LECTURE NOTES

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1. Second Order Elliptic Equations

When learning complex analysis, it was a remarkable fact that the real part, u, of a complex analytic function, because it satisfies the equation:

$$u_{xx} + u_{yy} = 0 = \Delta(u)$$

(Laplace equation) is real analytic, and furthermore, the oscillation of u in any given domain D, controls *all* the derivatives of u, of *any* order, in any subset $\tilde{D}$, compactly contained in D_-. For our discussion, an important consequence of this theory are the Schauder and Calderon-Zygmund estimates.

Heuristically, they say that if we have a solution of an equation

$$A_{ij}(x) D_{ij} u = f(x)$$

and $A_{ij}(x)$ is a small perturbation of the Laplacian in a given functional space, then $D_{ij}u$ is in the same functional space as A_{ij} and f. (For instance, if $[A_{ij}]$ is Hölder continuous and positive definite, we can transform it to the identity (the Laplacian) at any given point x_0, and will remain close to it in a neighborhood.)

One can give three, essentially different explanations of this phenomena.

a) Integral representations (Cauchy integral, for instance)**.** This gives rise to many of the modern aspects of real and harmonic analysis: fundamental solutions, singular integrals, pseudodifferential operators, etc..

b) Energy considerations. Harmonic functions, u, are local minimizers of the Dirichlet integral

$$E(v) = \int (\nabla v)^2 \, dx \ .$$

That is, if we change u to w, in $\tilde{D} \subset\subset D$ (that is, $u = w$ in $D \setminus \tilde{D}$), then

$$E(w)|_{\tilde{D}} \geq E(u)|_{\tilde{D}} \ .$$

This gives rise to the theory of calculus of variations (minimal surface, harmonic maps, elasticity, fluid dynamics).

One is mainly concerned, there, with equations (or systems) of the form

$$D_i F_i(\nabla u, X) = 0 \ . \tag{1.1}$$

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For instance, in the case in which u is a local minimizer of

$$E(u) = \int \mathcal{F}(\nabla u, X)\,dx$$

(1.1) is simply the Euler equation:

$$F_i = \nabla_p \mathcal{F} \ .$$

If we attempt to write (1.1) in second derivatives form, we get

$$F_{i,j}(\nabla u, X) D_{ij} u + \cdots = 0 \ .$$

This strongly suggests that in order for the variational problem to be "elliptic", like the Laplacian, $F_{i,j}$ should be positive definite, that is $\mathcal{F}$ should be strictly convex that in principle is in L^2 (finite energy). It also leads to the natural strategy of showing that ∇u is in fact Hölder continuous to then apply the (linear) Schauder theory.

That would imply that $D_{ij}u$ is c^α, thus ∇u is $C^{1,\alpha}$, and so on.

The passage from $\nabla u \in L^2$ to $\nabla u \in C^\alpha$, is, of course, no trivial matter. It is the celebrated De Giorgi theorem, that then evolved into the De Giorgi-Nash-Moser theory. In fact, the De Giorgi theorem is much more powerful than that. It considers a variational solution of the *linear* equation

$$D_i A_{ij}(x) D_j w = 0$$

but without assuming any regularity on the coefficients $A_{ij}(x)$, only ellipticity, and it proves that such a w is Hölder continuous.

Furthermore

$$\|w\|_{C^\alpha(B_{1/2})} \leq C \|w\|_{L^2(B_1)} \ .$$

In doing so, De Giorgi makes a *jump* of invariance classes.

(∗) From equations

$$D_i Q_{ij}(x) D_j u = 0$$

that are a small perturbation of the Laplacian, that is, that under dilations become asymptotically the Laplacian. We are now confronted with an equation that no matter how much we dilate, remains in the same class.

Finally, a third approach is

c) Comparison principle. Two solutions u_1, u_2 of $\Delta u = 0$ cannot "touch without crossing". That is, if $u_1 - u_2$ is positive it cannot become zero in some interior point, X_0, of D.

Again, heuristically, this is because the functions

$$F(D^2 u) = \Delta u = \mathrm{Trace}[D^2 u]$$

are monotone functions of the Hessian matrix $[D_{ij}u]$ and, thus, in some sense, we must have $F(D^2 u)$ ">" $F(D^2 u_2)$ at X_0 (or nearby).

The natural family of equations to consider, then, is

$$F(D^2 u) = 0$$

for F a strictly monotone function of $D^2 u$.

Such type of equations appear in differential geometry. For instance, the coefficients of the characteristic polynomial of the Hessian

$$P(\lambda) = \det(D^2u - \lambda I)$$

are such equations where D^2u is restricted to stay in the appropriate set of $\mathbb{R}^{n\times m}$. If λ_i denote the eigenvalues of D^2u

$$\begin{aligned}
C_1 &= \Delta u = \sum \lambda_i \qquad \text{(Laplace)}\\
C_2 &= \sum_{i\neq j} \lambda_k\lambda_k \ldots\\
&\vdots\\
C_n &= \prod \lambda_i = \det D^2u \qquad \text{(Monge-Ampere)}\ .
\end{aligned}$$

In the case of $C_n = \det D^2u = \prod \lambda_i$ is a monotone function of the Hessian provided that all λ_i's are positive. That is, provided that the function, u, under consideration is convex.

If $F(D^2u, X)$ is *uniformly elliptic*, that is, if F is strictly monotone as a function of the Hessian, or in differential form, if

$$F_{ij}(M) = D_{m_{ij}}F$$

is uniformly positive definite, then solutions of $F(D^2u) = 0$ are $C^{1,\alpha}$. As in the divergence case, first derivatives u_α satisfy an elliptic equation,

$$F_{ij}(D^2u)D_{ij}u_\alpha = 0 \tag{1.2}$$

now in non divergence form, and again with bounded measurable coefficients.

