \in L^2(\Omega)$$

$$\text{for all } v \in \mathfrak{K}$$

where

$$\mathfrak{K} = \{v \in W^{1,2}(\Omega) : v\big|_{\mathcal{M}} \geq \varphi,\ v\big|_{\partial\Omega\setminus\mathcal{M}} = g\}$$

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  where

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- For any (reasonable) initial condition $u = \varphi_0$ on $\Omega_0 = \Omega \times \{0\}$ the solution exists and is unique (Duvaut-Lions, 1986).

Known Results

- **Regularity of the solution**
  $\nabla v \in H^{\alpha,\alpha/2},\ 0 < \alpha < 1,$ on compact subsets of $\Omega_T \cup \mathcal{M}_T$
  - Athanasopoulos (1982)
  - Uraltseva (1985)
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- **Poon's Monotonicity Formula** Poon (1996): If u is a solution of the heat equation in a unit strip, the parabolic frequency

$$N_u(r) = \frac{r^2 \int_{\mathbb{R}^n} |\nabla u|^2(x,-r^2)\rho(x,-r^2)dx}{\int_{\mathbb{R}^n} u(x,-r^2)^2\rho(x,-r^2)dx}$$

  is monotone in $r \in (0,1)$. Here ρ denotes the backward heat kernel on $S_\infty = \mathbb{R}^n \times (-\infty, 0]$, i.e.

$$\rho(x,t) = (-4\pi t)^{-\frac{n}{2}} e^{\frac{x^2}{4t}}.$$

The class $\mathfrak{S}_\varphi(Q_1^+)$ consists of functions $v \in W_2^{2,1}(Q_1^+)$, with $\nabla v \in H^{\alpha,\alpha/2}(Q_1^+ \cup Q_1')$ for some $0 < \alpha < 1$, satisfying

$$\Delta v - \partial_t v = 0 \quad \text{in } Q_1^+$$

$$v - \varphi \geq 0, \quad -\partial_{x_n} v \geq 0, \quad (v-\varphi)\partial_{x_n} v = 0 \quad \text{on } Q_1',$$

and

$$(0,0) \in \Gamma(v) = \partial\{(x',t) \in Q_1' \mid v(x',0,t) > \varphi(x',t)\}.$$

Here $Q_1^+ = B_1^+ \times (-1,0]$ is the upper parabolic half-cylinder and $Q_1' = B_1' \times (-1,0]$ is the thin parabolic cylinder.

Reduction to Vanishing Obstacle

The difference $v(x,t) - \varphi(x',t)$ satisfies the Signorini conditions on Q_1' with zero obstacle, but at an expense of solving a nonhomogeneous heat equation instead of the homogeneous one.
This difference may be extended to the strip $S_1^+ = \mathbb{R}_+^n \times (-1, 0]$ by multiplying it by a suitable cutoff function ψ.
The resulting function will satisfy

$$\Delta u - \partial_t u = f(x,t) \quad \text{in } S_1^+,$$

with

$$f(x,t) = -\psi(x)[\Delta'\varphi - \partial_t\varphi] + [v(x,t) - \varphi(x',t)]\Delta\psi + 2\nabla v \nabla\psi.$$

For smooth enough φ, the function f is bounded in S_1^+!

Solutions in Half-Strips

A function u is in the class $\mathfrak{S}^f(S_1^+)$, for $f \in L_\infty(S_1^+)$, if $u \in W_2^{2,1}(S_1^+)$, $\nabla u \in H^{\alpha,\alpha/2}(S_1^+ \cup S_1')$, u has a compact support and solves

$$\Delta u - \partial_t u = f \quad \text{in } S_1^+,$$

$$u \geq 0, \quad -\partial_{x_n} u \geq 0, \quad u\partial_{x_n} u = 0 \quad \text{on } S_1',$$

and

$$(0,0) \in \Gamma(u) = \partial\{(x', t) \in S_1' : u(x', 0, t) > 0\}.$$

Generalized Monotonicity Formula

Problem Poon's monotonicity formula requires the function u to be caloric in an entire strip, and it is not immediately applicable to caloric functions in the unit cylinder Q_1.

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Hope N_v is "close" to being monotone.

Introduce quantities

$$h_u(t) = \int_{\mathbb{R}^n_+} u(x,t)^2 \rho(x,t)\,dx$$

$$i_u(t) = -t \int_{\mathbb{R}^n_+} |\nabla u(x,t)|^2 \rho(x,t)\,dx,$$

for any function u on S_1^+ for which they make sense. Poon's parabolic frequency function is given by

$$N_u(r) = \frac{i_u(-r^2)}{h_u(-r^2)}.$$

There are many substantial technical difficulties involved in working with this function directly. To overcome such difficulties, consider averaged versions of h_u and i_u:

$$H_u(r) = \frac{1}{r^2} \int_{-r^2}^{0} h_u(t)\,dt = \frac{1}{r^2} \int_{S_r^+} u(x,t)^2 \rho(x,t)\,dx\,dt$$

$$I_u(r) = \frac{1}{r^2} \int_{-r^2}^{0} i_u(t)\,dt = \frac{1}{r^2} \int_{S_r^+} |t||\nabla u(x,t)|^2 \rho(x,t)\,dx\,dt$$

