Hölder and Schauder estimates. Pointwise and semigroup strategies.

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Joint work with J.L. Torrea

Introduction

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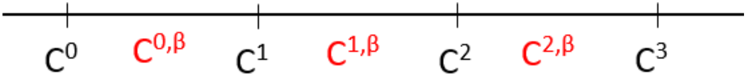

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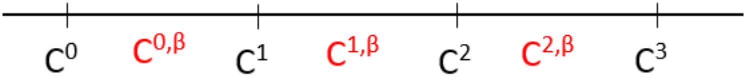

- Hölder spaces are basic in areas of functional analysis and relevant to solve partial differential equations (regularity results, existence of solutions for elliptic equations,...).

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Hermite Hölder spaces.

- Let $0 < \beta < 1$.
$C^{0,\beta}_{\mathcal{H}}(\mathbb{R}^n) = \{f : (1 + |\cdot|)^{\beta} f(\cdot) \in L^{\infty}(\mathbb{R}^n),$ and $\|f(\cdot + z) - f(\cdot)\|_{L^{\infty}(\mathbb{R}^n)} \leq A|z|^{\beta}\}$
with associated norm

$$\|f\|_{C^{\beta}_{\mathcal{H}}} = [f]_{M^{\beta}} + [f]_{C^{0,\beta}_{\mathcal{H}}},$$

where $[f]_{M^{\beta}} = \|(1 + |\cdot|)^{\beta} f(\cdot)\|_{\infty}$ and $[f]_{C^{\beta}_{\mathcal{H}}} = \sup_{|z|>0} \frac{\|f(\cdot+z)-f(\cdot)\|_{\infty}}{|z|^{\beta}}$.

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  where $[f]_{M^\beta} = \|(1 + |\cdot|)^\beta f(\cdot)\|_\infty$ and $[f]_{C_{\mathcal{H}}^{\beta}} = \sup_{|z|>0} \frac{\|f(\cdot+z) - f(\cdot)\|_\infty}{|z|^\beta}$.
- For $m \in \mathbb{N}$ and $0 < \beta < 1$, we say that $f \in C_{\mathcal{H}}^{m,\beta}(\mathbb{R}^n)$, if there exist the associated derivatives of order m and belong to $C_{\mathcal{H}}^{0,\beta}(\mathbb{R}^n)$ and
  $$\sum_{\substack{1\le j\le m \\ 1\le |i_1|,\dots,|i_j|\le n}} [A_{i_1}\dots A_{i_j} f]_{M^\beta} + [f]_{M^\beta} < \infty.$$

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Definition (Parabolic Hermite Hölder spaces.)

- Let $0<\alpha<1$. We say that $f\in C^{\alpha/2,\alpha}_{t,\mathcal{H}_x}$ if $f\in C^{\alpha/2,\alpha}$ and
  $$[f]_{M^\alpha} = \sup_{(t,x)\in\mathbb{R}^{n+1}} (1+|x|)^\alpha|f(t,x)| < \infty.$$
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- For $2 < \alpha < 3$, $f \in C^{\alpha/2,\alpha}_{t,\mathcal{H}_x}$, if $A_{\pm i}A_{\pm j} f$ and $\partial_t f$ belong to $C^{\alpha/2-1,\alpha-2}_{t,\mathcal{H}_x}$.

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and satisfies

$$\begin{cases} \partial_\tau v(\tau,x) = -\mathcal{H}v(\tau,x), \;\; \tau>0, x\in\mathbb{R}^n \\ v(0,x) = f(x), \;\; x\in\mathbb{R}^n. \end{cases}$$

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The **Poisson semigroup** associated with $\mathcal{H}$ can be written, for f in $L^\infty(\mathbb{R}^n)$, as

$$P_y^{\mathcal{H}} f(x) = e^{-y\sqrt{\mathcal{H}}} f(x) = \frac{y}{2\sqrt{\pi}} \int_0^\infty e^{-y^2/4\tau} e^{-\tau\mathcal{H}} f(x) \frac{d\tau}{\tau^{3/2}}$$

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and satisfies

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We define the space $\Lambda^{\alpha}_{\mathcal{H}}$ associated with the operator $\mathcal{H}$ as follows.

