Regularity estimates for elliptic equations involving fractional Neumann boundary conditions

by

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The student author, whose presentation of the scholarship herein was approved by the program of study committee, is solely responsible for the content of this dissertation. The Graduate College will ensure this dissertation is globally accessible and will not permit alterations after a degree is conferred.

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DEDICATION

I would like to dedicate this thesis to my parents Ed and Kim, my brother Sam, and my niece Ayamé. Their love and support has been critical in my journey to complete this work. I would also like to thank all of the friends I met along the way who have always been there for me throughout the graduate degree.

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ABSTRACT

We prove regularity estimates for elliptic equations involving fractional Neumann boundary conditions. In particular, we consider solutions which are harmonic in a Lipschitz domain with a fractional normal derivative boundary condition. To establish various estimates, we utilize the extension problem characterization for the normal derivative, and develop a De Giorgi type theory for our case. In total, we prove interior and boundary Schauder estimates for the aforementioned solutions.

CHAPTER 1. INTRODUCTION

The goal of this dissertation is to build a regularity theory for solutions to a class of equations involving a nonlocal boundary condition. In particular, we aim to make sense of the following fractional Neumann problem:

$$\begin{cases} \Delta u = 0, & \text{in } \Omega \\ \partial_\nu^\sigma u = f, & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded Lipschitz domain and ∂_ν^σ is defined as a fractional power of the classical normal derivative ∂_ν. In Chapter 2 we define the fractional normal derivative, relevant spaces in which we work, the notion of solution, and consider the well-posedness of the fractional Neumann problem (1.1). The definition of the fractional normal derivative is given spectrally in terms of so-called Steklov eigenfunctions, that is, let $\{s_k, \lambda_k\}$ be the family of Steklov eigenfunctions and eigenvalues solving

$$\begin{cases} \Delta s_k = 0, & \text{in } \Omega \\ \partial_\nu s_k = \lambda_k s_k, & \text{on } \partial\Omega. \end{cases}$$

Then the family of traces, $\operatorname{tr}(s_k) := \hat{s}_k$, on $\partial\Omega$ form an orthonormal basis of $L^2(\partial\Omega)$. If $g = \sum_{k=0}^\infty g_k \hat{s}_k \in L^2(\partial\Omega)$ we have

$$\partial_\nu g = \sum_{k=1}^\infty \lambda_k g_k \hat{s}_k.$$

Thus, for $0 < \sigma < 1$, we define

$$\partial_\nu^\sigma g = \sum_{k=1}^\infty \lambda_k^\sigma g_k \hat{s}_k.$$

As we will see later in this dissertation, this fractional normal derivative operator can be described in the case when $\partial\Omega$ is smooth as

$$\partial_\nu^\sigma g(x) = \int_{\partial\Omega} (g(x) - g(z))\, K_\sigma(x, z)\, dS_z$$

where the kernel K_σ enjoys the following estimate

$$K_\sigma(x,z) \sim \frac{1}{d(x,z)^{n-1+\sigma}},$$

where d is the geodesic distance between x and z on $\partial\Omega$. In particular, this is a nonlocal operator operator on $\partial\Omega$. Moreover, it interpolates between the identity operator ($\sigma = 0$) and the normal derivative ($\sigma = 1$). Therefore, it is interpreted as a fractional normal derivative of order $0 < \sigma < 1$. This will be made clear when we present the Schauder estimates in chapter 7.

1.1 Motivations and Applications

As motivation, we discuss various applications and examples where nonlocal equations on the boundary occur.

1.1.1 Radiative Transfer Equation

The process of radiative transfer is the physical phenomena of energy transfer via electromagnetic radiation. The propogation of this radiation through a given medium is described by absorbtion, emission, and scattering processes. The simplest example of a physical law of light scattering is that of Rayleigh scattering, which can be used to account for blue color of the sky. The radiative transfer equation in a medium free of absorbtion and emission in free space is given by

$$\begin{cases} \partial_t u + \theta \cdot \nabla_x u = \mathcal{I}(u) & \text{in } (0,T) \times \mathbb{R}^d \times \mathbb{S}^{d-1} \\ u = u_0 & \text{on } \{t=0\} \times \mathbb{R}^d \times \mathbb{S}^{d-1}. \end{cases}$$

We call $u = u(t,x,\theta)$ the radiation distribution,

$$\mathcal{I}(u) := \int_{\mathbb{S}^{d-1}} (u(\theta') - u(\theta)) b_s(\theta,\theta')\, d\theta'$$

the scattering operator, and b_s the angular scattering kernel. In the forward-peaked regime, the angular scattering kernel is approximated by

$$b_s\left(\theta,\theta'\right) = \frac{b(\theta,\theta')}{(1-\theta\cdot\theta')^{\frac{d-1}{2}+s}} \text{ where } s \in (0, \min\{1, \frac{d-1}{2}\})$$

and b has certain smoothness properties about the singularity. Note that, in general, $\mathcal{I}$ is a nonlocal operator on the boundary of the unit ball in d dimensions. Alonso and Sun [2] study this equation. In their paper, they consider a well-known angular scattering kernel called the Henyey-Greenstein kernel. When $d = 3$, we can write this kernel as

$$b_{HG}^{g}\left(\theta, \theta'\right) = \frac{1 - g^2}{(1 + g^2 - 2g\theta \cdot \theta')^{3/2}}$$

where $g \in (0, 1)$ is called the anisotropic factor. We call the case when g is very close to 1 the highly peaked regime. In particular,

$$\frac{b_{HG}^{g}(\theta, \theta')}{1 - g} \to \frac{1}{\sqrt{2}\,(1 - \theta \cdot \theta')^{3/2}} \text{ as } g \to 1.$$

To understand the scattering operator $\mathcal{I}$ as the nonlocal operator ∂_ν^σ, one can solve the Poisson problem in $B_1 \subseteq \mathbb{R}^d$

$$\begin{cases} \Delta w = 0, & \text{in } B_1 \\ w = f, & \text{on } \partial B_1 \end{cases}$$

to obtain

$$w(r, \theta) = c \int_{\partial B_1} \frac{1 - r^2}{(1 + r^2 - 2r\theta \cdot \theta')^{n/2}} f(\theta')\, dS_{\theta'}.$$

Notice that the kernel in the above expression is the Henyey-Greenstein kernel in the case that $n = 3$. Using the method of semigroups, formalized in Chapter 3, we can write

$$\partial_\nu^\sigma w(1, \theta) = \partial_\nu^\sigma f(\theta) = \int_{\partial B_1} \left(f(\theta) - f(\theta')\right) K_\sigma(\theta \cdot \theta')\, dS_{\theta'}$$

where the kernel K_σ has the following estimate:

$$K_\sigma(\theta, \theta') \sim \frac{1}{d(\theta, \theta')^{(n-1)+\sigma}}.$$

In particular, $\mathcal{I} = \partial_\nu^\sigma$ for the class of angular scattering kernels b_s arising from the process described above.

1.1.2 Dirichlet-to-Neumann on the Sphere

In [35], the authors present a fractional Laplacian defined on the unit sphere: $(-\Delta_{\mathbb{S}^{n-1}})^{\pm\sigma}$. The Laplacian on the sphere has eigenfunctions given by the spherical harmonics $\{Y_{k,\ell}\}$, which form an orthonormal basis of $L^2(\mathbb{S}^{n-1})$, with associated eigenvalues $\lambda_k = k(k+n-2)$. The fractional powers of the Laplacian on the sphere are given spectrally, that is

$$(-\Delta_{\mathbb{S}^{n-1}})^{\pm\sigma}\, u(x) = \sum_{k=0}^{\infty} \lambda_k^{\pm\sigma} \sum_{\ell=1}^{d_k} c_{k,\ell}(u) Y_{k,\ell}(x)$$

for $x \in \mathbb{S}^{n-1}$.

The fractional normal derivative we define arises as the fractional power of the Dirichlet-to-Neumann operator $L = \partial_\nu$

$$L^{\pm\sigma} u(x) = \partial_\nu^{\pm\sigma} u(x) = \sum_{k=0}^{\infty} k \sum_{\ell=1}^{d_k} c_{k,\ell}(u) Y_{k,\ell}(x)$$

for $x \in \mathbb{S}^{n-1}$. We present a detailed example of this fact in chapter 2. To see ∂_ν as a Dirichlet-to-Neumann operator, one can solve the Dirchlet problem

$$\begin{cases} \Delta w = 0, & \text{in } B_1 \\ w = u, & \text{on } \partial B_1, \end{cases}$$

as we did in the previous section, and compute $Lu = \partial_\nu w|_{\partial\Omega}$.

Moreover, it is shown in [35] that

$$(-\Delta_{\mathbb{S}^{n-1}})^{\sigma} = \left(\partial_\nu + \frac{n-2}{2}\right)^{2\sigma} + T_s$$

where T_s is a fractional integral operator. We remark that the eigenvalues, λ_k, of the fractional Laplacian on the sphere are essentially the square of those of the Dirichlet-to-Neumann map ∂_ν, which is why the power on the right hand side is 2σ. Therefore, to understand the fractional Laplacian on the sphere, it suffices to study fractional powers of the Dirichlet-to-Neumann map and T_s seperately.