The New Generalized Monotonicity Formula

Theorem 3 (D.-Garofalo-Petrosyan-To, 2017)

Let $\delta > 0$. Then there exists $C > 0$, depending only on δ and n, such that the function

$$\Phi_u(r) = \frac{1}{2} r e^{Cr^\delta} \frac{d}{dr} \log \max\{H_u(r), r^{4-2\delta}\} + \frac{3}{2}(e^{Cr^\delta} - 1)$$

is nondecreasing for $r \in (0, 1)$.

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is nondecreasing for $r \in (0,1)$.

Remark 4

$H_u(r) > r^{4-2\delta} \Rightarrow \Phi_u(r) \sim \frac{1}{2} r H_u'(r)/H_u(r) = 2N_u$, *when* $f = 0$.
The truncation of $H_u(r)$ with $r^{4-2\delta}$ controls the error terms caused by the right-hand-side f.

We then introduce parabolic rescalings

$$u_r(x,t) := \frac{u(rx, r^2 t)}{H_u(r)^{1/2}},$$

which solve a non-honmogeneous Signorini problem

$$\begin{aligned} \Delta u_r - \partial_t u_r = f_r(x,t) \quad & \text{in } S^+_{1/r} \\ u_r \geq 0, \quad -\partial_{x_n} u_r \geq 0, \quad u_r \partial_{x_n} u_r = 0 \quad & \text{on } S'_{1/r} \end{aligned}$$

Theorem 5

- *There is a subsequence* $r_j \to 0+$ *and a function* u_0 *in* $S_\infty^+ = \mathbb{R}_+^n \times (-\infty, 0]$ *such that*

$$\int_{S_R^+} (|u_{r_j} - u_0|^2 + |t||\nabla(u_{r_j} - u_0)|^2)\rho \to 0.$$

  We call any such u_0 *a blowup of* u *at the origin.*

- u_0 *is a nonzero global solution of the Signorini problem:*

$$\begin{aligned} \Delta u_0 - \partial_t u_0 = 0 \quad &\text{in } S_\infty^+ \\ u_0 \geq 0, \quad -\partial_{x_n} u_0 \geq 0, \quad u_0 \partial_{x_n} u_0 = 0 \quad &\text{on } S'_\infty \end{aligned}$$

  in the sense that it solves the Signorini problem in every Q_R^+.

- u_0 *is parabolically homogeneous of degree* κ*:*

$$u_0(\lambda x, \lambda^2 t) = \lambda^\kappa u_0(x, t), \quad (x, t) \in S_\infty^+, \ \lambda > 0$$

Homogeneous global solutions of homogeneity $1 < \kappa < 2$:

Let u be a nonzero κ-parabolically homogeneous solution of the Signorini problem in $S_\infty^+ = \mathbb{R}^n_+ \times (-\infty, 0]$ with $1 < \kappa < 2$. Then $\kappa = 3/2$ and

$$u(x,t) = C\,\mathrm{Re}(x' \cdot e + ix_n)_+^{3/2} \quad \text{in } S_\infty^+$$

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- Optimal regularity of the solution
- Smoothness of free boundary at regular point
- Structure of the singular set

The Fractional Heat Operator

We now turn our attention to the obstacle problem for the fractional heat operator

$$(\partial_t - \Delta)^s u(x,t) =$$
$$= \frac{s}{\Gamma(1-s)} \int_{-\infty}^{t} \int_{\mathbb{R}^n} (t-\tau)^{-s-1} G(x-z, t-\tau)[u(x,t) - u(z,\tau)] dz d\tau,$$

for $0 < s < 1$ and $u \in C^1(\mathbb{R}^n \times \mathbb{R}) \cap L^\infty(\mathbb{R}^n \times \mathbb{R})$.

Here $G(z,\tau) = (4\pi\tau)^{-\frac{n}{2}} e^{-\frac{|z|^2}{4\tau}}$ is the standard heat kernel and $\Gamma(z)$ is Euler Gamma function.

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The study of $(\partial_t - \Delta)^s$ was first proposed by M. Riesz in his fundamental paper *Intégrales de Riemann-Liouville et potentiels* (1938).

Motivation

An important motivation for the study of this nonlocal operator comes from the fact that it models a stochastic jump process with arbitrary distributions of both jump lengths and waiting times, such as the *continuous time random walk* (CTRW) introduced by Montroll and Weiss (1965).

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Klafter and Metzler (2000) describe such processes by means of the equation, nolocal both in space and time,

$$\eta(x,t) = \int_0^\infty \int_{\mathbb{R}} \Psi(z,\tau)\eta(x-z,t-\tau)dzd\tau.$$

This is an example of a *master equation*, introduced in 1973 by Kenkre, Montroll and Shlesinger.