Definition (Hermite Zygmund spaces)

Let $\alpha > 0$.

$$\Lambda^{\alpha}_{\mathcal{H}} = \left\{ g \in L^{\infty}(\mathbb{R}^n) : \left\| \partial_y^k P_y^{\mathcal{H}} g \right\|_{L^{\infty}(\mathbb{R}^n)} \leq C y^{-k+\alpha}, \;\; k = [\alpha] + 1, \;\; y > 0, \; C > 0 \right\},$$

with norm $\|g\|_{\Lambda^{\alpha}_{\mathcal{H}}} = C_1 + \|g\|_{\infty}$, where C_1 is the infimum of the constants C above.

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Definition (Hermite Zygmund spaces)

Let $\alpha > 0$.

$$\Lambda^{\alpha}_{\mathcal{H}} = \left\{ g \in L^{\infty}(\mathbb{R}^n) : \left\| \partial_y^k P_y^{\mathcal{H}} g \right\|_{L^{\infty}(\mathbb{R}^n)} \leq C y^{-k+\alpha}, \;\; k = [\alpha] + 1, \;\; y > 0, \; C > 0 \right\},$$

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$$\begin{aligned} \mathcal{P}_y^{\mathcal{L}} f(t,x) &= \frac{y}{2\sqrt{\pi}} \int_0^\infty e^{-y^2/4\tau} e^{-\tau\mathcal{L}} f(x) \frac{d\tau}{\tau^{3/2}} \\ &= \frac{y}{2\sqrt{\pi}} \int_0^\infty \int_{\mathbb{R}^n} e^{-y^2/4\tau} \frac{e^{-\frac{|z|^2}{4}\coth\tau} e^{-\frac{|2x-z|^2}{4}\tanh\tau}}{(2\pi\sinh 2\tau)^{n/2}} f(t-\tau, x-z)\, dz \frac{d\tau}{\tau^{3/2}}. \end{aligned}$$

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where $P_y^{\mathcal{H}} g(x)$ is the Poisson semigroup associated with the operator $\mathcal{H} = -\Delta_x + |x|^2$. In this case,

$$\boxed{f \in \Lambda_{\mathcal{L}}^{\alpha}(\mathbb{R}^{n+1}) \Leftrightarrow g \in \Lambda_{\mathcal{H}}^{\alpha}(\mathbb{R}^n)}$$

Proposition

Let $\alpha > 0$. If $f \in \Lambda_{\mathcal{L}}^{\alpha}(\mathbb{R}^{n+1})$, then for every $0 < \beta < \alpha$, $f \in \Lambda_{\mathcal{L}}^{\beta}(\mathbb{R}^{n+1})$.

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Theorem

Let $0 < \alpha < 1$. Then

$$C^{\alpha/2,\alpha}_{t,\mathcal{H}_x} = \Lambda^{\alpha}_{\mathcal{L}},$$

with equivalence of norms. That is, $f \in \Lambda^{\alpha}_{\mathcal{L}}$ if, and only if,

$$\|f(\cdot - \tau, \cdot - z) - f(\cdot, \cdot)\|_{L^{\infty}(\mathbb{R}^{n+1})} \leq C(|\tau|^{1/2} + |z|)^{\alpha}$$

and $[f]_{M^{\alpha}} = \sup_{(t,x)\in\mathbb{R}^{n+1}} (1 + |x|)^{\alpha} |f(t,x)| < \infty.$

Theorem

1 *Suppose that* $0 < \alpha < 2$. *Then* $f \in \Lambda^{\alpha}_{\mathcal{L}}$ *if and only if there exists a constant* $C > 0$ *such that*

$$\|f(t-\tau, x-z) + f(t-\tau, x+z) - 2f(t,x)\|_{L^\infty(\mathbb{R}^{n+1})} \leq C(|\tau|^{1/2} + |z|)^\alpha,$$

for all $(\tau, z) \in \mathbb{R}^{n+1}$ *and* $(1+|x|)^\alpha f \in L^\infty(\mathbb{R}^{n+1})$. *In this case, if* C_2 *denotes the least constant* C *for which the inequality above is true, then* $\|f\|_{\Lambda^\alpha_{\mathcal{L}}} := [f]_{M^\alpha} + C_2$.