1.1.3 Applications to Probability

It is well known that harmonic functions satisfy the mean value property, and, consequently, have a probabilistic interpretration. Moreover, one can consider the PDE intepretation of a long jump random walk. We outline the construction in [36]. Let $K : \mathbb{R}^n \to [0, \infty)$ be even and such that

$$\sum_{k \in \mathbb{Z}^n} K(k) = 1.$$

Now, consider a random walk on the scaled lattice $h\mathbb{Z}^n$ for small $h > 0$. The long jump feature is introduced by allowing particles in this random walk to jump arbitrarily far with some probability, that is, $K(x - y)$ is the probability that $hx \in h\mathbb{Z}^n$ jumps to $hy \in h\mathbb{Z}^n$. Let $u(x, t)$ be the probability that a particle is at the point $x \in h\mathbb{Z}^n$ at time $t \in \tau\mathbb{Z}$ and let $\alpha \in (0, 2)$. If we suppose $\tau = h^\alpha$ and

$$K(x) = |x|^{-(n+\alpha)}$$

we obtain the following expression by letting $\tau = h^\alpha \to 0$:

$$\partial_t u(x, t) = \int_{\mathbb{R}^n} \frac{u(x + y, t) - u(x, t)}{|y|^{n+\alpha}}.$$

It is then shown that the singular integral on the right hand side is related to the fractional Laplacian of order α. In particular, one has

$$\partial_t u = - (-\Delta)^{\alpha/2} u.$$

This observation leads us to a probabilistic interpretation of the fractional Neumann problem. In chapter 2, we show that the fractional normal derivative in the upper half plane is precisely the fractional Laplacian. If we interpret the fractional Laplacian as in the above setting, we can surmise that a particle reaching the boundary experiences instantaneous, long jumps along the flat boundary. Moreover, we can intepret the general fractional normal derivative ∂_ν^σ in the same way. A particle in Ω undergoes Brownian motion, as solutions to the fractional Neumann problem are harmonic in the interior, and when the particle reaches the boundary, it experiences long jumps as described above.

1.1.4 Quasi-geostropic equation on the sphere and Lipschitz boundaries

The quasi-geostrophic equation (QG) is an equation used to model atmospheric quasi-geostrophic motion, that is, the motion of wind in an atmosphere where the Coriolis force and pressure gradient forces experience an inertial effect. The dissipative quasi-geostrophic equation has the form

$$\partial_t u + v \cdot \nabla u = -\left(-\Delta\right)^{\sigma} u, \text{for } x \in \mathbb{R}^n,\ t > 0.$$

The fractional Laplacian represents fractional diffusion, and v is a velocity field. In the paper [12], L. Caffarelli and A. Vasseur prove regularity estimates for the critical case $\sigma = 1/2$. Note that the equation is defined in a flat setting. When $n = 2$, the QG can be interpreted physically as modelling temperature evolution on a 2-dimensional surface experiencing a 3-dimensional quasi-geostrophic flow. For example, one can think of this as a toy model for the interaction between the atmosphere and surface of the ocean. We remark that the fractional normal derivative we define is exactly $(-\Delta)^{\sigma/2}$ in the upper half plane.

Let us now consider the spherical case. In the paper [3], the authors prove global well-posedness of the QG on the two-dimensional sphere given by

$$\theta_t + v \cdot \nabla_{\mathbb{S}^{n-1}} \theta = -\left(-\Delta_{\mathbb{S}^{n-1}}\right)^{\sigma} \theta, \text{for } x \in \mathbb{S}^{n-1},\ t > 0,$$

and v is a velocity field under a constraint. Recall that there is a relationship between ∂_ν^σ and $(-\Delta_{\mathbb{S}^{n-1}})^{\sigma/2}$ as seen in previous sections. The surface of the Earth is Lipschitz, so a more realistic model for quasi-geostrophic flow would be defined on a Lipschitz domain. In this way, we propose the following model: let Ω be a bounded domain interpreted interpeted as the Earth and $\partial\Omega$ the surface of the Earth. Then an analogue of the QG equations in this model is

$$\partial_t u + v \cdot \nabla u = -\partial_\nu^{\sigma/2} u, \text{for } x \in \mathbb{R}^n,\ t > 0.$$

1.2 Description of Results

1.2.1 Chapter 2

In this chapter, we make sense of the fractional Neumann problem

$$\begin{cases} \Delta u = 0 & \text{in } \Omega \\ \partial_\nu^\sigma u = f & \text{on } \partial\Omega, \end{cases}$$

where $0 < \sigma < 1$ and Ω is a bounded Lipschitz domain.

We define ∂_ν^σ spectrally. To that end, we consider a sequence of Steklov eigenfunctions, s_k with eigenvalues λ_k, solving the problem

$$\begin{cases} \Delta s_k = 0 & \text{in } \Omega \\ \partial_\nu s_k = \lambda_k s_k & \text{on } \partial\Omega. \end{cases}$$

We then discuss the relevant spaces in which we work. Denote $\{\mathrm{tr}(s_k) = \hat{s}_k\}$. The idea of considering these functions is that one can define $\partial_\nu^\sigma \hat{s}_k = \lambda_k^\sigma \hat{s}_k$ on $\partial\Omega$. It has also been shown that the $\hat{s_k}$ form an orthonormal basis of $L^2(\partial\Omega)$. So, we can expand u in this basis to define the fractional normal derivative.

After this, we define and analyze the fractional trace spaces $H^\sigma(\pm\partial\Omega)$ for $0 < \sigma < 1$. In particular, we define

$$H^\sigma(\partial\Omega) := \{u = \sum_{k=0}^{\infty} u_k s_k : \sum_{k=1}^{\infty} (1 + \lambda_k)^{2\sigma} \, u_k^2 < \infty\}.$$

Note that these align with the classical fractional Sobolev spaces when $\partial\Omega$ is smooth [4]. We then analyze the behavior of ∂_ν^σ on these trace spaces. We further define the space $H_0^\sigma(\partial\Omega)$ to be the subspace of $H^\sigma(\partial\Omega)$ defined on all $u \in L^2(\partial\Omega)$ for which $\int_{\partial\Omega} u \, dS = 0$, or, equivalently, $u_0 = 0$ in its Steklov expansion. The importance of considering this final space is that it allows us to prove uniqueness of solutions to the fractional Neumann problem, since solutions, in general, will differ up to a constant.

Once we have defined all the relevant spaces, we define the fractional normal derivative ∂_ν^σ spectrally. That is,

$$\partial_\nu^\sigma u = \sum_{k=0}^{\infty} \lambda_k^\sigma u_k s_k \text{ on } \partial\Omega$$

where $u_k = \langle u, s_k \rangle_{L^2(\partial\Omega)}$. We then analyze the behavior of ∂_ν^σ on $H^s(\partial\Omega)$, namely, we show that $\partial_\nu^\sigma : H^s(\partial\Omega) \to H^{s-\sigma}(\partial\Omega)$.

In the case that $-\sigma < 0$, we need to modify the defintion. That is, we define

$$\partial_\nu^{-\sigma} u = \sum_{k=1}^{\infty} \lambda_k^{-\sigma} u_k \hat{s}_k \in H^{s+\sigma}(\partial\Omega).$$

We have a similar relationship, as in the case $\sigma > 0$, given by $\partial_\nu^{-\sigma} : H^s(\partial\Omega) \to H^{s+\sigma}(\partial\Omega)$.

Finally, we prove that, under suitable assumptions, the fractional Neumann problem admits a unique weak solution.

1.2.2 Chapter 3

In Chapter 3, we characterize the operator ∂_ν^σ as a nonlocal operator of order σ as we discussed in the initial section. To do this, we appeal to the so-called method of semigroups. One can define, for g in a fractional trace space, the semigroup generated by ∂_ν as

$$e^{-t\partial_\nu} g := \sum_{k=0}^{\infty} e^{-t\lambda_k} g_k s_k$$

where the above sum is understood in a fractional trace space. One shows that this is a semigroup of bounded linear operators on $L^2(\partial\Omega)$. Let us assume that $\partial\Omega$ is smooth for the following description. In the case that $\partial\Omega$ is not smooth, the following computations must be understood in the weak sense, which we consider in chapter 3. We first show that the semigroup can be written as integration against a heat kernel W_t:

$$e^{-t\partial_\nu} g(x) = \int_{\partial\Omega} W_t(x,z) g(z)\, dS_z,$$

where the kernel W_t has Poisson-type estimates

$$W_t(x,z) \sim \frac{t}{(t^2 + d(x,z)^2)\, n/2}.$$

Here, $d(x,z)$ is the geodesic distance between x and z on $\partial\Omega$.

Recall the numerical identity with the gamma function:

$$\lambda_k^\sigma = \frac{1}{\Gamma(-\sigma)} \int_0^\infty \left(e^{-t\lambda_k} - 1\right) \frac{dt}{t^{1+\sigma}}.$$

Using the initial defintion for $e^{-t\partial_\nu}$ along with the numerical identity above, we obtain

$$\partial_\nu^\sigma g(x) = \int_0^\infty \left(e^{-t\partial_\nu} g(x) - g(x)\right) \frac{dt}{t^{1+\sigma}}$$

in the case that $\sigma > 0$.

Finally, we derive a pointwise integro-differential formula for ∂_ν^σ. To do this, we define a new kernel, K_σ, in terms of W_t, that is

$$K_\sigma(x,z) = C_\sigma \int_0^\infty W_t(x,z) \frac{dt}{t^{1+\sigma}}.$$

Moreover, we can rearrange

$$\partial_\nu^\sigma g(x) = \int_0^\infty \left(e^{-t\partial_\nu} g(x) - g(x)\right) \frac{dt}{t^{1+\sigma}}$$

to get

$$\partial_\nu^\sigma g(x) = \int_{\partial\Omega} (g(x) - g(z))\, K_\sigma(x,z)\, dS_z$$

where K_σ has the following estimates:

$$K_{\sigma(x,z)} \sim \frac{1}{d(x,z)^{n-1+\sigma}}.$$

This shows that ∂_ν^σ is a nonlocal operator of order σ. When $\sigma < 0$, we define $K_{-\sigma}$ as

$$\frac{1}{\Gamma(\sigma)} \int_0^\infty W_t\,(x,z) \frac{dt}{t^{1-\sigma}},$$

and a similar computation gives

$$\partial_\nu^{-\sigma} g(x) = \int_{\partial\Omega} K_{-\sigma}(x,z) f(x)\, dS_z.$$

Moreover, the kernel $K_{-\sigma}$ has estimates

$$K_{-\sigma}(x,z) \sim \frac{1}{d(x,z)^{n-1-\sigma}}.$$

We conclude the chapter with a remark on scaling. In particular, we define

$$u_\lambda(\overline{x}) = u(\lambda\overline{x})$$

for $\overline{x} \in \frac{1}{\lambda}\Omega$, where $\frac{1}{\lambda}\Omega$ is all those $\overline{x}$ for which $\overline{x} = x/\lambda$ for some $x \in \Omega$. In the case that $\Omega = B_\lambda$, we have $\frac{1}{\lambda}\Omega = B_1$. For $\overline{x} = x/\lambda$ at the boundary of $\frac{1}{\lambda}\Omega$, we get the scaling

$$\partial_{\nu,\lambda} u_\lambda(\overline{x}) = \lambda \partial_\nu u(\lambda\overline{x})$$

and we say the normal derivative ∂_ν scales like λ. Analagously, we show, using the semigroup characterization, that

$$(\partial_{\nu,\lambda})^{\pm\sigma} u_\lambda(\overline{x}) = \lambda^{\pm} \left(\partial_\nu^\sigma u\right)(\lambda\overline{x}),$$

and we say that $\partial_\nu^{\pm\sigma}$ scales like $\lambda^{\pm\sigma}$.