Master equations are presently receiving increasing attention by mathematicians, also thanks to the work of Caffarelli and Silvestre (2014), who established the Hölder continuity of viscosity solutions to *generalized master equations*

$$Lu(x,t) = \int_{\mathbb{R}^n}\int_0^\infty K(x,t;z,\tau)(u(x,t) - u(x-z,t-\tau))dzd\tau = 0,$$

under suitable assumptions on the kernel K. With the choice

$$\begin{aligned} K(x,t;z,\tau) &= K(z,\tau) \\ &= \frac{s}{\Gamma(1-s)}\tau^{-s-1}G(z,\tau) = \frac{s(4\pi)^{-\frac{n}{2}}}{\Gamma(1-s)}\tau^{-s-1-n/2}e^{-\frac{|z|^2}{4\tau}} \end{aligned}$$

it is an easy exercise to verify that K verifies such assumptions, and therefore the fractional heat equation fits the framework of the generalized master equations.

The Obstacle Problem

We are interested in the nonlocal obstacle problem

$$\min\{u - \psi, (\partial_t - \Delta)^s u\} = 0.$$

The function ψ is the *obstacle*.

As briefly discussed before, its elliptic counterpart

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has a rich history.

In 2007 Caffarelli and Silvestre introduced a remarkable extension procedure which allows to convert problems involving the fractional Laplacian $(-\Delta)^s$ acting on functions of $x \in \mathbb{R}^n$, into ones involving a **local degenerate elliptic** operator acting on functions of the variable $X = (x, y) \in \mathbb{R}^{n+1}_+ = \mathbb{R}^n_x \times \mathbb{R}^+_y$.

For a given $u \in \mathrm{Dom}(-\Delta)^s$ one considers the function $U(x, y)$ that solves the so-called extension problem with $a = 1 - 2s \in (-1, 1)$

$$\begin{cases} \mathrm{div}_X(y^a \nabla_X U) = 0 & x \in \mathbb{R}^n, y > 0, \\ U(x, 0) = u(x). \end{cases}$$

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Then, one has the following weighted *Dirichlet-to-Neumann* relation:

$$-\frac{2^{-a}\Gamma(\frac{a+1}{2})}{\Gamma(\frac{1-a}{2})} \lim_{y\to 0^+} y^a \frac{\partial U}{\partial y}(x,y) = (-\Delta)^s u(x).$$

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This is a thin obstacle problem since now the obstacle is confined to the thin manifold $M = \mathbb{R}^n \times \{0\}$ which bounds the thick space $\mathbb{R}^{n+1}_+$.

Statement of the Parabolic Problem

Stinga and Torrea and - indipendently - Nystrom and Sande (Ca. 2015) showed that, at a local level, the nonlocal obstacle problem

$$\min\{u - \psi, (\partial_t - \Delta)^s u\} = 0$$

is equivalent to the following lower-dimensional obstacle problem for the degenerate parabolic operator $\mathcal{L}_a = y^a \frac{\partial V}{\partial t} - \mathrm{div}_X(y^a \nabla_X V)$:

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is equivalent to the following lower-dimensional obstacle problem for the degenerate parabolic operator $\mathcal{L}_a = y^a \frac{\partial V}{\partial t} - \operatorname{div}_X(y^a \nabla_X V)$:

$$\begin{cases} \mathcal{L}_a V = 0 & \text{in } Q_1^+, \\ V(x,0,t) \geq \psi(x,t), & \text{for } (x,t) \in Q_1', \\ -\lim\limits_{y \to 0^+} y^a \frac{\partial V}{\partial y}(x,y,t) \geq 0, & \text{for } (x,t) \in Q_1', \\ \lim\limits_{y \to 0^+} y^a \frac{\partial V}{\partial y}(x,y,t) = 0, & \text{on } \{(x,t) \in Q_1' \mid V(x,0,t) > \psi(x,t)\}. \end{cases}$$

This is a *thin obstacle problem* since now the obstacle ψ lives on the thin manifold Q_1 in space-time $\mathbb{R}^n \times (-1,0)$.

The pioneering work of Chiarenza and Serapioni

The equation $\mathcal{L}_a V = 0$ is a special case of the class of degenerate parabolic equations in divergence form

$$\partial_t(\omega(X)V) = \mathrm{div}(A(X)\nabla V),$$

where $\omega(X)$ is a Muckenhoupt A_2-weight which controls the degeneracy of the matrix-valued function $A(X)$.

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If we take $\omega(X) = |y|^a$, then we have $\omega \in A_2(\mathbb{R}^{n+1})$ since $|a| < 1$. Using the Chiarenza-Serapioni result and the Signorini conditions we know that local solutions to the thin obstacle problem satisfy a parabolic Harnack inequality and are therefore Hölder continuous up to the thin set $(\mathbb{R}^n \times \{0\}) \times (-1, 0)$.