The $a_{ij}(x)D_{ij}u_\alpha = 0$ corresponding to De Giorgi type theorem, is due to Krylov and Safanov, and states again that solutions of such an equation are Hölder continuous.

We point out that, again this result has "jumped" invariance classes. Unfortunately, this is not enough to "bootstrap", that is, we still cannot use Schauder estimates, as in the divergence case: The coefficients, $A_{ij}(x) = F_{ij}(D^2u)$, depend on second derivatives. If we managed to prove that D^2u is Hölder continuous, then from equation 1.1 $D_\alpha u$ would be $C^{2,\alpha}$, i.e., u would be $C^{3,\alpha}$ and we could improve and improve, as long as $F(\cdot)$ is very smooth.

To prove this, once more convexity reappears. If $F(D^2u)$ is concave (or convex) then all pure second derivatives are super (or sub) solutions of the linearized operator. This, together with the fact that D^2u lies in the surface $F(D^2u) = 0$, implies the Hölder continuity of D^2u, and, by the bootstrapping argument via Schauder estimates, u is as smooth as F allows.

2. De Giorgi

Theorem 1. *Let u be a solution of $D_ia_{ij}D_ju = 0$ in B_1 of $\mathbb{R}^n$ with $0 < \lambda I \leq a_{ij}(x) \leq \Lambda I$ (i.e., a_{ij} is uniformly elliptic). Then $u \in C^\alpha(B_{1/2})$ with*

$$\|u\|_{C^\alpha(B_{1/2})} \leq C\|u\|_{L^2(B_1)}$$

$(\alpha = \alpha(\lambda, \Lambda, n)$.

Proof. The proof is based on the interplay between Sobolev inequality, that says that $\|u\|_{L^{2+\varepsilon}}$ is controlled by $\|\nabla u\|_{L^2}$ and the energy inequality, that says that in turn, u *being a solution of the equation* means that $\|\nabla u_\theta\|_{L^2}$ is controlled by $\|u_\theta\|_{L^2}$ *for every truncation* θ: $u_\theta = (u-\theta)^+$.

We recall the Sobolev and energy inequalities:

Sobolev. *If v is supported in B_1, then*

$$\|v\|_{L^p(B_1)} \leq C\|\nabla v\|_{L^2(B_1)}$$

for some $p(n) > 2$.

If we are not too picky we can prove it by representing $V(x)$ as

$$v(x_0) = \int_{B_1} \frac{\nabla v(x) \cdot x_0 - x}{|x - x_0|^n}\, dx = \nabla v * G\ .$$

(Check why!). Since G belongs "almost" to $L^{n/n-1}$, any $P < 2n/n-2$ would do. $p = 2n/n-2$ requires another proof. □

Energy inequality. *If $u \geq 0$, $D_i a_{ij} D_j u \geq 0$ and $\varphi \in C_0^\infty(B_1)$ then*

$$\int_{B_1} (\nabla \varphi u)^2\, dx \leq C \sup |\nabla\varphi|^2 \int_{B_1 \cap \operatorname{supp}\varphi} u^2\ .$$

(Note that there is a loss going from one term to the other: $\nabla\varphi u$ versus u.)

Proof. We multiply Lu by $\varphi^2 u$. Since everything is positive we get (after integrating by parts)

$$-\int \nabla^T(\varphi^2 u) A \nabla u \geq 0\ .$$

We have to transfer a φ from the left ∇ to the right ∇.

We use a Cauchy-type inequality whenever we have a term of the form

$$\int \nabla^T \varphi u\ A\ u(\nabla\varphi) \leq \varepsilon \int \nabla^T(\varphi u)\ A\ \nabla(\varphi u) + \frac{1}{\varepsilon}\int |\nabla\varphi|^2 u^2 \|A\|\ .$$

(Try it!!). □

Now, the proof of Theorem 1 is split in two parts:

Step 1. (From an L^2 to an L^∞ bound.)

Lemma 1. *If $\|u^+\|_{L^2(B_1)}$ is small enough $(< S(n, \lambda, \Lambda))$, then*

$$\sup_{B_{1/2}} u^+ \leq 1\ .$$

Proof. We will consider a sequence of truncations

$$\varphi_k u_k$$

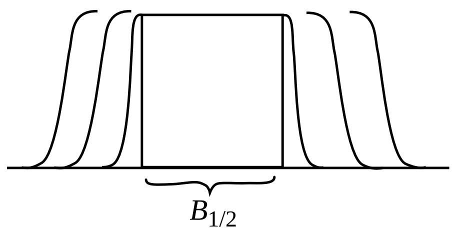
where φ_k is a sequence of shrinking cut off functions converging to $\chi_{B_{1/2}}$.

More precisely:

$$\varphi_k := \begin{cases} 1 & \text{for} \ \ |x| \le 1 + 2^{-(k+1)} \\ 0 & \text{for} \ \ |x| \ge 1 + 2^{-k} \end{cases}$$

and we only ask $|\nabla \varphi_k| \le C\ 2^k$ everywhere.

Note that $\varphi_k \equiv 1$ on $\operatorname{supp} \varphi_{k+1}$



while u_k is a sequence of monotone truncations converging to $(u-1)^+$:

$$u_k = [u - (1 - 2^{-k})]^+ \quad (\text{in} \ \ |x| \le 1 + 2^{-(k+1)}) \ .$$

Note that where $u_{k+1} > 0$, $u_k > 2^{-(k+1)}$.

Therefore *if* $(\varphi_{k+1} u_{k+1}) > 0$, $(\varphi_k u_k) > 2^{-(k+1)}$.