1.2.3 Chapter 4

The main goal of this chapter is to define and understand the extension problem characterization in our setting. The extension problem characterization allows us to localize our problem at the cost of introducing another variable.

We begin this chapter by defining vector-valued spaces in which our problem can be formulated. We then use a procedure outlined in [27] to prove that the trace space of a vector-valued Sobolev space, in which our extension problem is framed, aligns precisely with the trace spaces defined in Chapter 2. Furthermore, we prove various density results.

After discussing the preliminaries, we consider two related extension problems: one on $\partial\Omega$ where the nonlocal operator is defined, and one on $\overline{\Omega}$ which localizes the problem. Fix $a = 1 - 2\sigma$. The extension problem on top of $\partial\Omega$ is

$$\begin{cases} y^a \partial_\nu w = \left(y^a w_y\right)_y & \text{for } x \in \partial\Omega\, y > 0 \\ w(x,0) = u(x) & \text{for } x \in \partial\Omega. \end{cases}$$

We define the natural Sobolev spaces and weak solutions. The advantage of considering this extended problem is that we have the characterization

$$-\lim_{y\to 0^+} y^a w_y = c_\sigma \partial_\nu^\sigma u.$$

Therefore, studying regularity of solutions to the extension problem at $y = 0$ gives us a way to obtain regularity for solutions to the fractional Neumann problem. We also analyze various properties of solutions to the extension problem. It turns out, however, that this perspective is not enough to localize our problem because ∂_ν is, in fact, a nonlocal operator. As intuition for this fact, we remark that $\partial_\nu = (-\Delta)^{1/2}$ in the flat case $\Omega = \mathbb{R}^{n+1}_+$ with $\partial\Omega = \mathbb{R}^n$.

Instead, we consider the extension problem on top of $\overline{\Omega}$ by remembering that our ambient Steklov eigenfunctions satisfy a harmonic condition on the interior. To that end, we consider

$$\begin{cases} \Delta_x U(x,y) = 0, & \text{for } x \in \Omega\, y \geq 0 \\ y^a \partial_\nu U = (y^a U_y)_y & \text{for } x \in \partial\Omega\, y > 0 \\ U(x,0) = u(x) & \text{for } x \in \partial\Omega. \end{cases}$$

We perform a similar analysis as on the boundary extension, but we notice that the weak formulation for solutions is now local. Using this characterization, we are going to prove regularity for u by proving regularity for U all the way up to the boundary. The extension problem is very general, see, for example, the papers [10] and [34].

1.2.4 Chapter 5

In this chapter, we prove global regularity estimates for solutions to the fractional Neumann problem. The idea is to prove a nonlocal L^2 to L^∞ estimate for these solutions by using a De Girogi type iteration. That is, if $f \in L^q(\partial\Omega)$ for q depending on dimension and σ, then the solution u to the fractional Neumann problem is bounded on $\overline{\Omega}$.

To that end, we begin by proving a fractional Sobolev inequality which will be used in the proof of the above theorem. The fractional Sobolev inequality states that solutions to the fractional Neumann problem satisfy

$$\|u\|_{L^r(\partial\Omega)} \leq C\|\partial_\nu^{\sigma/2}u\|_{L^2(\partial\Omega)}$$

for some $r > 2$ depending on σ. The proof of this embedding makes extensive use of the extension problem characterization on $\overline{\Omega}$. Indeed, we prove a lemma that allows us to work in the flattened case where we can appeal to the fractional Sobolev embedding for the fractional Laplacian. We also utilize the fact that solutions to the extension problem on $\overline{\Omega}$ minimize an energy functional that is associated to our problem. We then modify a De Giorgi argument where the idea is to obtain a nonlinear reccurence relationship on successive energy levels. The nonlinear relationship is obtained using the interplay of the fractional Sobolev embedding and an energy inequality.

1.2.5 Chapter 6

In the penultimate chapter, we prove regularity results for a flattened, harmonic-like extension problem which will be used in the proceeding chapter to perform Schauder estimates. The equation we consider is

$$\begin{cases} \Delta_x U = 0 & \text{in } (-1,1)\times(B_1^+ \cup T_1) \\ (y^a U_y)_y = y^a \partial_{x_n} U & \text{on } (-1,1)\times T_1, \end{cases}$$

where $B_1^+ = B_1 \cap \{x_n > 0\}$ is the half ball in $\mathbb{R}^n$ and $T_1 = B_1 \cap \{x_n = 0\}$ the face in $\mathbb{R}^{n-1}$.

The first step is to localize the L^2 to L^∞ estimate of chapter 5 in the case of this harmonic-like extension problem. In particular, we prove that if the energy satisfies

$$\int_{-1}^{1} |y|^a \left(\int_{B_2^+} U^2\,dx + \int_{T_2} U^2\,dx' \right) dy < \varepsilon_0$$

for some $\varepsilon > 0$, then

$$U \leq 2 - \lambda \text{ on } T_{1/2} \times (-1/2, 1/2)\,.$$

In summary, this result allows us to work with U locally when proving oscillation decay.

Once we have this local result, we prove osillation decay with a critical density argument. We first prove a compact embedding in our setting, and, consequently, prove a lemma akin to that of

Fabes' lemma. Using this Fabes' lemma, we can prove the critical density estimate. That is, if U exceeds a fixed constant with positive density in a half cylinder $(-1,1)\times(B_1^+\cup T_1)$, it must be that U is bounded below in a subcylinder $(-1/2,1/2)\times(B_{1/2}^+\cup T_{1/2})$. Once we have this result, it is routine to prove that the oscillation of U decays as the half cylinders shrink. We then prove, as a corollary, that U is Hölder continuous at the origin. Finally, the Hölder continuity at the origin allows us to bootstrap regularity via incremental quotients. If we fix a unit tangential vector e in $B_1^+\times T_1$, we can define

$$U_h(x,y)=\frac{U(x+e,y)-U(x,y)}{|h|^{\alpha}}.$$

We can show that U_h is a solution and hence U inherits further regularity. Iterating this argument allows us to conclude that U is smooth in x. Further, we obtain an estimate on U_y, which is useful for the Schauder estimates in chapter 7.

1.2.6 Chapter 7

In this final chapter, we prove Schauder estimates on solutions to the extension problem, which, in turn, give the Schauder estimates for the fractional problem: if $f\in C^{\alpha}(\partial\Omega)$ and $\partial\Omega$ is Lipschitz, then $u\in C^{\alpha+\sigma}(\overline{\Omega})$. We first show that the harmonic-like solutions well approximate solutions to the extension problem on $\overline{\Omega}$. Then, we can transfer the regularity of the harmonic-like solutions to those of the flattened extension problem.

Solutions to the extension problem on some flattened portion of $\overline{\Omega}$ satisfy

$$\int_0^1 y^a\left[\int_{B_1^+} A(x)\nabla_x U\nabla_x\Psi\,dx+\int_{T_1} U_y\Psi_y\Phi\,dx'\right]dy=\int_{T_1}\Psi(x',0)f(x')\Phi\,dx',$$

the coefficients $A(x)$ are, at worst, uniformly elliptic, bounded, and measurable. We prove, using compactness, that, if $A(x)$ is close to I, and f is close to 0, then there is a harmonic-like function that well approximates U. The intuition is that U is very close to harmonic under these assumptions. From here, there is a first linear approximation to the solution by using the linear part of the harmonic-like approximator. Then we rescale and iterate this argument to find a sequence of constants that will converge to the constant that approximates the solution at the origin with $C^{\alpha+\sigma}$ rate. Moreover, we conclude that the solution u to the fractional Neumann problem is $C^{\alpha+\sigma}(\overline{\Omega})$ provided $\partial_\nu^\sigma u = f \in C^\alpha(\partial\Omega)$.

CHAPTER 2. FRACTIONAL NEUMANN PROBLEM

In this chapter, we aim to make sense of the following problem with fractional Neumann boundary condition

$$\begin{cases} \Delta u = 0, & \text{in } \Omega \\ \partial_\nu^\sigma u = f, & \text{on } \partial\Omega \end{cases} \tag{2.1}$$

for Ω a bounded Lipschitz domain in $\mathbb{R}^n$ with $n \geq 2$, $0 < \sigma < 1$, and $f \in L^2(\partial\Omega)$ has mean zero: $\int_{\partial\Omega} f\, dS = 0$. In particular, we define the operator ∂_ν^σ as the fractional power of the usual normal derivative operator, the relevant spaces in which we work, and a notion of solution to (2.1).

2.1 Steklov eigenfunctions and trace spaces

In this section we describe the main properties of Steklov eigenfunctions and fractional trace spaces. All the details about Steklov expansions can be found in [4], see also [5]. We provide examples, mainly for the case of the upper half space.