Related Results

The parabolic nonlocal obstacle problem

$$\min\{u - \psi, (-\Delta)^s u + u_t\} = 0$$

has been treated by Caffarelli and Figalli (2013) and Barrios, Figalli, and Ros-Oton (2018). However, even if the stationary versions are the same, this problem is fundamentally different from the one we are considering.

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In recent work Athanasopoulos, Caffarelli and Milakis (2018) establish the optimal regularity of solutions, as well as $C^{1,\alpha}$ regularity of the free boundary at certain non-singular points for solutions to

$$\min\{u - \psi,\ (\partial_t - \Delta)^s u\} = 0,$$

using the correspondence with the local degenerate problem.

Proceeding as in the case $a = 0$ we subtract the obstacle and multiply by a cut-off $\zeta(X) = \zeta^\star(|X|) \in C_0^\infty(B_1)$, $0 \le \zeta \le 1$, and then consider the new function

$$U(X,t) = \zeta(X)(V(X,t) - \psi(x,t)).$$

The function U solves the following problem in the space-time strip S_1^+ in thick space

$$\begin{cases} \mathcal{L}_a U = y^a F & \text{in } S_1^+, \\ & \\ & \\ & \end{cases}$$

Reduction to zero obstacle and globalization

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If $\psi \in C^2_{x,t}$, then not only $F \in L^\infty(S_1^+)$ but also $F_t \in L^\infty(S_1^+)$!
This allows us to prove the crucial fact

$$U_t \in L^\infty(S_1^+)$$

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With this information we can bring $y^a U_t$ to the right-hand side and then, setting $F - U_t \ \longrightarrow \ F$, consider the *elliptic* problem for the function $u(X) = U(X, \bar{t})$ **at each fixed time-level** $\bar{t}$:

$$\begin{cases} \operatorname{div}_X(y^a \nabla_X u) = y^a F & \text{in } B_1^+, \\ u(x,0) \geq 0, & \text{for } x \in B_1', \\ -\lim\limits_{y\to 0^+} y^a \frac{\partial u}{\partial y}(x,y) \geq 0, & \text{for } x \in B_1', \\ \lim\limits_{y\to 0^+} y^a \frac{\partial U}{\partial y}(x,y) = 0, & \text{on the set } \ \{x \in B_1' \mid u(x,0) > 0\} \end{cases}$$

Regularity of Solutions

This problem was studied by Caffarelli, Salsa and Silvestre under the assumption that $F \in C^{0,1}(B_1^+)$.

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This result is sharp:

$$u_0(X) = u_0(x,y) = \left(x_n + \sqrt{x_n^2 + y^2}\right)^{\frac{1-a}{2}} \left(x_n - \frac{1-a}{2}\sqrt{x_n^2 + y^2}\right)$$

solves the thin obstacle problem with $F \equiv 0$, and of course u_0 satisfies (0.8) at best!

Since our $F \in L^\infty(B_1^+)$, we cannot use the results of Caffarelli, Salsa Silvestre directly. But we can use an improved elliptic monotonicity formula due Caffarelli, De Silva and Savin (2017), which allows to obtain

$$u \in C^{1,\frac{1-a}{2}}(B_{1/2} \cup (B'_{1/2} \times \{0\}))$$

when $F \in L^\infty(B_1^+)$. Using the fact that the estimates are uniform in $\bar{t} \in (-1, 0)$, we prove that

$$\nabla_x U \in \mathbb{H}^{\frac{1-a}{2},\frac{1-a}{4}}(S_1^+ \cup (S_1' \times \{0\}))$$

($\mathbb{H}^{\alpha,\alpha/2}$ = intrinsic parabolic Hölder classes)

In addition, thanks to some delicate $W^{2,2}$ estimates, we show that

$$|\nabla U_{x_i}| \in L^2(S_1^+, y^a \overline{\mathcal{G}}_a(X,t) dXdt).$$

Here, we have denoted by

$$\overline{\mathcal{G}}_a(X,t) = \mathcal{G}_a(X,0,|t|), \qquad t<0,$$

the Neumann fundamental solution of the backward operator $\mathcal{L}_a^\star = y^a \frac{\partial}{\partial t} + \mathrm{div}_X(y^a \nabla_X)$ with pole at $0 = (0,0,0)$.

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One has the remarkable formula

$$\overline{\mathcal{G}}_a(X,t) = \frac{(4\pi)^{-\frac{n}{2}}}{2^a \Gamma(\frac{a+1}{2})} |t|^{-\frac{n+a+1}{2}} e^{-\frac{|X|^2}{4|t|}}.$$

One-parameter Almgren-Poon type monotonicity formula

Let

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One of our main tools is the following result:

Theorem 6 (Banerjee-D.-Garofalo-Petrosyan, 2019)

Suppose that $|F(X,t)| \le C_\ell |(X,t)|^{\ell-2}$ for every $(X,t) \in S_1^+$, for $\ell \ge 2$ and some constant $C_\ell > 0$. Then, for every $\sigma \in (0,1)$ there exist a constant $C > 0$, depending on n, a, C_ℓ and σ, such that the function

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$$r \mapsto \Phi_{\ell,\sigma}(U,r) \overset{def}{=} \frac{r}{2} e^{Cr^{1-\sigma}} \frac{d}{dr} \log \max\left\{ H(U,r),\ r^{2\ell-2+2\sigma} \right\} + 2(e^{Cr^{1-\sigma}} - 1)$$

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is monotone nondecreasing on $(0,1)$.