We will now show that, if $\|u\|_{L^2(B_1)} = A_0$ is small enough then

$$A_k = \int_{B_1} (\varphi_k u_k)^2 \to 0 \ .$$

In particular $(u-1)^+|_{B_{1/2}} = 0$ a.e.. This is done through a (*non linear*!!) recurrence relation for A_k.

We have

Sobolev inequality.

$$\left[\int (\varphi_{k+1} u_{k+1})^p \right]^{2/p} \le C \int (\nabla \varphi_{k+1} u_{k+1})^2 \ .$$

But, from Hölder

$$\int (\varphi_{k+1} u_{k+1})^2 \le \left[\int (\varphi_{k+1} u_{k+1})^p \right]^{2/p} \cdot |\{\varphi_{k+1} u_{k+1} > 0\}|^{\varepsilon}$$

so we get

$$A_{k+1} \le C \int [\nabla(\varphi_{k+1} u_{k+1})]^2 \cdot |\{\varphi_{k+1} u_{k+1} > 0\}|^{\varepsilon}$$

We now control the RHS by A_k.

From **energy** we get

$$\int \nabla (\varphi_{k+1} u_{k+1})^2 \le C\ 2^{2k} \int_{\operatorname{supp} \varphi_{k+1}} u_{k+1}^2$$

(But $\varphi_k \equiv 1$ on $\operatorname{supp} \varphi_{k+1}$)

$$\leq C\ 2^{2k} \int (\varphi_k u_k)^2 = C\ 2^{2k} A_k\ .$$

To control the term:

$$|\{\varphi_{k+1} u_{k+1} > 0\}|^\varepsilon \leq |\{\varphi_k u_k > 2^{-k}\}|^\varepsilon$$

We use Chebyshev's inequality

$$\leq 2^{4k\varepsilon} \left(\int (\varphi_k u_k)^2 \right)^\varepsilon .$$

So we get

$$A_{k+1} \leq C\ 2^{4k} (A_k)^{1+\varepsilon}\ .$$

Then, for $A_0 = \delta$ small enough $A_k \to 0$ (prove it).

Corollary 1. *If u is a solution of $Lu = 0$ in B_1, then*

$$\|u\|_{L^\infty(B_{1/2})} \leq C\ \|u\|_{L^2(B_1)}$$

Step 2. Oscillation decay:
Let $\operatorname{osc}_D u = \sup_D u - \inf_D u$

Theorem 2. *If u is a solution of $Lu = 0$ in B_1 then $\exists\ \sigma(\lambda, \Lambda, n) < 1$ such that*

$$\operatorname{osc}_{B_{1/2}} u \leq \sigma \operatorname{osc}_{B_1} u\ .$$

The proof is based on the following lemma.

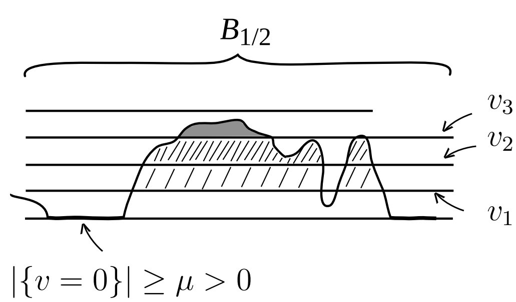
Lemma 2. *Let $0 \leq v \leq 1$, $Lv \geq 0$ in B_1. Assume that $B_{1/2} \cap \{v = 0\}| = \mu$ ($\mu > 0$) Then $\sup_{B_{1/4}} v \leq 1 - \varepsilon(\mu)$.*

Idea of the proof. We will consider a dyadic sequence of truncations

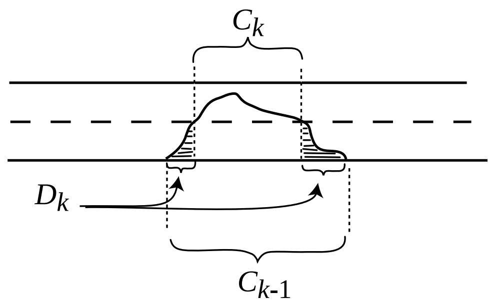
$$v_k = [v - (1 - 2^{-k})]^+$$

and their renormalizations

$$w_k = 2^k v_k$$



We will be interested in the set $C_k = \{v_k > 0\}$. Its complement $A_k = \{v_k = 0\}$ and the transition: $D_k = [C_k - C_{k-1}]$



We will show that in a finite number of steps, k_0, $k_0(\lambda, \Lambda, \mu)$,

$$|C_{k_0}| = 0 \ .$$

Then $\varepsilon(\mu) = 2^{-k_0}$.

Note that

a) $A_0 = \mu$

b) By the energy inequality, since $|w_k|_{B_1} \leq 1$,

$$\int_{B_{1/2}} |\nabla w_k|^2 \leq C$$

c) If C_k gets small enough

$$4 \int (w_k)^2 \leq |C_k| < \delta \ ,$$

and $2w_k|_{B_{1/4}} \leq 1$, by Step 1 we would be done (prove this).

□

We contend now that, since $\|w_k\|_{H^1}$ is bounded and $|\{w_k = 0\}| = |A_k| \geq \mu$, w_k "needs some room" to go from 0 to 1/2, that is we should have an inequality of the type

$$|D_k| \geq C(|A_k|, |C_{k+1}|)$$

The precise statement is as follows

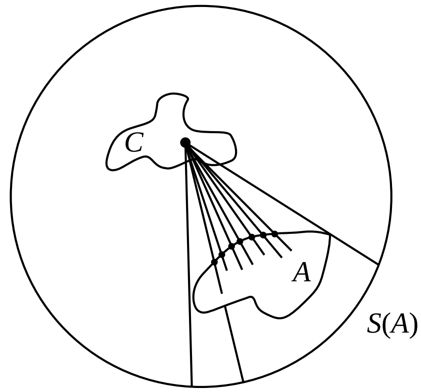
Sublemma. *Let* $0 \leq w \leq 1$

$$\begin{aligned} a &= |A| = |\{w = 0\} \cap B_{1/2}| \\ c &= |C| = |\{w = 1\} \cap B_{1/2}| \\ d &= |D| = |\{0 < w < 1\} \cap B_{1/2}| \ . \end{aligned}$$