2.1.1 Steklov eigenfunctions

Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$. Consider the Steklov eigenvalue problem for a harmonic function $s \in H^1(\Omega)$ with eigenvalue $\lambda \in \mathbb{R}$:

$$\begin{cases} \Delta s = 0, & \text{in } \Omega, \\ \partial_\nu s = \lambda s, & \text{on } \partial\Omega, \end{cases}$$

in the weak sense, that is, s satisfies

$$\lambda \int_{\partial\Omega} s\phi \, dS = \int_\Omega \nabla s \nabla \phi dx$$

for every $\phi \in H^1(\Omega)$. It is well known that there exists a sequence of eigenvalues $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots$, each one of finite multiplicity, with corresponding harmonic Steklov eigenfunctions $\{s_k\}_{k\geq 0} \subset H^1(\Omega)$, such that $\lambda_k \nearrow \infty$, as $k \to \infty$. The natural inner product associated to this problem is

$$\langle u, v\rangle_\partial := \int_\Omega \nabla u \nabla v \, dx + \int_{\partial\Omega} uv \, dS, \quad \text{for } u, v \in H^1(\Omega). \tag{2.2}$$

It can be seen that this ∂-inner product is equivalent to the usual inner product in $H^1(\Omega)$ [5, Corollary 6.2]. Let

$$\mathcal{H}(\Omega) := \big\{u \in H^1(\Omega) : u \text{ is harmonic in } \Omega\big\}.$$

We have $u \in \mathcal{H}(\Omega)$ if and only if

$$\int_\Omega \nabla u \nabla \varphi \, dx = 0, \quad \text{for every } \varphi \in H_0^1(\Omega),$$

namely, if and only if u is orthogonal to $H_0^1(\Omega)$ with respect to the ∂-inner product. This observation gives the following orthogonal decomposition:

$$H^1(\Omega) = \mathcal{H}(\Omega) \oplus_\partial H_0^1(\Omega).$$

The Steklov eigenfunctions $\{s_k\}_{k\geq 0}$ form an orthogonal basis of $\mathcal{H}(\Omega)$ with respect to the ∂-inner product. We normalize each s_k so that their traces have $L^2(\partial\Omega)$-norm equal to 1, that is, if we set

$$\hat{s}_k(x) := \operatorname{tr}(s_k)(x), \quad \text{for } x \in \partial\Omega,$$

then

$$\int_{\partial\Omega} \hat{s}_k^2 \, dS = 1, \quad \text{for } k \geq 0.$$

In particular, the first eigenfunction is $s_0 = 1/|\partial\Omega|^{1/2}$. Solutions to the Steklov eigenvalue problem satisfy

$$\int_\Omega |\nabla s_k|^2 \, dx = \int_{\partial\Omega} \partial_\nu s_k s_k \, dS = \lambda_k \int_{\partial\Omega} \hat{s}_k^2 \, dS = \lambda_k,$$

and $\|s_k\|_\partial^2 = 1 + \lambda_k$. So, $\{s_k/(1+\lambda_k)^{1/2}\}_{k\geq 0}$ is a ∂-orthonormal basis of $\mathcal{H}(\Omega)$. For $u \in H^1(\Omega)$ consider the series

$$\mathcal{P}u(x) := \sum_{k=0}^\infty \frac{\langle u, s_k\rangle_\partial}{1+\lambda_k} s_k(x). \tag{2.3}$$

The operator $\mathcal{P}$ is the ∂-orthogonal projection of $H^1(\Omega)$ onto $\mathcal{H}(\Omega)$. If a function u has an expansion of the form

$$u(x) = \sum_{k=0}^{\infty} u_k s_k(x), \quad \text{for } x \in \Omega,$$

for some coefficients u_k, then

$$u \in \mathcal{H}(\Omega) \quad \text{if and only if} \quad \sum_{k=0}^{\infty} (1+\lambda_k)|u_k|^2 < \infty. \tag{2.4}$$

Now, $\{\hat{s}_k\}_{k\geq 0}$ is an orthonormal basis of $L^2(\partial\Omega)$. Given $u \in H^1(\Omega)$ we have $\mathrm{tr}(u) \in L^2(\partial\Omega)$, so that

$$\mathrm{tr}(u) = \sum_{k=0}^{\infty} \langle \mathrm{tr}(u), \hat{s}_k \rangle_{L^2(\partial\Omega)} \hat{s}_k = \sum_{k=0}^{\infty} \frac{\langle u, s_k \rangle_\partial}{1+\lambda_k} \hat{s}_k, \quad \text{in } L^2(\partial\Omega). \tag{2.5}$$

The second identity above follows because $\mathrm{tr}(u) = \mathrm{tr}(\mathcal{P}u)$ and $\mathcal{P}u$ is harmonic. See [4, Section 4].

Example 2.1.1 (The sphere)**.** *In the case when Ω is the unit ball B_1 of $\mathbb{R}^n$, the boundary $\partial\Omega$ is the unit sphere $\mathbb{S}^{n-1}$ and the Steklov eigenproblem reads*

$$\begin{cases} \Delta s = 0, & \text{in } B_1, \\ \nabla s \cdot X = \lambda s, & \text{on } \mathbb{S}^{n-1}. \end{cases}$$

The spherical harmonics are denoted by $Y_{k\ell}(X)$, $X \in \mathbb{S}^{n-1}$, $k \geq 0$, $\ell = 1, \dots, d_k$, where d_k is the dimension of the eigenspace at level k. It is well known that the family $\{Y_{k\ell}\}$ is an orthonormal basis of $L^2(\mathbb{S}^{n-1})$. The Steklov eigenfunctions are the harmonic extensions of $Y_{k\ell}$ to the interior of the ball. For $x = rX \in \overline{B}_1$, $X \in \mathbb{S}^{n-1}$, $0 \leq r \leq 1$, we have $s_{k\ell}(x) = r^k Y_{k\ell}(X)$. Then $s_{k\ell}$ are harmonic in B_1 and, by Euler's Lemma, they satisfy the Neumann condition with eigenvalues $\lambda_{k\ell} = k$.

Example 2.1.2 (The upper half space)**.** *Let Ω be the upper half space*

$$\mathbb{R}^n_+ = \{(x', x_n) \in \mathbb{R}^n : x' \in \mathbb{R}^{n-1}, x_n > 0\}.$$

Then $\partial\Omega = \mathbb{R}^{n-1}$ and the Steklov eigenproblem is

$$\begin{cases} \Delta_{\mathbb{R}^{n-1}} s + \partial_{x_n x_n} s = 0, & \text{in } \mathbb{R}^n_+, \\ -\partial_{x_n} s(x', 0) = \lambda s(x', 0), & \text{on } \mathbb{R}^{n-1}, \end{cases}$$

where $\Delta_{\mathbb{R}^{n-1}}$ denotes the Laplacian on $\mathbb{R}^{n-1}$. It is readily seen that in this case we have a continuum of Steklov eigenfunctions indexed by $\xi' \in \mathbb{R}^{n-1}$:

$$s_{\xi'}(x) = s_{\xi'}(x', x_n) = e^{-x_n|\xi'|}e^{-ix'\cdot\xi'}, \quad (x', x_n) \in \overline{\mathbb{R}^n_+}.$$

The Steklov spectrum is continuous:

$$-\partial_{x_n} s_{\xi'}(x', 0) = |\xi'| s_{\xi'}(x', 0), \qquad x' \in \mathbb{R}^{n-1}.$$

Observe that the traces are $s_{\xi'}(x', 0) = e^{-ix'\cdot\xi'}$. Hence the Steklov eigendecomposition is nothing but the Fourier inversion formula on $\mathbb{R}^{n-1}$:

$$g(x') = \frac{1}{(2\pi)^{(n-1)/2}} \int_{\mathbb{R}^{n-1}} \widehat{g}(\xi') e^{ix'\cdot\xi'}\, d\xi'. \tag{2.6}$$

2.1.2 Trace spaces $H^{\pm s}(\partial\Omega)$, for $s \geq 0$

A function $g \in L^2(\partial\Omega)$ with Steklov expansion

$$g = \sum_{k=0}^{\infty} g_k \hat{s}_k, \tag{2.7}$$

is said to belong to $H^s(\partial\Omega)$, $s \geq 0$, whenever

$$\|g\|^2_{H^s(\partial\Omega)} \equiv \|g\|^2_{L^2(\partial\Omega)} + [g]^2_{H^s(\partial\Omega)} := \sum_{k=0}^{\infty} (1 + \lambda_k^{2s}) g_k^2 < \infty. \tag{2.8}$$

We certainly have $H^0(\partial\Omega) = L^2(\partial\Omega)$ as Hilbert spaces.

Lemma 2.1.3 (Auchmuty)**.** *If $u \in H^1(\Omega)$ then $\operatorname{tr}(u) \in H^{1/2}(\partial\Omega)$. Conversely, if $g \in H^{1/2}(\partial\Omega)$ then there exists a unique harmonic extension $\mathcal{E}g \in \mathcal{H}(\Omega)$ of g to Ω such that $\operatorname{tr}(\mathcal{E}g) = g$ on $\partial\Omega$. Hence the space $H^{1/2}(\partial\Omega)$ is the class of all traces of H^1-functions in Ω, so it coincides with the usual fractional Sobolev space on $\partial\Omega$ (see [18]).*

Proof. We give it here for completeness, see [4, p. 899]. If $u \in H^1(\Omega)$ we can write $u = \mathcal{P}u + v$, with $\mathcal{P}u$ harmonic as in (2.3) and $v \in H^1_0(\Omega)$. By the ∂-orthonormality of $\{s_k/(1+\lambda_k)^{1/2}\}$ and the $L^2(\partial\Omega)$-orthonormality of $\hat{s}_k$ in the expansion of $\operatorname{tr}(u)$ in (2.5), we get

$$\|u\|^2_{H^1(\Omega)} \sim \|u\|^2_\partial = \sum_{k=0}^{\infty} \frac{\langle u, s_k\rangle^2_\partial}{1+\lambda_k} = \sum_{k=0}^{\infty} (1+\lambda_k) \frac{\langle u, s_k\rangle^2_\partial}{(1+\lambda_k)^2} = \|\operatorname{tr}(u)\|^2_{H^{1/2}(\partial\Omega)}. \tag{2.9}$$