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is monotone nondecreasing on $(0,1)$. In particular, it exists

$$\Phi_{\ell,\sigma}(U,0^+) \overset{def}{=} \lim_{r\to 0^+} \Phi_{\ell,\sigma}(U,r).$$

The Free Boundary

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Let $\sigma \in (0,1)$, $\ell \geq 4$ and $\kappa = \Phi_{\ell,\sigma}(U, 0+)$ be such that $\kappa < \ell - 1 + \sigma$. Then

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Definition 7

The set $\Lambda_\psi(u) = \{x \in \mathbb{R}^n \,:\, u(x) = 0\}$ is the coincidence set, and its boundary $\Gamma_\psi(u) = \partial\Lambda_\psi(u)$ is the free boundary.

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An important consequence of the gap theorem is:

The set of free boundary points which have minimal frequency $\kappa = \frac{3-a}{2}$ is a relatively open subset of the free boundary.

The regular free boundary

Definition 8

We define the regular part of the free boundary as the collection of all free boundary points $(X_0, t_0) = (x_0, 0, t_0)$ *at which*

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Theorem 9

The regular free boundary is locally a $\mathbb{H}^{\alpha,\alpha/2}$ *hypersurface.*

Idea of proof:

- Assume $(0, 0)$ is a free boundary point;
- $U(\cdot, 0)$ is a solution to the *elliptic* thin obstacle problem with bounded right-hand-side (since $\partial_t U$ is bounded);
- The elliptic Almgren frequency at $x = 0$ can be shown to be $\frac{3-a}{2}$;
- The regularity of the free boundary follows from the elliptic theory and from the boundedness of $\partial_t U$.

The singular set

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A free boundary point $X_0 = (x_0, 0, t_0)$ is singular if

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$$\lim_{r\to 0+} \frac{\mathcal{H}^{n+1}(\Lambda(v) \cap Q_r(X_0))}{\mathcal{H}^{n+1}(Q_r(X_0))} = 0.$$

We denote the set of singular points by $\Sigma(v)$ and call it the singular set. We can further classify singular points according to the homogeneity of their blowup, by defining

$$\Sigma_\kappa(v) := \Sigma(v) \cap \Gamma_\kappa^{(\ell)}(v), \quad \kappa \leq \ell.$$

Theorem 10 (Characterization of singular points)

Let F be such that $|F(X,t)| \leq M\|(X,t)\|^{\ell-2}$ in S_1^+ and $|\nabla_X F(X,t)| \leq L\|(X,t)\|^{\ell-3}$ in $Q_{1/2}^+$, $\ell \geq 3$ and $0 \in \Gamma_\kappa^{(\ell)}(u)$ with $\kappa < \ell - 1 + \sigma$.

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(iii) $\kappa = 2m$, $m \in \mathbb{N}$.

We now recall the quantity

$$H(U,r) = \frac{1}{r^2} \int_{S_r^+} U^2 \, \overline{\mathcal{G}}_a y^a dXdt,$$

and introduce

$$D(U,r) = \frac{1}{r^2} \int_{S_r^+} |t||\nabla U|^2 \, \overline{\mathcal{G}}_a y^a dXdt.$$

To study the singular set we first prove the following

Theorem 11

Suppose that $\ell \geq 2$ is such that for some constant $C_\ell > 0$ one has $|F(X,t)| \leq C_\ell |(X,t)|^{\ell-2}$ for every $(X,t) \in S_1^+$.
For $\kappa \in (0,\ell)$ we define the parabolic κ-Weiss type functional

$$\mathcal{W}_\kappa(U,r) \overset{def}{=} r^{-2\kappa}\left\{D(U,r) - \frac{\kappa}{2} H(U,r)\right\}.$$

Then, for any $0 < \sigma \leq \ell - \kappa$ there exists $C > 0$ depending on n, a, ℓ, C_ℓ such that the function $r \longrightarrow \mathcal{W}_\kappa(U,r) + Cr^{2\sigma}$ is monotonically nondecreasing in $(0,1)$, and therefore the limit

$$\mathcal{W}_\kappa(U,0^+) \overset{def}{=} \lim_{r\to 0^+} \mathcal{W}_\kappa(U,r)$$

exists.

Monneau type monotonicity formula in Gaussian space

A direct consequence of the Weiss monotonicity formula is the main tool to analyze singular points.