Then if $\int |\nabla w|^2 \leq C_0$,

$$|D| =\geq C_1(|A|\,|C|)^2 \ .$$

Proof. For x_0 in C we reconstruct w integrating along all rays that go from x_0 to a point in A



$$1 = w(x_0) = \int w_r \, dr$$

or

$$|A| \leq \text{Area } S(A) < \int_D \frac{|\nabla w(y)|}{|x_0 - y|^{n-1}} \, dy$$

$$\left(w_r \, dr \, d\sigma \leq \frac{|\nabla w| r^{n-1} \, dr \, d\sigma}{r^{n-1}} \right)$$

Integrating x_0 on C

$$|A| \, |C| \leq \int_D |\nabla w(y)| \left(\int_C \frac{dx_0}{|x_0 - y|^{n-1}} \right) dy$$

Among all C with the same measure $|C|$, the integral in x_0 is maximized by the ball of radius $|C|^{1/n}$

$$\int_C \leq |C|^{1/n} \, .$$

So

$$|A| \, |C| \leq |C|^{1/n} \left(\int_D |\nabla w|^2 \right)^{1/2} |D|^{1/2} \, .$$

Since $\int |\nabla w|^2 \leq C_1$, the proof is complete. □

Proof of the theorem. We iterate this argument with $2(w_k \Lambda \frac{1}{2}) = w$. If C_k stays bigger than δ after a finite number of steps $k + 0 = k(\delta, \mu)$, we get

$$\sum |D_k| \geq |B_{1/2} \qquad \text{(which is \textit{impossible}.)}$$

So for some $k < k_0$, $|C_k| \leq \delta$ that makes $|C_{k+1}| = 0$ from the first part of the proof.

Corollary 2. $\operatorname{osc}_{B_{2^{-k}}} u \leq \lambda^k \operatorname{osc}_{B_1} u$.

Corollary 3. *$u \in C^\alpha(B_{1/2})$ with $\lambda = 2^{-\alpha}$ (defines α).*

Corollary 4. *If $\|u\|_{L^\infty(\mathbb{R}^n)} \leq C$ (for any C) $\Longrightarrow$ u is constant (Liouville-type theorem).*

Note. This argument in Lemma 1 is very useful when two quantities of different homogeneity compete with each other: area and volume (in a minimal surface) or area and harmonic measure, or harmonic measure and volume as in free boundary problems.

3. Krylov Safonov

Let u be a nonnegative solution of $Lu = \sum a_{ij} D_{ij} u = 0$ in $B_1 \subset \mathbb{R}^n$ with

$$\lambda \,\mathrm{Id} \leq a_{ij} \leq \Lambda \,\mathrm{Id}$$

Then, in $B_{1/2}$:

$$\sup_{B_{1/2}} u \leq C \inf_{B_{1/2}} u$$

with $C = C(\lambda, \Lambda, n)$.

Proof. We start by remarking that an equation $Lu = 0$ as above is totally infinitesimal in nature.

Further, since no regularity of the coefficients a_{ij} is required. The only information that $Lu = 0$ is giving us is that at every point the largest eigenvalue of $D^2 u$, $\mu_{\max}$ must be nonnegative, the smaller $\mu_{\min}$ must be nonpositive and

$$\mu_{\max} \sim -\mu_{\min} \ .$$

Therefore, the passage from the infinitesimal to the global is a very delicate issue that depends on a very special equation that has simultaneously divergence and nondivergence structure, the Monge-Ampere equation:

$$MA(u) = \det D^2 u = f(x) \ .$$

The MA equation,

$$\det D^2 u = \prod \mu_j$$

is elliptic only when all μ_j are positive (or negative), that is when u *is convex* (or concave).

On the other hand, $\det D^2 u$ is the Jacobian of the map

$$x \to y(x) = \nabla \ u(x)$$

and as such it has a hidden divergence structure. This is reflected in the celebrated Alexandrov-Bakelman-Pucci theorem.

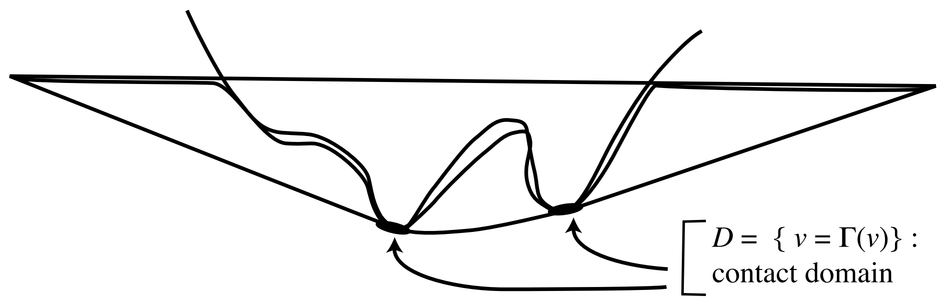
Theorem 3. *Let v be a solution of $a_{ij} D_{ij} v = f$ in B_1. Suppose that $v \geq 0$ on $\partial B - 1$. Then*

$$\sup (v^-)^n \leq C \int_{??} (f^+)^n \, dx \leq \|f^+\|^n_{L^n} \ .$$

Remark 1. The domain of integration, ??, will be very important for us and will be specified below.

Remark 2. This Theorem plays, in some sense, the role of the Sobolev inequality for De Giorgi's theorem, asserting that a combination of second derivatives controls v.