Conversely, suppose that $g \in H^{1/2}(\partial\Omega)$. Then, by (2.4) and (2.8), the function

$$\mathcal{E}g(x) := \sum_{k=0}^{\infty} \langle g, \hat{s}_k\rangle_{L^2(\partial\Omega)} s_k(x) = \sum_{k=0}^{\infty} (1+\lambda_k)^{1/2} \langle g, \hat{s}_k\rangle_{L^2(\partial\Omega)} \frac{s_k(x)}{(1+\lambda_k)^{1/2}}, \quad \text{for } x \in \Omega,$$

belongs to $\mathcal{H}(\Omega)$, with $\|\mathcal{E}g\|_\partial = \|g\|_{H^{1/2}(\partial\Omega)}$. Moreover $\mathrm{tr}(\mathcal{E}g) = g$. Thus $\mathcal{E}$ is a linear isometry from $H^{1/2}(\partial\Omega)$ to $\mathcal{H}(\Omega)$. □

The spaces $H^s(\partial\Omega)$ defined as above coincide with the usual spaces defined via complex interpolation when $\partial\Omega$ is sufficiently smooth, see [4, Theorems 5.1, 5.2]. The space $H^{-s}(\partial\Omega)$ is defined as the completion of $L^2(\partial\Omega)$ with respect to the norm in (2.8) with $-s$ in place of s. Observe that $H^s(\partial\Omega) \subset L^2(\partial\Omega) \subset H^{-s}(\partial\Omega)$. Any function $f \in L^2(\partial\Omega)$ defines a continuous linear functional F_f on $H^s(\partial\Omega)$ via

$$F_f(g) = \int_{\partial\Omega} fg\, dS, \quad \text{for every } g \in H^s(\partial\Omega).$$

Also, if $f = \sum_k f_k \hat{s}_k$ and $g = \sum_k g_k \hat{s}_k$, then $F_f(g) = \sum_k f_k g_k = \langle f, g\rangle_{L^2(\partial\Omega)}$. We now prove that $H^{-s}(\partial\Omega)$ coincides with the dual of $H^s(\partial\Omega)$.

Lemma 2.1.4. *Let $T \in \left(H^{s/2}(\partial\Omega)\right)'$. Then for any $u \in H^{s/2}(\partial\Omega)$, we have*

$$T(u) = \sum_{k=0}^{\infty} u_k T\left(\hat{s}_k\right).$$

Proof. By the Riesz representation theorem, there exists $v \in H^{s/2}(\partial\Omega)$ for which

$$T(u) = \langle v, u\rangle_{H^{s/2}(\partial\Omega)} = \sum_{k=0}^{\infty} (1+\lambda_k)^s \langle v, \hat{s}_k\rangle_{L^2(\partial\Omega)} \langle u, \hat{s}_k\rangle_{L^2(\partial\Omega)} = \sum_{k=0}^{\infty} T(\hat{s}_k) u_k$$

for any $u \in H^{s/2}(\partial\Omega)$. □

This leads us to the following characterization of the dual: define

$$H^{-s/2}(\partial\Omega) := \{T \in \left(H^{s/2}(\partial\Omega)\right)' \mid \sum_{k=0}^{\infty} (1+\lambda_k)^{-s} T\left(\hat{s}_k\right)^2 < \infty\}$$

for $0 < s < 1$. Then $H^{-s/2}(\partial\Omega) \subseteq \left(H^{s/2}(\partial\Omega)\right)'$ via the definition

$$T(\phi) = \sum_{k=0}^{\infty} T\left(\hat{s_k}\right) \phi_k := \langle T, \phi\rangle_{L^2(\partial\Omega)}$$

for all $\phi \in H^{s/2}(\partial\Omega)$. In particular, we have the following equivalence:

Theorem 2.1.5. *The spaces* $\left(H^{s/2}(\partial\Omega)\right)' = H^{-s/2}(\partial\Omega)$ *are isometrically isomorphic.*

Proof. We have already seen that $H^{-s/2}(\partial\Omega)$ embeds continuously into $\left(H^{s/2}(\partial\Omega)\right)'$ since, for $T \in H^{-s/2}(\partial\Omega)$, we have

$$\begin{aligned}
\|T\| &= \sup_{\|\phi\|_{H^{s/2}}=1} \left| \sum_{k=0}^{\infty} T\left(\hat{s}_k\right) \phi_k \right| \\
&\leq \sup_{\|\phi\|_{H^{s/2}}=1} \left(\sum_{k=0}^{\infty} \left(1+\lambda_k\right)^{-s} T\left(\hat{s}_k\right)^2 \right)^{1/2} \left(\sum_{k=0}^{\infty} \left(1+\lambda_k\right)^{s} \phi_k^2 \right)^{1/2} \\
&= \|T\|_{H^{-s/2}}.
\end{aligned}$$

Suppose $\mathcal{F} \in \left(H^{s/2}(\partial\Omega)\right)'$. By the Riesz representation theorem, there is a unique $v \in H^{s/2}(\partial\Omega)$ for which

$$\mathcal{F}(\phi) = \langle v, \phi \rangle_{H^{s/2}(\partial\Omega)} = \sum_{k=0}^{\infty} \left(1+\lambda_k\right)^{s} v_k \phi_k.$$

Define $u_k = \left(1+\lambda_k\right)^s v_k = \mathcal{F}\left(\hat{s}_k\right)$. Then $\mathcal{F} \in H^{-s/2}(\partial\Omega)$ since

$$\|\mathcal{F}\|^2_{H^{-s/2}(\partial\Omega)} = \sum_{k=0}^{\infty} \left(1+\lambda_k\right)^{-s} u_k^2 = \sum_{k=0}^{\infty} \left(1+\lambda_k\right)^{s} v_k^2 = \|v\|^2_{H^{s/2}(\partial\Omega)}.$$

Furthermore, we have

$$\begin{aligned}
\|\mathcal{F}\| &= \sup_{\|\phi\|_{H^{s/2}}=1} \left| \sum_{k=0}^{\infty} u_k \phi_k \right| \\
&= \sup_{\|\phi\|_{H^{s/2}}=1} \left| \sum_{k=0}^{\infty} \left(1+\lambda_k\right)^{s} v_k \phi_k \right| \\
&\geq \langle v, \frac{v}{\|v\|_{H^{s/2}(\partial\Omega)}} \rangle_{H^{s/2}(\partial\Omega)} \\
&= \|v\|_{H^{s/2}(\partial\Omega)} \\
&= \|\mathcal{F}\|_{H^{-s/2}}(\partial\Omega)
\end{aligned}$$

proving the claim. ∎

Example 2.1.6 (The upper half space)**.** *In the case of the Steklov expansions in the upper half space* $\mathbb{R}^n_+$ *we find out that* $H^s(\mathbb{R}^{n-1})$ *is the space of functions* $g \in L^2(\mathbb{R}^{n-1})$ *such that*

$|\xi'|^s\widehat{g}(\xi') \in L^2(\mathbb{R}^{n-1})$. *In other words, we require* $(-\Delta_{\mathbb{R}^{n-1}})^{s/2}g \in L^2(\mathbb{R}^{n-1})$ *and* $[g]_{H^s(\mathbb{R}^{n-1})} = \|(-\Delta_{\mathbb{R}^{n-1}})^{s/2}g\|_{L^2(\mathbb{R}^{n-1})}$. *This is the usual fractional Sobolev space on* $\mathbb{R}^{n-1}$ *where the norm is given explicitly by*

$$\|g\|^2_{H^s(\mathbb{R}^{n-1})} = \|g\|^2_{L^2(\mathbb{R}^{n-1})} + \int_{\mathbb{R}^{n-1}}\int_{\mathbb{R}^{n-1}} \frac{(g(x')-g(y'))^2}{|x'-y'|^{n-1+s}}\,dx'\,dy'.$$

2.1.3 The space $H_0^s(\partial\Omega)$

We will now consider a useful subspace of $H^s(\Omega)$: the space $H_0^\sigma(\partial\Omega)$ where u has mean 0, that is,

$$\int_{\partial\Omega} u\,dS = 0.$$

This space will be used in defining the notion of weak solution to the fractional Neumann problem.

We define, for $-1 < s < 1$, the space

$$H_0^s(\partial\Omega) := \{u \in H^s(\partial\Omega) : \int_{\partial\Omega} u\,dS = 0 \text{ if } s > 0; \text{ or } \langle u, 1\rangle = 0 \text{ if } s < 0\}.$$

One can equip $H_0^s(\partial\Omega)$ with three equivalent norms. Indeed, consider the following norms defined on $H_0^s(\partial\Omega)$:

$$\|u\|_1 = \sum_{k=1}^{\infty} \lambda_k^{2s}u_k^2,\ \|u\|_2 = \sum_{k=1}^{\infty} (1+\lambda_k)^{2s}\,u_k^2, \text{ and } \|u\|_3 = \sum_{k=1}^{\infty}\left(1+\lambda_k^{2s}\right)u_k^2.$$

We remark that $\|\cdot\|_1$ is indeed a norm since $u_0 = 0$ under the mean 0 assumption.

Lemma 2.1.7. *The norms* $\|\cdot\|_1$ *and* $\|\cdot\|_2$ *are equivalent.*

Proof. It suffices to find constants $C, D > 0$ for which

$$(1+\lambda_k)^{2s} < C\lambda_k^{2s}$$

and

$$\lambda_k^{2s} < D\,(1+\lambda_k)^{2s}$$

for all $k \geq 1$. Observe

$$\left(\frac{1+\lambda_k}{\lambda_k}\right)^{2s} \to 1$$

as $k \to \infty$, so there is some C for which

$$\left(\frac{1+\lambda_k}{\lambda_k}\right)^{2s} \leq C$$

for all $k \geq 1$. The exact same argument applied to the reciprocal concludes the proof. □

We now show that the norm $\|u\|_3$ is equivalent to the previous two.