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2. *Suppose that* $\alpha > 2$. *Then* $f \in \Lambda_{\mathcal{L}}^{\alpha}$ *if and only if*

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In this case the following equivalence holds

$$\|f\|_{\Lambda_{\mathcal{H}}^{\alpha}} \sim \sum_{i,j=1}^{n} \left(\|A_{\pm i} A_{\pm j} f\|_{\Lambda_{\mathcal{L}}^{\alpha-2}}\right) + \|\partial_t f\|_{\Lambda_{\mathcal{L}}^{\alpha-2}}.$$

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3. *For* $0 < \alpha < 3$, $\alpha \notin \mathbb{N}$, *the spaces* $C^{\alpha/2,\alpha}_{t,\mathcal{H}_x} = \Lambda^{\alpha}_{\mathcal{L}}$, *with equivalence of norms.*

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Example: $\Lambda^1_{\mathcal{H}} \not\subset Lip$.

There exists a function $g \in \Lambda^1_{\mathcal{H}}(\mathbb{R})$, but so that $\sup_{x,z\in[0,1]} |g(x+z)-g(x)| \leq C|z|$ fails for all C.

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- Choose $g(x) = h(x)\varphi(x)$. Then, $|(1+|x|)g(x)| \leq C$ and by the Mean Value Theorem

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- If g would satisfy $|g(x+z) - g(x)| \leq C|z|$, then for $x, z \in [0,1]$ we would have $|h(x+z) - h(x)| \leq C|z|$. But the Weierstrass function doesn't satisfy Lipschitz condition.

For $\lambda, \beta > 0$, the following identities are true:

$$\lambda^\beta = \frac{1}{c_\beta} \int_0^\infty \left(e^{-t\lambda^{1/2}} - 1 \right)^{[2\beta]+1} \frac{dt}{t^{1+2\beta}}, \tag{1}$$

where $c_\beta = \int_0^\infty \left(e^{-\tau} - 1 \right)^{[2\beta]+1} \frac{d\tau}{\tau^{1+2\beta}}$, and

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If we replace λ by $\mathcal{L}$, we have the formal definitions:

$$\mathcal{L}^{\beta} f(t,x) = \frac{1}{c_\beta} \int_0^\infty \left(e^{-\tau\mathcal{L}^{1/2}} - I \right)^{[2\beta]+1} f(t,x) \frac{d\tau}{\tau^{1+2\beta}},$$

where $c_\beta = \int_0^\infty \left(e^{-\tau} - 1 \right)^{[2\beta]+1} \frac{d\tau}{\tau^{1+2\beta}}$. Also, for $\beta > 0$,

$$\mathcal{L}^{-\beta} f(t,x) = \frac{1}{\Gamma(2\beta)} \int_0^\infty e^{-\tau\mathcal{L}^{1/2}} f(t,x) \frac{d\tau}{\tau^{1-2\beta}}.$$

The **Fourier-Hermite transform** of a function $f \in L^1(\mathbb{R}^{n+1})$ can be defined as

$$\mathcal{F}(f)(\rho,\mu) = \int_{\mathbb{R}^{n+1}} f(t,x)e^{-i\rho t}h_\mu(x)dtdx, \ \ \rho \in \mathbb{R}, \ \mu \in \mathbb{N}_0^n,$$

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- As $\mathcal{F}(\mathcal{L}f)(\rho,\mu) = \mathcal{F}((\partial_t + \mathcal{H})f)(\rho,\mu) = (i\rho + 2|\mu| + n)\mathcal{F}(f)$, we have