Proof of the Lemma. We consider $-v = \min(v, 0)$. We extend it by zero to B_2 and form its convex envelope $\Gamma(v)$ in B_2, that is, $\Gamma(v)(x)$ is the supremum of $\ell(x)$ where ℓ ranges over all linear functions that lie blow v^- in B_2, the set where v agrees with $\Gamma(v)$ will be called the "contact domain" and will be denoted by D.



We now consider the gradient map

$$x \longrightarrow y(x) = \nabla\Gamma(v)$$

and estimate its volume by above and below:

By above:

$$\text{Vol }(\nabla\Gamma(B_1)) = \int_{B_1} \det D^2\Gamma\ .$$

But $\int_{B_1} \det D^2\Gamma(v)$ is connected to f^+ by the following two observations

a) $\det D^2\Gamma(v)$ is supported on the contact set D, since through any other point Γ contains at least a segment (prove it). In particular we will take $?? = D$.

b) At a contact point

$$0 \leq D^2\Gamma(v) \leq D^2 v$$

Therefore

$$\det D^2\Gamma(v) \leq \prod_{j=1}^{n} \mu_j \leq (\mu_{\max})^n \leq C(f^+)^n$$

(μ_j denotes the eigenvalues of $D^2 v$). Recall that $a_{ij}D_{ij}v = f$ means that $\mu_{\max} \sim -\mu_{\min} + f$, which gives the inequality on the right.

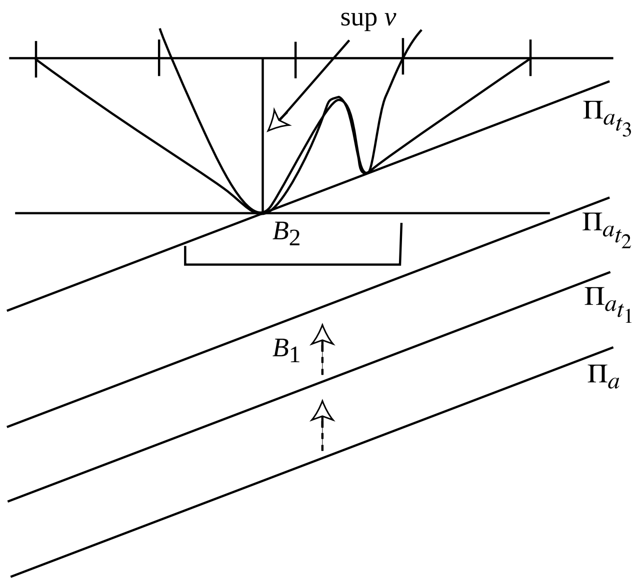
Therefore, we can estimate by above:

$$\text{Vol }(\nabla\Gamma(B_1)) \leq \int_D (f^+)^n\, dx\ .$$

Now the estimate by below: Consider any plane Π:

$$\Pi = \{y = \sum w_j x_j + a\} \text{ with slope, } |w| \leq \frac{\sup(v^-)}{4}\ .$$

Translate it far down the constant very negative so that it stays fully below the graph of $\Gamma(v)$, and then raise it continuously: $(a_t = a + t)$.



The plane Π_{a_t}, for t large enough will cross the graph of $\Gamma(v)$, *away* from the edge of B_2. In fact, inside B_1, since $D \subset B_1$. Therefore any such $w \in \nabla\Gamma(B_1)$ which only means that the ball of radius $\frac{\sup(v^-)}{4}$ is contained in $\nabla\Gamma(B_1)$. Therefore

$$\left(\frac{\sup(v^-)}{4}\right)^n \leq C \text{ Vol } (\nabla\Gamma(B_1)) \leq C \int_D (f^+)^n \ .$$

This completes the proof with ?? $= D$ the contact set of Γ with v. $\square$

We now go back to our u nonnegative solution of

$$Lu = 0 \ .$$

We will assume that $u(0) = 1$ and show that $\sup_{B_{1/2}} u \leq C_0$. This is done in two parts: In the first we only assume that u is a supersolution, (we call it $\bar{u}$, $L\bar{u} \leq 0$) and show that:

Lemma 3. *If $\bar{u} \geq 0$, and $\bar{u}(0) = 1$, then $\bar{u}$ belongs to a "weak L^ε", for some small ε:*

$$|\{\bar{u} > t\}| \leq C\ t^{-\varepsilon} \quad \textit{for} \quad \varepsilon = \varepsilon(\lambda, \Lambda, n) \ .$$

For that we need a sublemma:

Sublemma. (first rough version) $\bar{u}$ *as above, then*

$$|\{\bar{u} < 2\} \cap B_1| \geq \theta > 0 \qquad (\theta = \theta(\lambda, \Lambda, n)) \ .$$

(**Remark:** Note the similarity with De Giorgi theorem.)

Proof. We consider

$$v = \bar{u} - 2(1 - |x|^2)$$

and we apply the ABP Theorem, note that $\bar{u}(0) = 1$ means $v^-(0) = 1$, so

$$\underset{v^-(0)}{\underset{\|}{\mathbf{1}}} \le \sup_{B_1} (v^-)^n \le C \int_D ((Lv)^+)^n$$

We note that

a) Since $L\bar{u} \le 0$, $Lv \le C$.

b) On the contact set D, $0 \ge \Gamma(v) = v$, therefore $\bar{u} < 2$.

We thus get

$$1 \le C|\{\Gamma(v) = v\}| \le C|\{u < 2\}| \ .$$

□

Remark. On the set where $\Gamma(v) = v$, v has a "global" tangent plane, ℓ, by below (that of $\Gamma(v)$), and thus, $\bar{u}$ has a global tangent paraboloid $P = \ell + 2 - 2|x|^2$ by below, giving control on first and second derivatives from below.