Lemma 2.1.8. *The norms $\|\cdot\|_1$ and $\|\cdot\|_3$ are equivalent.*

Proof. The direction $\|u\|_1 \leq C\|u\|_3$ is trivial. To see the reverse inequality, consider the following quotient

$$\frac{1+\lambda_k^{2s}}{\lambda_k^{2s}} = \frac{1}{\lambda_k^{2s}} + 1 \to 1.$$

Hence we obtain the desired bound.

□

Now, we have three equivalent characterizations for the space $H_0^s(\partial\Omega)$. We can write a similar argument as in the previous section to conclude the dual space of

$$H_0^s(\partial\Omega) = \{u \in L^2(\partial\Omega) : \int_{\partial\Omega} u\, dS = 0, \sum_{k=1}^{\infty} \lambda_k^{2s} u_k^2 < \infty\}$$

is precisely

$$H_0^{-s}(\partial\Omega) := \{T \in (H_0^{\sigma}(\partial\Omega))' : \sum_{k=1}^{\infty} \lambda_k^{-2s} T\left(\hat{s}_k\right)^2 < \infty\}.$$

Now, we can translate between the spaces $H_0^s(\partial\Omega)$ and $H^s(\partial\Omega)$ via the following mapping::

Theorem 2.1.9. *Given $u \in H^s(\partial\Omega)$, the function*

$$s(u) = u - u_0 s_0 = \sum_{k=1}^{\infty} u_k s_k$$

is in $H_0^s(\partial\Omega)$.

2.2 The fractional normal derivative ∂_ν^σ and its inverse $\partial_\nu^{-\sigma}$

We now define the fractional powers of the normal derivative $\partial_\nu^{\pm\sigma}$ in terms of Steklov eigenfunctions.

2.2.1 Definition of ∂_ν^σ

If the boundary of Ω is C^1 and $u \in C^1(\overline{\Omega})$ then $\partial_\nu u(x) = \nabla u(x) \cdot \nu(x)$. When the Steklov eigenfunctions s_k are smooth we have $\partial_\nu s_k = \lambda_k \hat{s}_k$, for each $k \geq 0$. Let

$$u = \sum_{k=0}^{\infty} u_k s_k \in \mathcal{H}(\Omega), \tag{2.10}$$

where the coefficients are given by $u_k = \frac{\langle u, s_k \rangle_\partial}{1+\lambda_k}$. Then the (linear extension of the) normal derivative of u is given by

$$\partial_\nu u = \sum_{k=1}^{\infty} \lambda_k u_k \hat{s}_k, \quad \text{on } \partial\Omega.$$

We define the fractional normal derivative of order σ of a harmonic function $u \in \mathcal{H}(\Omega)$ as in (2.10) by

$$\partial_\nu^\sigma u := \sum_{k=1}^{\infty} \lambda_k^\sigma u_k \hat{s}_k, \quad \text{on } \partial\Omega. \tag{2.11}$$

Notice that $\partial_\nu^1 = \partial_\nu$ and $\partial_\nu^{\sigma_1} \circ \partial_\nu^{\sigma_2} = \partial_\nu^{\sigma_1+\sigma_2}$. Also, $\partial_\nu^\sigma(1) = 0$. We have

$$\|\partial_\nu^\sigma u\|^2_{H^s(\partial\Omega)} = \sum_{k=1}^{\infty} (1+\lambda_k^{2s}) \lambda_k^{2\sigma} \left| \frac{\langle u, s_k \rangle_\partial}{1+\lambda_k} \right|^2,$$

so that ∂_ν^σ is a continuous map from $\mathcal{H}(\Omega)$ to $H^{1/2-\sigma} \subseteq H^s(\partial\Omega)$, for $s \leq 1/2 - \sigma$. Moreover, the norm in $H^{\sigma/2}(\partial\Omega)$ can be expressed in terms of ∂_ν^σ. Indeed, for $f, g \in H^{\sigma/2}(\partial\Omega)$ we have

$$\int_{\partial\Omega} (fg + f\partial_\nu^\sigma \mathcal{E}g)\, dS = \langle f, g + \partial_\nu^\sigma \mathcal{E}g \rangle_{L^2(\partial\Omega)} = \sum_{k=0}^{\infty} (1+\lambda_k^\sigma) f_k g_k = \langle f, g \rangle_{H^{\sigma/2}(\partial\Omega)}. \tag{2.12}$$

For details in the case $\sigma = 1$ see [4, Theorems 6.1 and 7.1].

We can look at ∂_ν^σ as an operator acting directly on functions g living on the boundary $\partial\Omega$, without considering its harmonic extension $\mathcal{E}g$ as in the previous paragraph. Let $g \in H^s(\partial\Omega)$, $s \in \mathbb{R}$, with Steklov expansion (2.7). We define

$$\partial_\nu^\sigma g := \sum_{k=1}^{\infty} \lambda_k^\sigma g_k \hat{s}_k.$$

This definition coincides with the one above in terms of $\mathcal{E}g$ when $g \in H^s(\partial\Omega)$ for $s \geq 0$. Then $\partial_\nu^\sigma : H^s(\partial\Omega) \to H^{s-\sigma}(\partial\Omega)$ and $[g]_{H^s(\partial\Omega)} = [\partial_\nu^\sigma g]_{H^{s-\sigma}(\partial\Omega)}$, for $s \in \mathbb{R}$. In particular, (2.12) becomes

$$\langle f, g \rangle_{H^{\sigma/2}(\partial\Omega)} = \langle f, g \rangle_{L^2(\partial\Omega)} + \langle \partial_\nu^{\sigma/2} f, \partial_\nu^{\sigma/2} g \rangle_{L^2(\partial\Omega)}.$$

When $g \in H^s(\partial\Omega)$ and $s - \sigma \geq 0$ we have that $\partial_\nu^\sigma g$ is still a function in a fractional trace space $H^{s-\sigma}(\partial\Omega)$. When $s - \sigma < 0$ we get that $\partial_\nu^\sigma g$ is a generalized function, so that $\partial_\nu^\sigma g$ can be seen as a distribution acting on $H^{\sigma-s}(\partial\Omega)$ via the usual pairing:

$$\langle \partial_\nu^\sigma g, h\rangle = \sum_{k=1}^{\infty} \lambda_k^\sigma g_k h_k, \quad \text{for any } h \in H^{\sigma-s}(\partial\Omega),\ h_k = \langle h, \hat{s}_k\rangle_{L^2(\partial\Omega)}. \tag{2.13}$$

Observe that $|\langle \partial_\nu^\sigma g, h\rangle| \leq [g]_{H^s(\partial\Omega)}[h]_{H^{\sigma-s}(\partial\Omega)}$. Let us make explicit the borderline cases. For $g \in H^{\sigma/2}(\partial\Omega)$ with Steklov expansion (2.7), we have that $\partial_\nu^\sigma g \in H^{-\sigma/2}(\partial\Omega)$. If $g \in H^\sigma(\partial\Omega)$ then $\partial_\nu^\sigma g \in L^2(\Omega)$ and $[g]_{H^\sigma(\partial\Omega)} = \|\partial_\nu^\sigma g\|_{L^2(\partial\Omega)}$. Thus,

$$H^\sigma(\partial\Omega) = \big\{g \in L^2(\partial\Omega) : \partial_\nu^\sigma g \in L^2(\partial\Omega)\big\}, \quad \text{when } 0 \leq \sigma \leq 1.$$

Lemma 2.2.1 (Poincaré inequality)**.** *Let $g \in H^\sigma(\partial\Omega)$ for some $\sigma \geq 0$. Then*

$$\int_{\partial\Omega} \big(g - (g)_{\partial\Omega}\big)^2\, dS \leq C \int_{\partial\Omega} |\partial_\nu^\sigma g|^2\, dS,$$

where $(g)_{\partial\Omega} = \frac{1}{|\partial\Omega|}\int_{\partial\Omega} g\, dS$, and $C > 0$ depends only on σ and $\partial\Omega$.

Proof. As $g - (g)_{\partial\Omega} = \sum_{k=1}^\infty c_k \hat{s}_k$ in $L^2(\partial\Omega)$, for some coefficients c_k, and the sequence of Steklov eigenvalues is non decreasing, we have

$$\int_{\partial\Omega} (g - (g)_{\partial\Omega})^2\, dS = \sum_{k=1}^\infty c_k^2 \leq \frac{1}{\lambda_1^{2\sigma}} \sum_{k=1}^\infty \lambda_k^{2\sigma} c_k^2 = C \int_{\partial\Omega} |\partial_\nu^\sigma g|^2\, dS.$$

□

2.2.2 Definition of $\partial_\nu^{-\sigma}$

Observe that $\partial_\nu^\sigma : H^s(\partial\Omega) \to H_0^{s-\sigma}(\partial\Omega)$ by definition. In a completely analogous way as we did before, we define the negative powers $\partial_\nu^{-\sigma} g$ for $g \in H_0^s(\partial\Omega)$ with expansion (2.7) (the sum starting at $k = 1$) as

$$\partial_\nu^{-\sigma} g = \sum_{k=1}^\infty \lambda_k^{-\sigma} g_k \hat{s}_k \in H_0^{s+\sigma}(\partial\Omega).$$

Moreover, $\|\partial_\nu^{-\sigma} g\|_{H_0^{s+\sigma}(\partial\Omega)} \leq \|g\|_{H_0^s(\partial\Omega)}$.