Theorem 12

Assume that for some $\ell \geq 3$ the function F satisfies the bounds $|F(X,t)| \leq C_\ell |(X,t)|^{\ell-2}$ and $|\nabla F(X,t)| \leq C_\ell^\star |(X,t)|^{\ell-3}$ in S_1^+. Suppose that $0 \in \Sigma_\kappa(U)$ with $\kappa = 2m < \ell$, for $m \in \mathbb{N}$. For any parabolically κ-homogeneous polynomial p_κ in S_∞ we define the Monneau type functional

$$\mathcal{M}_\kappa \overset{def}{=} \frac{1}{r^{2\kappa+2}} \int_{S_r^+} (U - p_\kappa)^2 \, \overline{\mathcal{G}}_a y^a, \qquad r \in (0,1).$$

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Then, for any $0 < \sigma < \ell - \kappa$ there exists a constant $C > 0$, depending on $n, a, \ell, C_\ell, \sigma$, such that the function $r \to \mathcal{M}_\kappa + Cr^\sigma$ is monotonically nondecreasing on $(0,1)$.

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We show that at a singular point of homogeneity $\kappa = 2m$ such homogeneous blowup must be a parabolically κ-homogeneous polynomial p_κ satisfying

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Monneau monotonicity formula also implies another important piece of information: *The continuous dependence of the blowup from the free boundary points.*

Definition 13

We say that a $(d+1)$ *dimensional manifold* $\mathcal{S} \subset \mathbb{R}^n \times \mathbb{R}$ *for* $d = 0, ..., n-1$ *is space-like of class* $C^{1,0}$ *if locally, after a rotation of coordinates, one can represent it as a graph*

$$(x_{d+1}, ..., x_n) = g(x_1, ..., x_d, t)$$

where $g, \nabla_x g$ *are continuous.*

Likewise, a n-dimensional manifold $\mathcal{S} \subset \mathbb{R}^n \times \mathbb{R}$ *is time-like of class* C^1 *if it can be locally represented as*

$$t = g(x_1, ..., x_n)$$

where g *is* C^1.

Structure of the singular set

Combining these results with a parabolic Whitney type extension theorem we are able to establish the rectifiable structure of the singular set

Theorem 14

Let $F \in H^{\ell,\ell/2}(Q_1)$, $\ell \geq 3$. *Then, for any* $\kappa = 2m < \ell$, $m \in \mathbb{N}$, *we have* $\Gamma_\kappa(U) = \Sigma_\kappa(U)$.

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Let $F \in H^{\ell,\ell/2}(Q_1)$, $\ell \geq 3$. Then, for any $\kappa = 2m < \ell$, $m \in \mathbb{N}$, we have $\Gamma_\kappa(U) = \Sigma_\kappa(U)$. Moreover, for every $d = 0, 1, \ldots, n-2$, the set $\Sigma_\kappa^d(U)$ is contained in a countable union of $(d+1)$-dimensional space-like $C^{1,0}$ manifolds and $\Sigma_\kappa^{n-1}(v)$ is contained in a countable union of $(n-1)$-dimensional time-like C^1 manifolds.

Wall of Finite Thickness

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$$\frac{\partial u}{\partial \nu} = 0.$$

- $\varphi(x) \geq u(x)$
  It is reasonable to assume that the outflow through the wall is proportional to the difference in pressure:

$$-\frac{\partial u}{\partial \nu} = k(u - \varphi),$$

where $k > 0$ measures the conductivity of the wall.

Remarks

- If the conductivity $k = 0$, no fluid enter or leaves the chamber and the pressure u is a solution to the Neumann problem.

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- If the conductivity $k \to \infty$, in the limit one recovers the Signorini boundary conditions. Duvaut and Lions showed that if u_k is the solution corresponding to the conductivity k, then u_k converges weakly in L^2 to the solution to the thin obstacle problem.

Here we consider a local version of the problem, posed in the upper half ball

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This allows an alternate interpretation of the problem as a boundary temperature control problem, as derived by Duvaut and Lions. The same model also describes the flux of electricity through semi-conducting walls.

Boundary temperature control

Assume that a continuous medium occupies a region Ω in $\mathbb{R}^n$, with boundary Γ and outer unit normal ν.

Given a reference temperature $h(x)$, for $x \in \Gamma$, it is required that the temperature at the boundary $u(x, t)$ deviates as little as possible from $h(x)$.

Thermostatic controls are placed on the boundary to inject an appropriate heat flux when necessary. The controls are regulated as follows:

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Thermostatic controls are placed on the boundary to inject an appropriate heat flux when necessary. The controls are regulated as follows:

- If $u(x,t) = h(x)$, no correction is needed and therefore the heat flux is null.
- If $u(x,t) \neq h(x)$, a quantity of heat proportional to the difference between $u(x,t)$ and $h(x)$ is injected.