Now we want to iterate this argument at every scale to obtain the L^ε estimate. For that, we need two tools. One, the possibility of localizing better the set where $\bar{u}$ is bounded. The second a Calderon Zygmund type lemma for iteration.

An improved sublemma 1. $\bar{u}$ as above, then for any cube contained in Q_1 with sides of size $1/8$

$$(Q_{1/8}(x_0)) - |\{\bar{u} < \tau\} \cap Q_{1/8}(x_0)| \ge \theta > 0$$

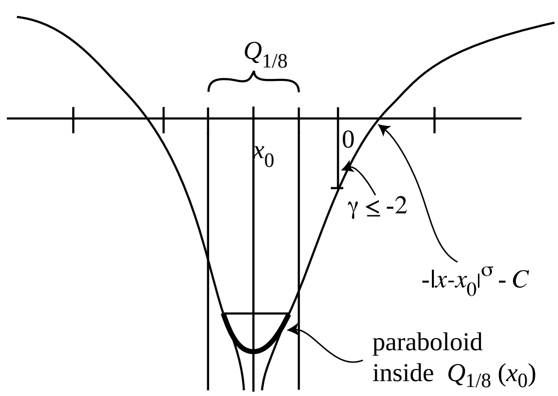
for some τ, θ (λ, Λ, n).

Proof. For $\sigma = \sigma(\lambda, \Lambda, n)$ large, $\gamma = -|x|^{-\sigma}$ is a supersolution of $L\gamma \le 0$.

Then instead of the auxiliary function $v = \bar{u} - 2(1 - |x|^2)$, we use

$$v = \bar{u} + \tilde{\gamma}(x - x_0) \ ,$$

with $\tilde{\gamma}$ being γ where we remove the singularity at x_0 by chopping off a little piece of the tip and replace it with a paraboloid:



Then $Lv \le 0$ outside $Q_{1/8}(x_0)$ and $Lv \le C$ inside $Q_{1/8}(x_0)$.

Then $\Gamma(v) = v$ can occur only inside $Q_{1/8}$, i.e., $D \subset Q_{1/8}$ and as before

$$-v(0) \le \int_D Lv \le C|D| = C|D \cap Q_{1/8}|$$

The second ingredient is a Calderón-Zygmund type lemma:

Lemma 4. *Let $A \subset B \subset Q_1$ be two measurable sets with the following properties*

a) $|A| \leq \delta$

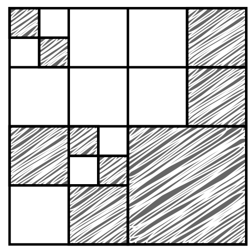
b) *Whenever* $\frac{|A \cap Q_s(x)|}{|Q_s|} > \delta$, *this implies that* for any $Q_{2s}(y)$ *that contains* $Q_s(y) \subset B$

$$Q_{2s}(y) \subset B$$

Then

$$|A| \leq \delta |B| \ .$$

Proof. We make a CZ decomposition: We split Q_1 into 2^n cubes $Q_{1/2}$



If $\frac{|A \cap Q_{1/2}(x)|}{|Q_{1/2}|} > \delta$, we keep it, if not we keep subdividing. This way we build a sequence of diadic disjoint cubes Q_j that contain A (a.e.). Each one of the predecessors A_j^* (the last cube we did not choose) has the property that

a) $\frac{|A \cap Q_j^*|}{??} < \delta$

b) $Q_j^* \subset B$.

We may assume the Q_j^* are disjoint. Since they still cover A, we get

$$|A| = \sum |A \cap Q_j^*| \leq \delta \sum |Q_j^*| \leq \delta |B| \ .$$

We are now ready to prove the L^ε estimate.

Proof. Consider

$$A_k = |\{\bar{u} > t^k\}| \ .$$

We will apply the previous lemma to

$$B = A_k$$
$$A = A_{k+1}$$
$$C = ??$$

and

$$u^* = \frac{\bar{u}}{t^k} \ .$$

Then, we know from the lemma that $|A_k| \leq |A_1| \leq (1-\theta) = \delta$. Let's check that A_k, A_{k+1} satisfy the A–B conditions. Indeed if

$$\frac{|A_{k+1} \cap Q|}{|Q|} > \delta$$

that means that
$$|\{u^* > t\} \cap Q| \geq (1-\theta)|Q|$$
or
$$|\{u^* < t\} \cap Q| \leq \theta|Q|$$
that means that there *cannot* be any point nearby (in $4Q$) where u^* is less than 1. (If not we contradict the lemma.) Thus $Q^* \subset A_k = B$.

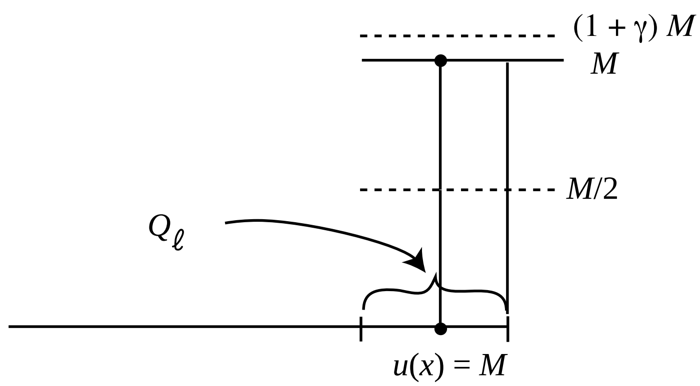
By applying this argument inductively, we get that
$$|A_k| \leq (1-\theta)^k = (t^k)^{-\varepsilon}$$
for ε chosen so $t^{-\varepsilon} = (1-\theta)$. The final step in the proof uses only that $u \in L^\varepsilon_w$ and u is a subsolution (we will call it $\underline{u}$):

Lemma 5. *Suppose that* $0 \leq \underline{u}$ *in* Q_2, $L\underline{u} \geq 0$ *and* $\|\underline{u}\|_{L^\varepsilon_w} \leq 1$. *Then*
$$\sup_{Q_{1/2}} \underline{u} \leq M_0 \ .$$

Proof. Suppose that $\sup_{Q_{1/2}} \underline{u} = M_0$. We want to find an apriori bound on M_0. The idea is the following:

If $\underline{u}(x_0) =$ a generic constant M, and we tube around x_0, a large enough cube $Q_{\ell(M)}$ then $\underline{u}$ must cross $(1+\gamma)M$ in Q, if γ is chosen small enough.