2.3 The fractional Neumann problem

Now that the fractional operator ∂_ν^σ has been properly defined, let us describe the solutions to the fractional Neumann problem

$$\begin{cases} \Delta u = 0, & \text{in } \Omega, \\ \partial_\nu^\sigma u = f, & \text{on } \partial\Omega. \end{cases} \tag{2.14}$$

2.3.1 Existence of weak solutions and basic estimates

In order to define the concept of weak solution to (2.14), we multiply the equation by a test function φ, integrate by parts and use the spectral definition of the fractional normal derivative to get

$$\begin{aligned} \int_\Omega \nabla u \nabla \varphi \, dx &= \int_{\partial\Omega} \varphi \partial_\nu u \, dS = \int_{\partial\Omega} \varphi \partial_\nu^{1-\sigma} \partial_\nu^\sigma u \, dS \\ &= \int_{\partial\Omega} \varphi \partial_\nu^{1-\sigma} f \, dS = \int_{\partial\Omega} \left(\partial_\nu^{(1-\sigma)/2}\varphi\right)\left(\partial_\nu^{(1-\sigma)/2} f\right) dS. \end{aligned}$$

We are looking for solutions $u \in H^1(\Omega)$, so we take $\varphi \in H^1(\Omega)$. Then the traces of u and φ on $\partial\Omega$ are in $H^{1/2}(\partial\Omega)$. This implies that $\partial_\nu^\sigma u \in H_0^{1/2-\sigma}(\partial\Omega)$. Hence we need to take the Neumann boundary data f in $H_0^{1/2-\sigma}(\partial\Omega)$. In this case, $\partial_\nu^{1-\sigma} f \in H_0^{-1/2}(\partial\Omega)$ and the second to last integral above is well defined. On the other hand, $\partial_\nu^{(1-\sigma)/2}\varphi \in H_0^{\sigma/2}(\partial\Omega)$, so for the last integral above to be well defined we need $\partial_\nu^{(1-\sigma)/2} f \in H_0^{-\sigma/2}(\partial\Omega)$, namely, $f \in H_0^{1/2-\sigma}(\partial\Omega)$ again.

Definition 2.3.1. Let $f \in H_0^{1/2-\sigma}(\partial\Omega)$. We say that a function $u \in H^1(\Omega)$ is a weak solution to (2.14) whenever

$$\int_\Omega \nabla u \nabla \varphi \, dx = \langle \partial_\nu^{1-\sigma} f, \varphi \rangle = \langle \partial_\nu^{(1-\sigma)/2} f, \partial_\nu^{(1-\sigma)/2} \varphi \rangle, \tag{2.15}$$

for every $\varphi \in H^1(\Omega)$.

By the comments above, the pairings appearing in (2.15) are all well defined. In particular, if

$$f = \sum_{k=1}^\infty f_k \hat{s}_k \in H_0^{1/2-\sigma}(\partial\Omega), \tag{2.16}$$

and we write

$$\operatorname{tr}(\varphi)=\sum_{k=0}^{\infty}\varphi_k\hat{s}_k, \quad \text{on } \partial\Omega, \tag{2.17}$$

then the pairings in (2.15) are both equal to

$$\sum_{k=1}^{\infty}\lambda_k^{1-\sigma}f_k\varphi_k=\sum_{k=1}^{\infty}\big(\lambda_k^{(1-\sigma)/2}f_k\big)\big(\lambda_k^{(1-\sigma)/2}\varphi_k\big).$$

Lemma 2.3.2. *Let f be as in (2.16) and let us define*

$$u(x):=\mathcal{E}(\partial_\nu^{-\sigma}f)(x)=\sum_{k=1}^{\infty}\frac{1}{\lambda_k^{\sigma}}f_k s_k(x), \quad \textit{for } x\in\Omega. \tag{2.18}$$

Then $u\in\mathcal{H}(\Omega)$ is a weak solution to (2.14), which is unique up to an additive constant. Moreover

$$\|u\|_{H^1(\Omega)}^2\sim\|u\|_\partial^2=\int_\Omega|\nabla u|^2\,dx+\int_{\partial\Omega}u^2\,dS=\|\operatorname{tr}(u)\|_{H^{1/2}(\partial\Omega)}^2=[f]_{H^{1/2-\sigma}(\partial\Omega)}^2. \tag{2.19}$$

Proof. Recall that the norm $\|\cdot\|_\partial$ is equivalent to the usual $H^1(\Omega)$ norm and that $\{s_k/(1+\lambda_k)^{1/2}\}$ is ∂-orthonormal in $H^1(\Omega)$. Therefore, to show that $u\in H^1(\Omega)$ we need to see that

$$\|u\|_\partial^2=\sum_{k=1}^{\infty}(1+\lambda_k)\frac{1}{\lambda_k^{2\sigma}}f_k^2<\infty.$$

But now,

$$\sum_{\lambda_k\geq 1}\frac{1+\lambda_k}{\lambda_k^{2\sigma}}f_k^2\leq 2\sum_{k=1}^{\infty}\lambda_k^{1-2\sigma}f_k^2=2[f]_{H^{1/2-\sigma}(\partial\Omega)}^2,$$

which is finite because $f\in H^{1/2-\sigma}(\partial\Omega)$. Next we have to check that

$$\int_\Omega\nabla u\nabla\varphi\,dx=\sum_{k=1}^{\infty}\lambda_k^{1-\sigma}f_k\varphi_k,$$

for every $\varphi\in H^1(\Omega)$ with trace as in (2.17). As we already saw, any function $\varphi\in H^1(\Omega)$ can be written in a unique way as $\varphi=\mathcal{P}\varphi+\psi$, where $\mathcal{P}\varphi\in\mathcal{H}(\Omega)=\overline{\operatorname{span}\{s_k:k\geq 0\}}^{\partial}$ and $\psi\in H_0^1(\Omega)$, with $\mathcal{P}\varphi$ and ψ orthogonal with respect to the ∂-inner product. By the orthogonality of $\hat{s}_k$ and the fact that $\operatorname{tr}(\mathcal{P}\varphi)=\operatorname{tr}(\varphi)$, we have $\mathcal{P}\varphi=\sum_{k=0}^{\infty}\varphi_k s_k$. Therefore, since $\operatorname{tr}(\psi)=0$ on $\partial\Omega$ and ψ is orthogonal to $u,\varphi\in\mathcal{H}(\Omega)$ with respect to the ∂-inner product,

$$\int_\Omega\nabla u\nabla\varphi\,dx=\int_\Omega\nabla u\nabla(\mathcal{P}\varphi)\,dx+\int_\Omega\nabla u\nabla\psi\,dx$$

$$= \int_\Omega \nabla u \nabla (\mathcal{P}\varphi)\, dx + \int_\Omega \nabla u \nabla \psi \, dx + \int_{\partial\Omega} u\psi \, dS$$
$$= \int_\Omega \nabla u \nabla (\mathcal{P}\varphi)\, dx + \langle u, \psi \rangle_\partial = \int_\Omega \nabla u \nabla (\mathcal{P}\varphi)\, dx.$$

Notice now that, by the definition of s_k and the orthonormality of $\hat{s}_k$,

$$\int_\Omega \nabla u \nabla s_k \, dx = \int_{\partial\Omega} u \lambda_k \hat{s}_k \, dS = \lambda_k^{1-\sigma} f_k,$$

so that by the previous identity and by using linearity and the density of $\hat{s}_k$ we finally have

$$\int_\Omega \nabla u \nabla \varphi \, dx = \int_\Omega \nabla u \nabla (\mathcal{P}\varphi)\, dx = \int_{\partial\Omega} u \left(\sum_{k=0}^{\infty} \lambda_k \varphi_k \hat{s}_k \right) dS = \sum_{k=1}^{\infty} \lambda_k^{1-\sigma} f_k \varphi_k.$$

The uniqueness holds up to an additive constant because $\partial_\nu^\sigma 1 = 0$. The estimate (2.19) follows from the fact that $\|u\|_\partial = \|\operatorname{tr}(u)\|_{H^{1/2}(\Omega)}$ (see (2.9)) and the definition of u. □

The proof showed that it is enough to take $\varphi \in \mathcal{H}(\Omega)$ in Definition 2.3.1.

Definition 2.3.3. Let $f \in H_0^{1/2-\sigma}(\partial\Omega)$ be as in (2.16). We say that a function $u \in \mathcal{H}(\Omega)$ is a weak solution to (2.14) if and only if (2.15) holds for every $\varphi \in \mathcal{H}(\Omega)$.

The following simple estimate follows by taking $\varphi = u$ in the weak definition.

Lemma 2.3.4. *Let $u \in \mathcal{H}(\Omega)$ be a weak solution to (2.14) with $f \in H_0^{1/2-\sigma}(\partial\Omega)$. Then $\operatorname{tr}(u) \in H^{1/2}(\partial\Omega)$, $\partial_\nu^\sigma u \in H_0^{1/2-\sigma}(\partial\Omega)$ and*

$$\int_\Omega |\nabla u|^2 \, dx = [f]^2_{H^{1/2-\sigma}} = \|\partial_\nu^{1/2-\sigma} f\|^2_{L^2(\partial\Omega)} = \|\partial_\nu^{1/2} u\|^2_{L^2(\partial\Omega)} = [\operatorname{tr}(u)]^2_{H^{1/2}(\partial\Omega)}.$$

CHAPTER 3. METHOD OF SEMIGROUPS

In this chapter, we give a rigorous argument for the nonlocality and order of the operator ∂_ν^σ for $0 < \sigma < 1$. The arguments presented utilize the method of semigroups. In particular, we define the semigroup generated by ∂_ν and use relevant semigroup formulas to present a pointwise integro-differential formula for the operator $\partial_\nu^{\pm\sigma}$ which, in the case that $\partial\Omega$ is smooth, has useful kernel estimates.