The boundary condition can be written as

$$-\frac{\partial u}{\partial \nu} = \Phi(u),$$

where

$$\Phi(u) = \begin{cases} k_-(u-h) & \text{if } u < h \\ 0 & \text{if } u = h \\ k_+(u-h) & \text{if } u > h \end{cases}$$

Statement of the problem

In our current setting the problem becomes

$$\begin{cases} \Delta u = 0 \text{ in } B_1^+ \\ u = g \text{ on } (\partial B_1)^+ \\ u_{x_n} = k_+(u^+)^{p-1} - k_-(u^-)^{p-1} \text{ on } \Gamma \end{cases}$$

where $g \in C^{2,\alpha}\left(\overline{B_1}\right)$ is the given boundary datum, $p > 1$, and

$$\begin{aligned} (\partial B_1)^+ &= \{x \in \partial B_1 \mid x_n > 0\} \\ \Gamma &= \{x \in B_1 \mid x_n = 0\} \\ u^+ &= \max\{u, 0\}, \ u^- = -\min\{u, 0\} \geq 0. \end{aligned}$$

Variational Formulation

We seek to minimize

$$J(v) = \frac{1}{2}\left(\int_{B_1} |\nabla v|^2 + \int_\Gamma \tilde{k}_-(v^-)^p + \int_\Gamma \tilde{k}_+(v^+)^p\right)$$

over all $v \in W^{1,2}(B_1)$, $v - g \in W_0^{1,2}(B_1)$ for a given boundary datum $g \in C^{2,\alpha}\left(\overline{B_1}\right)$.

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Note: $\tilde{k}_\pm = 2k_\pm/p$.

- Allen-Lindgren-Petrosyan (2015)
  Studied minimizers of

$$J_a(v) = \int_{B_1^+} |\nabla v|^2 x_n^a + 2\int_\Gamma \left(k_-(v^-)^1 + k_+(v^+)^1\right)$$

with $a \in (-1, 1)$.
Proved optimal regularity of the minimizer u: For $K \Subset B^+ \cup \Gamma$

$$u \in C^{0,1-a}(K) \qquad \text{if } a \geq 0,$$

$$u \in C^{1,-a}(K) \qquad \text{if } a < 0,$$

as well as separation of the two free boundaries $\partial\{u > 0\} \cap \Gamma$ and $\partial\{u < 0\} \cap \Gamma$ when $a \geq 0$.

- Allen (2016)
  Considered the problem

$$
\begin{aligned}
\operatorname{div}\left(x_n^a \nabla u(x', x_n)\right) &= 0 \text{ in } B_1^+,\\
\lim_{x_n \to 0} x_n^a u_{x_n}(x', x_n) &= -k u^+(x, 0) \text{ on } \Gamma,
\end{aligned}
$$

  with $k > 0$.
  The main objective is the study of the singular points of the free boundary.

Comparison with Signorini problem

New difficulties:

1. Non-homogeneous boundary condition $\Rightarrow$ this problem does not admit global homogeneous solutions of any degree.
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   Allowing for both constants $k^+, \ k^-$ to be finite (even when one of the two vanishes) de facto destroys the one-phase character of the problem.

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Redeeming feature:

The non-homogeneous character of the boundary condition allows to employ bootstrap arguments to prove regularity.

Joint work with T. Backing and R. Jain.

Theorem 15 (Existence and Uniqueness)

There exists a unique minimizer $u \in \{v \in W^{1,2}(B_1) \mid v - g \in W_0^{1,2}(B_1)\}$ *for the energy* $J(v)$.

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Theorem 16 (Regularity of Solutions)

Let $g \in C^{2,\alpha}(\overline{B_1})$ *and let* $k_\pm$ *be non-negative, finite and non-equal constants. Let* u *be the unique minimizer of the energy* $J(v)$. *Then*

- $u \in C^{p-1,\alpha}(\overline{B^+_{1/2}})$ for every $\alpha < p-1$, if p is an *integer*.
- $u \in C^{\lfloor p-1 \rfloor,\alpha}(\overline{B^+_{1/2}})$ for every $\alpha < p-1-\lfloor p-1 \rfloor$, if p is *not an integer*.

Additionally, if $k_- = k_+$ *or if* g *does not change sign, then* $u \in C^\infty(\overline{B^+_{1/2}})$.

Optimal regularity

Assume $p \geq 2$, p integer. Then, our regularity result ensures $u \in C^{p-1,\alpha}$ for all $\alpha < 1$. Is this optimal?

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Proof. Argument by contradiction, based on comparison principle and construction of suitable barrier.

Open question: Case $p = 2$: Backing-D.-Jain; Case $p > 2$: D.-Krummel.

Free boundary: Regular set

The regular set of the free boundary is defined as

$$\mathcal{R} = \{(x',0) \in \Gamma \mid u(x',0) = 0,\ \nabla u(x',0) \neq 0\}$$

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Open problem: Higher regularity of the free boundary.

A perturbed Almgren's Frequency Functional

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Theorem 19

Let $p \geq 2$, u be a solution, and let $F(u) = k_-(u^-)^p + k_+(u^+)^p$. Then the perturbed Almgren Frequency Functional

$$\tilde{N}(r,u) = r\frac{\int_{B_r^+} |\nabla u|^2 + \frac{2}{p}\int_\Gamma F(u)}{\int_{(\partial B_r)^+} u^2}$$

is monotone increasing in $r \in (0,1)$.