Let's see why:



We look simultaneously in $Q_{\ell(M)/2}$ at the sets
$$A = \{u \geq M/2\} \cap Q_{\ell(M)/2}$$
$$B = \{u < M/2\} \cap Q_{\ell(M)/2}$$
Obviously
$$|A| + |B| = |Q_{\ell(M)/2}| \ .$$
From the L^ε estimate that we are assuming
$$|A| \leq \left(\frac{M}{2}\right)^{-\varepsilon}$$
(independently of Q).

We want to make both A and $B < \frac{1}{2}|Q_{\ell(M)/2}|$ to get a contradiction.

For $|A|$, then, we need

$$\left(\frac{M}{2}\right)^{-\varepsilon} \le \left|\frac{\ell(M)}{2}\right|^n .$$

That is $\ell(M) = M^{-\hat{\varepsilon}}$ a small *negative power of* M will do. For B, we look at

$$w = \frac{(1+\gamma)M - \underline{u}}{\gamma M} .$$

Then on $Q_{\ell(M)}$:

a) $w \ge 0$ since we are assuming (by contradiction) that $u \le (1+\gamma)M$
b) $Lw \le 0$, since $L\underline{u} \ge 0$
c) $w(x_0) = 1$

From the L^{ε} estimate of Lemma.....,

$$\left|\{w > t\} \cap Q_{\ell(M)/2}\right| \le t^{-\varepsilon}|Q_{\ell(M)/2}| .$$

If we choose $t = \frac{(\frac{1}{2}+\gamma)}{\gamma}$, we get

$$|B| = \left|\left\{\underline{u} \le \frac{M}{2}\right\} \cap Q_{\ell(M)/2}\right| \le \left[\frac{\gamma}{\frac{1}{2}+\gamma}\right]^{\varepsilon}|Q_{\ell(M)/2}|$$

We choose γ small so that $[\frac{\gamma}{\frac{1}{2}+\gamma}]^{\varepsilon} < \frac{1}{2}$ (independently of M and we get a *contradiction.*

Recapitulating: For $\gamma, \hat{\varepsilon}$ small enough $(\gamma, \hat{\varepsilon}(\lambda, \Lambda, n))$, if $u(x_0) = M$ then,

$$\sup_{Q_{M^{-\hat{\varepsilon}}}} u \ge (1+\gamma)M$$

We are ready to *complete the proof*: Let $x_0 \in Q_{1/4}(0)$ and $u(x_0) = M_0$ large. By repeating the argument above we can find a sequence of points x_j, such that

$$M_j = u(x_j) = (1+\gamma)M_{j-1} = (1+\gamma)^j M_0$$

and

$$u(x_{j+1}) \in Q_(x_j)$$

with

$$\ell(Q_j) = (M_j)^{-\hat{\varepsilon}} = (1+\gamma)^{-\hat{\varepsilon}\, j} M_0^{-\hat{\varepsilon}} .$$

That is

$$|x_{j+1} - x_j| \le M_0^{-\hat{\varepsilon}}(1+\gamma)^{-j\hat{\varepsilon}} .$$

If M_0 is large enough, the sequence x_j stays in $Q_{1/2}$ and $u(x_j) \to \infty$. *A contradiction.*

4. Evans-Krylov

Theorem 4. *Let u be a solution of $F(D^2u) = 0$, with F uniformly elliptic and concave (or convex). Then for some $\alpha(\lambda, \Lambda, n)$,*

$$\|u\|_{C^{2,\alpha}(B_{1/2})} \leq C\|u\|_{L^\infty(B_1)} \ .$$

Proof. a) We prove first that u is $C^{1,1}$. We may assume that $F_{ij}(0) = \delta_{ij}$. In particular,

$$\text{tran } (M) \geq F(M)$$

so $\Delta u \geq 0$.

Also, if $F(M) = 0$ any solution v of $F(D^2v) = 0$ satisfies

$$F_{ij}(M)(D_{ij}v) \geq 0$$

or more generally, if u, v are solutions

$$F_{ij}(D^2u)D_{ij}v \geq 0$$

it follows that given a solution u, the second order incremental quotient

$$\delta_{h,e} = u(x + he) + u(x - he) - u(x)$$

is a subsolution if

$$F_{ij}(D^2u(x))D_{ij}(\delta) \geq 0$$

Corollary 5. (from weak Harnack)

$$\|D_{\alpha\alpha}u\|_{L^\infty(B_{1/2})} \leq C\|u\|_{L^\infty(B_1)}$$

Proof. Δu is a subsolution, bounded by below, and in L^ε_w (from its divergence structure). From the weak Harnack inequality Δu is bounded. This implies that $u \in W^{2,\rho}$ and thus, the weak Harnack applies to $(D_{\alpha\alpha}u)^+$ for all α.

$(D_{\alpha\alpha}u)^-$ is controlled then by the fact that $\Delta u \geq 0$.

b) We are now ready to prove the $C^{2,\alpha}$ estimate.