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$$\mathcal{F}(\mathcal{P}_\tau f)(\rho,\mu) = \mathcal{F}(e^{-\tau\mathcal{L}^{1/2}}f)(\rho,\mu) = e^{-\tau(i\rho+2|\mu|+n)^{1/2}}\mathcal{F}(f)(\rho,\mu) \text{and}$$
$$\mathcal{F}(\mathcal{L}^{\pm\beta}f)(\rho,\mu) = (i\rho + 2|\mu| + n)^{\pm\beta}\mathcal{F}(f)(\rho,\mu),$$

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Lemma

(a) *Let* $0 < 2\beta < \alpha$ *and* $f \in \Lambda^\alpha_\mathcal{L}$ *then we have* $\mathcal{L}^\beta f(t,x) \le C < \infty$, $(t,x) \in \mathbb{R}^{n+1}$.

(b) *For every* $\beta > 0$ *and* $f \in Ł^\infty(\mathbb{R}^{n+1})$ *we have* $\mathcal{L}^{-\beta}f(t,x) \le C < \infty$, *for all* $(t,x) \in \mathbb{R}^{n+1}$.

Theorem (Hölder estimates)

Let $0 < 2\beta < \alpha$. *If* $f \in \Lambda_{\mathcal{L}}^{\alpha}$, *then* $\mathcal{L}^{\beta} f \in \Lambda_{\mathcal{L}}^{\alpha-2\beta}$, *and*

$$\|\mathcal{L}^{\beta} f\|_{\Lambda_{\mathcal{L}}^{\alpha-2\beta}} \leq C \|f\|_{\Lambda_{\mathcal{L}}^{\alpha}}.$$

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Theorem (Schauder estimates)

Let $\beta > 0$.

(i) *If $f \in \Lambda^{\alpha}_{\mathcal{L}}$, with $\alpha > 0$, then $\mathcal{L}^{-\beta} f \in \Lambda^{\alpha+2\beta}_{\mathcal{L}}$ and*

$$\|\mathcal{L}^{-\beta} f\|_{\Lambda^{\alpha+2\beta}_{\mathcal{L}}} \leq C\|f\|_{\Lambda^{\alpha}_{\mathcal{L}}}$$

(ii) *If $f \in L^{\infty}(\mathbb{R}^{n+1})$, then $\mathcal{L}^{-\beta} f \in \Lambda^{\beta}_{\mathcal{L}}(\mathbb{R}^{n+1})$ and*

$$\|\mathcal{L}^{-\beta} f\|_{\Lambda^{\beta}_{\mathcal{L}}} \leq C\|f\|_{\infty}.$$

For the multiplier operator of the Laplace transform on the spaces $\Lambda^{\alpha}_{\mathcal{L}}$, we have:

Theorem

Let a be a bounded function on $[0, \infty)$ and consider

$$m(\lambda) = \lambda^{1/2} \int_0^{\infty} e^{-s\lambda^{1/2}} a(s)ds, \ \ \lambda > 0.$$

Then, for every $\alpha > 0$, the multiplier operator $m(\mathcal{L})$ is bounded from $\Lambda^{\alpha}_{\mathcal{L}}$ into itself.

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These results allow as to prove the regularity of other operators.

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Now we consider the **Parabolic Riesz transforms of order m, $m \geq 1$,** defined by

$$R_\delta = (A^{\delta_1}_{\pm 1} A^{\delta_2}_{\pm 2} \dots A^{\delta_n}_{\pm n})\mathcal{L}^{-m/2} \quad \text{and} \quad R_m = \partial_t^m \mathcal{L}^{-m},$$

where $\delta = (\delta_1, \dots, \delta_n) \in \mathbb{N}_0^n$ such that $|\delta| = \delta_1 + \cdots + \delta_n = m$,

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Theorem

Let $m \in \mathbb{N}$. The Riesz transforms of order m, R_m and R_{δ}, where $|\delta| = m$, are bounded from $\Lambda^{\alpha}_{\mathcal{L}}$ into itself, for every $\alpha > 0$.

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Thank you for your attention!!