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Since $\tilde{N}(r,u) \geq 0$, we immediately have

Corollary 20

There exists $\lim_{r\to 0^+} \tilde{N}(r,u) = \mu \in [0,\infty)$.

Some consequences

To fix ideas, in the following we will always assume $p = 2$.
Recall

$$N(r,u) = \frac{r\int_{B_r^+} |\nabla u|^2}{\int_{\partial B_r^+} u^2}, \qquad F(u) = k_-(u^-)^2 + k_+(u^+)^2,$$

and

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$$N(r,u) \geq \frac{\tilde{N}(r,u) - Cr}{1 + Cr}.$$

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Hence, there exists $\lim_{r \to 0^+} N(r,u) = \mu$.

From now assume $\nabla u(0) = 0$. We introduce

$$\varphi(r) = \varphi(r; u) = \fint_{(\partial B_r)^+} u^2.$$

Corollary 21

Let $0 \leq \lim_{r\to 0^+} \tilde{N}(r) = \mu < \infty$. *Then the function* $r \mapsto r^{-2\mu}\varphi(r)$ *is nondecreasing for* $0 < r < 1$. *In particular,*

$$\varphi(r) \leq r^{2\mu}\varphi(1) \leq r^{2\mu} \sup_{B_1} |u|^2.$$

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Corollary 22

Let $0 < r < 1$. *Then for any* $\delta > 0$ *there exists* $R_0 = R_0(\delta) > 0$ *such that for all* $r, R \leq R_0$

$$\varphi(R) \leq \left(\frac{R}{r}\right)^{2(\mu+\delta)} \varphi(r).$$

Combining the two previous results we obtain

Corollary 23

For all $x \in B_{r/2}$,

$$|u(x)| \leq r^{\mu} \sup_{B_1} |u|.$$

Analysis of the blow-ups

The final step to obtain regularity estimate around free boundary points with vanishing gradient is to study blow-up sequences. Define

$$v_r(x) = \frac{u(rx)}{[\varphi(u,r)]^{1/2}}.$$

Note: $\|v_r\|_{L^2(\partial B_1)} = 1$.
Using the previous results, we see that $\{v_r\}$ are equibounded in H^1_{loc} and, thanks to the regularity estimates, also in $C^{1,\alpha}$.

Thus, there exists a uniformly convergent subsequence on every compact subset of $\mathbb{R}^n$ such that $v_j \to v^*, \quad \nabla v_j \to \nabla v^*$.
Note: $\|v_r\|_{L^2(\partial B_1)} = 1 \Rightarrow$ the blow-up is nontrivial.
As $r_j \to 0^+$,

$$\tilde{N}(r_j, u) = \tilde{N}(1, v_j) \to \tilde{N}(1, v^*) = \mu.$$

Since $\mu = \lim_{r_j \to 0^+} N(1, v^*)$, v^* is homogenous of degree μ.

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Moreover,

$$[u(rx)]_y = r u_{x_n}(rx) = r\left(k_+ u^+ - k_- u^-\right).$$

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Letting $r \to 0$, we find that v^* satisfies

$$\begin{cases} \Delta v^* = 0 & \text{in } B_1^+ \\ v^*_{x_n} = 0 & \text{on } \Gamma \end{cases}$$

We can evenly reflect v^* and consider the solution in B_1.

Combining these results, and applying the classical theory we can finally conclude

Theorem 24

Let u be a solution with $\nabla_{x'} u(0) = 0$, and let v^* be its blow-up limit. Then v^* is an homogeneous harmonic polynomial of degree $\mu \geq 2$.

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We now introduce

- Weiss-type functional

$$W(r,u) = \frac{H(r,u)}{r^{n-1+2\mu}}(N(r,u) - \mu),$$

  where $H(r,u) = \int_{(\partial B_r)^+} u^2$

- Monneau-type functional

$$M(r,u) = \frac{1}{r^{n-1+2\mu}} \int_{(\partial B_r)^+} (u - p_\mu)^2,$$

  p_μ harmonic polynomial, homogeneous of degree μ and even in x_n.

Both functionals are quasi-monotone as functions of r.

Consequences

- **Nondegeneracy**: There exists a constant $c > 0$ such that, for $r < 1$

$$\sup_{B_r^+} |u| \geq c r^{\mu}.$$

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$$\sup_{B_r^+} |u| \geq cr^{\mu}.$$

- **Uniqueness of the homogeneous blow-ups:** There exists a unique non-zero harmonic polynomial p_μ, homogeneous of degree μ and even in x_n, such that

$$v_r^{(\mu)}(x) = \frac{u(rx)}{r^{\mu}} \to p_\mu(x).$$

This give essentially μ-differentiability at singular points:

$$u(x) = p_\mu(x) + o(|x|^{\mu}).$$

Future developments

- Structure of the free boundary (joint with Backing and Jain)

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- Parabolic case

Thank you for your attention!