A couple of preliminaries

b_1) **Renormalization:**

If u is a solution of $F(D^2u)$, then $\bar{u}$ $\lambda u(\mu x)$ is again a solution of

$$\widetilde{F}(D^2\bar{u}) = 0$$

with $\widetilde{F}$ being just a dilation of F (if $\lambda = \mu^{-2}$, $\widetilde{F} = F$, or if F is homogeneous $\widetilde{F} = F$). The important fact is that the structural conditions of F remain the same. Therefore, it is enough to prove, that if u is a $C^{1,1}$ solution in B_1, then the oscillation of D^2u decreases a fixed amount when we go from B_1 to B_ρ for some fixed $0 < \rho < 1$.

b_2) Second observation is that if F is uniformly elliptic and both M and $M + N$ satisfy $F(M) = F(M + N) = 0$ then $\lambda \leq \frac{\|N^+\|}{\|N^-\|} \leq \Lambda$, and n particular

$$\|N\| \sim \|N^+\| \sim \|N^-\| \ .$$

We now proceed as follows.

Let us normalize the situation so that

$$\text{diam}(D^2u(B_1)) = 1$$

We want to show that for some $0 < \rho_0$,

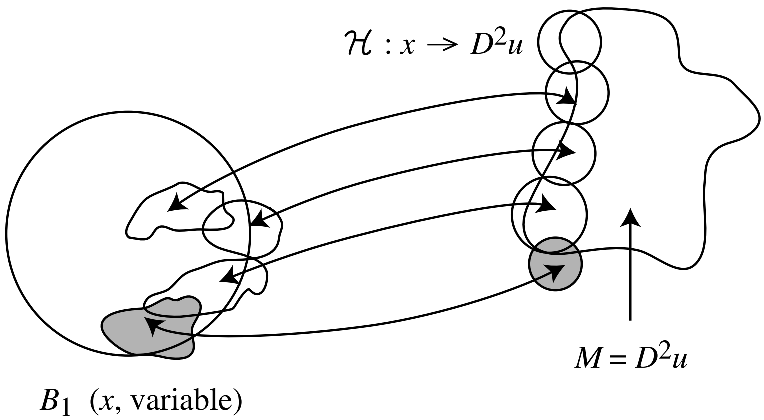
$$\operatorname{diam}(D^2u(B_{\rho_0})) \leq 1/2 \ .$$

Let us call $\Gamma = \{D^2u(B_1)\}$. Then diam $\Gamma = 1$, that means that $\exists\ M_0,\ M_1 = M_0 + N$, two matrices in Γ with

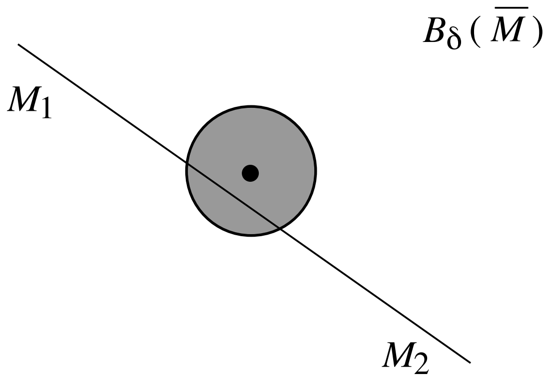
$$\|M_0 - M_1\| = 1 \ .$$

At this point, we do two coverings of Γ by balls B_j, and $\hat{B}_j$ with finite overlapping. The first family, of radius δ, the second of radius $\varepsilon \ll \delta$, both the be chosen.

The number of $B_j \sim \delta^{-n^2}$ and that of $\hat{B}_j \sim \varepsilon^{-n^2}$. We first inspect the inverse image of the B_j ($\mathcal{H}$ the Hessian map)



Since $\mathcal{H}^{-1}(B_j)$ covers B_1, there exists one B_j, B^0, such that $|\mathcal{H}^{-1}(B^0)|\delta^{n\times m}$ (in B_1 of x). But in $\mathcal{H}(B_1)$ we have that



since diam $\mathcal{H}(B_1) = 1 \geq 1/2$, there exists M_2 such that

$$\|M_2 - \bar{M}\| \geq \frac{1}{4}$$

(note that we use diam $\geq 1/2$ instead of 1).

In particular, from b_2

$$\|M_2 - \bar{M}\|^+ \geq\sim \frac{1}{4} = \theta > 0$$

we have that if

$$M_2 = D^2 u(x_2)$$

and

$$\bar{M} = D^2 u(\bar{x})$$

for some α

$$D^2_{\alpha\alpha} u(x_2) \geq D^2_{\alpha\alpha} u(\bar{x}) + \theta \ .$$

If we choose $\delta \ll \theta$, we have further that

$$D^2_{\alpha\alpha} u(x_2) \geq D^2_{\alpha\alpha} u(x) + \frac{\theta}{2}$$

for any x in $\mathcal{H}^{-1}(B_\delta)$ and therefore

$$\sup_{B_1} D^2_{\alpha\alpha} u(x) \geq \sup_{\mathcal{H}^{-1}(B_\delta)} \left(D^2_{\alpha\alpha} u + \frac{\theta}{2} \right) \ .$$

But $D^2_{\alpha\alpha} u$ is a subsolution and this implies that

$$\sup_{B_{1/2}} D_{\alpha\alpha} u(x) \leq \sup_{B_1} D_{\alpha\alpha} u - C\ \theta \ .$$

We now inspect the covering by balls of radius ε.

If $\varepsilon \ll C\ \theta$, there is at least one ball of the $\hat{B}_j$ that we do not need anymore to cover $\mathcal{H}(B_{1/2})$.

As long as diam $\mathcal{H}(B_{1/2}) \geq 1/2$ we may repeat the argument. After a finite number of steps, if diam remains above $1/2$ we run out of $\hat{B}_j$. A contradiction.

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