3.1 Semigroup generated by ∂_ν

We first make sense of the expression $e^{-t\partial_\nu}g$. Let $g \in H^s(\partial\Omega)$. For $t \geq 0$, we define

$$e^{-t\partial_\nu} := \sum_{k=0}^{\infty} e^{-t\lambda_k} g_k \hat{s}_k$$

where the above sum is understood in $H^s(\partial\Omega)$. It is routine to show that $e^{-t\partial_\nu} : H^s(\partial\Omega) \to H^s(\partial\Omega)$ satisfies the semigroup property. Let $t_1, t_2 \geq 0$. Then

$$e^{-t_1\partial_\nu}\left(e^{-t_2\partial_\nu} g\right) = \sum_{k=0}^{\infty} e^{-t_1\lambda_k} \langle e^{-t_2\partial_\nu}, \hat{s}_k\rangle \hat{s}_k.$$

Observe

$$\langle e^{-t_2\partial_\nu} g, \hat{s}_k\rangle = \sum_{j=0}^{\infty} e^{-t_2\lambda_j} g_j \langle \hat{s}_j, \hat{s}_k\rangle = e^{-t_2\lambda_k} g_k.$$

Hence

$$e^{-t_1\partial_\nu}\left(e^{-t_2\partial_\nu} g\right) = \sum_{k=0}^{\infty} e^{-(t_1+t_2)\lambda_k} g_k \hat{s}_k = e^{-(t_1+t_2)\partial_\nu} g.$$

We have that $\left[e^{-t\partial_\nu} g\right]_{H^s(\partial\Omega)} \leq [g]_{H^s(\partial\Omega)}$ for all $t \geq 0$ since

$$\left[e^{-t\partial_\nu} g\right]_{H^s(\partial\Omega)} = \sum_{k=0}^{\infty} \lambda_k^{2s} e^{-2t\lambda_k} g_k^2 \leq \sum_{k=0}^{\infty} \lambda_k^{2s} g_k^2 = [g]_{H^s(\partial\Omega)}.$$

Also, $e^{-t\partial_\nu} g \to g$ in $H^s(\partial\Omega)$, as $t \to 0^+$, by the dominated convergence theorem as

$$\|e^{-t\partial_\nu} g - g\|_{H^s(\partial\Omega)}^2 = \sum_{k=0}^{\infty} (1 + \lambda_k^{2s})(e^{-t\lambda_k} - 1)^2 g_k^2.$$

Moreover,

$$\partial_t(e^{-t\partial_\nu} g) = -\partial_\nu \left(e^{-t\partial_\nu}\right),$$

for every $t > 0$, in $L^2(\partial\Omega)$. Hence if g is smooth enough, $v(x,t) \equiv e^{-t\partial_\nu} g(x)$ satisfies

$$\begin{cases} \partial_t v = -\partial_\nu v, & \text{for } x \in \partial\Omega,\ t > 0, \\ v(x,0) = g(x), & \text{for } x \in \partial\Omega \end{cases}.$$

We prove the first part of the following lemma.

Lemma 3.1.1. *Let $g \in L^2(\partial\Omega)$. Then for any $h \in L^2(\partial\Omega)$ we have*

$$\langle e^{-t\partial_\nu} g, h\rangle_{L^2(\partial\Omega)} = \iint_{\partial\Omega} W_t(x,z) g(x) h(z)\, dS_x\, dS_z,$$

where $W_t(x,z) := \sum_{k=0}^{\infty} e^{-t\lambda_k} \hat{s}_k(x) \hat{s}_k(z)$, for $t > 0$, $x, z \in \partial\Omega$. Moreover,

$$e^{-t\partial_\nu} 1(x) = \int_{\partial\Omega} W_t(x,z)\, dS_z = 1, \quad \text{for every } x \in \partial\Omega,\ t \geq 0.$$

Suppose that $\partial\Omega$ is a smooth manifold and denote by $d(x,z)$ the geodesic distance between the points x and z on $\partial\Omega$. In this case, the kernel $W_t(x,z)$ satisfies the following Poisson-type estimates (see [20, Theorem 4.4], also [14, 28]):

(1) $0 \leq W_t(x,z) \leq C \dfrac{t}{(t^2 + d(x,z)^2)^{n/2}}$, *for all $t \geq 0$, $x, z \in \partial\Omega$;*

(2) There is $R_0 > 0$ such that $W_t(x,z) \geq c \dfrac{t}{(t^2 + d(x,z)^2)^{n/2}}$, *for $d(x,z) + t \leq R_0$;*

for some positive constants C, c.

Proof. Let $g, h \in L^2(\partial\Omega)$ with Steklov expansions

$$\sum_{k=0}^{\infty} g_k \hat{s}_k \text{ and } \sum_{k=0}^{\infty} h_k \hat{s}_k,$$

respectively. Then by the $L^2(\partial\Omega)$ orthonormality of the $\{\hat{s}_k\}$, we have

$$\langle e^{-t\partial_\nu} g, h\rangle_{L^2(\partial\Omega)} = \sum_{k=0}^{\infty} e^{-t\lambda_k} \langle g, \hat{s}_k\rangle_{L^2(\partial\Omega)} \langle h, \hat{s}_k\rangle_{L^2(\partial\Omega)} = \iint_{\partial\Omega} W_t(x,z) g(x) h(z)\, dS_x\, dS_z.$$

Moreover,

$$e^{-t\partial_\nu}1(x)=\sum_{k=0}^{\infty}e^{-t\lambda_k}\langle 1,\hat{s}_k\rangle_{L^2(\partial\Omega)}\hat{s}_k\,(x)=\sum_{k=0}^{\infty}e^{-t\lambda_k}\hat{s}_k(x)\int_{\partial\Omega}s_k(x)\,dS_z=\int_{\partial\Omega}W_t(x,z)\,dS_z,$$

and

$$\sum_{k=0}^{\infty}e^{-t\lambda_k}\langle 1,\hat{s}_k\rangle_{L^2(\partial\Omega)}\hat{s}_k\,(x)=e^0\,|\partial\Omega|^{1/2}\left(\frac{1}{|\partial\Omega|^{1/2}}\right)=1.$$

□

We just give a short remark about the nonnegativity of the kernel $W_t(x,z)$. Though $v(x,t)=e^{-t\partial_\nu}g(x)$ is defined in principle for $x\in\partial\Omega$, it can be trivially extended into Ω as a harmonic function in x, that we still denote by $v(x,t)$, for every $t\geq 0$. If $p\geq 1$ then, integrating by parts,

$$\begin{aligned}\frac{d}{dt}\|v(\cdot,t)\|_{L^{2p}(\partial\Omega)}^{2p}&=-2p\int_{\partial\Omega}v^{2p-1}\partial_\nu v\,dS\\&=-2p\int_\Omega v^{2p-1}\Delta v\,dx-2p\int_\Omega\nabla(v^{2p-1})\nabla v\,dx\\&=-2p(2p-1)\int_\Omega(v^{p-1})^2|\nabla v|^2\,dx\leq 0.\end{aligned}$$

Hence, by taking $p\to\infty$, we get $\|e^{-t\partial_\nu}g\|_{L^\infty(\partial\Omega)}\leq\|g\|_{L^\infty(\partial\Omega)}$. Thus, if g and Ω are smooth,

$$\inf_{(x,t)\in\partial\Omega\times(0,\infty)}e^{-t\partial_\nu}g(x)=\inf_{x\in\partial\Omega}g(x).$$

Therefore, $e^{-t\partial_\nu}g\geq 0$ for any $g\geq 0$, and so the kernel $W_t(x,z)$ is nonnegative.

3.2 Semigroup formulas for $\partial_\nu^{\pm\sigma}$

Recall that if $g\in H^s(\partial\Omega)$, $s\in\mathbb{R}$, is of the form (2.7) then $\partial_\nu^\sigma g\in H^{s-\sigma}(\partial\Omega)$ is given in general by (2.13).

Lemma 3.2.1. *Let $g\in H^s(\partial\Omega)$, $s\in\mathbb{R}$. Then*

$$\partial_\nu^\sigma g=\frac{1}{\Gamma(-\sigma)}\int_0^\infty\left(e^{-t\partial_\nu}g-g\right)\frac{dt}{t^{1+\sigma}},$$

in the sense that, for any $h \in H^{\sigma-s}(\partial\Omega)$,

$$\langle \partial_\nu^\sigma g, h\rangle = \frac{1}{\Gamma(-\sigma)} \int_0^\infty \left(\langle e^{-t\partial_\nu} g, h\rangle - \langle g, h\rangle\right) \frac{dt}{t^{1+\sigma}}, \tag{3.1}$$

and the integral is absolutely convergent.

Proof. Since $\partial_\nu^\sigma : H^s(\partial\Omega) \to H^{s-\sigma}(\partial\Omega)$, $e^{-t\partial_\nu} : H^s(\partial\Omega) \to H^s(\partial\Omega)$ and $H^s(\partial\Omega) \subset H^{s-\sigma}(\partial\Omega)$, all the pairings appearing in the formula above are well defined. Recall the numerical identity with the Gamma function

$$\lambda_k^\sigma = \frac{1}{\Gamma(-\sigma)} \int_0^\infty \left(e^{-t\lambda_k} - 1\right) \frac{dt}{t^{1+\sigma}}.$$

Observe that

$$\int_0^\infty \sum_{k=0}^\infty |e^{-t\lambda_k} - 1||g_k||h_k| \frac{dt}{t^{1+\sigma}} = |\Gamma(-\sigma)| \sum_{k=0}^\infty \lambda_k^\sigma |g_k||h_k| < \infty.$$

The conclusion follows from Fubini's Theorem. □

Lemma 3.2.2. *Let* $f \in H_0^s(\partial\Omega)$, $s \in \mathbb{R}$*. Then* $\partial_\nu^{-\sigma} f \in H_0^{s+\sigma}(\partial\Omega)$ *and*

$$\partial_\nu^{-\sigma} f = \frac{1}{\Gamma(\sigma)} \int_0^\infty e^{-t\partial_\nu} f \frac{dt}{t^{1-\sigma}},$$

in the sense that for any $h \in H_0^{-s-\sigma}(\partial\Omega)$ *we have*

$$\langle \partial_\nu^{-\sigma} f, h\rangle = \frac{1}{\Gamma(\sigma)} \int_0^\infty \langle e^{-t\partial_\nu} f, h\rangle \frac{dt}{t^{1-\sigma}}, \tag{3.2}$$

and the integral is absolutely convergent.

Proof. The computation is analogous the one in the proof of Lemma 3.2.1. In this case we have to use the identity with the Gamma function

$$\lambda_k^{-\sigma} = \frac{1}{\Gamma(\sigma)} \int_0^\infty e^{-t\lambda_k} \frac{dt}{t^{1-\sigma}},$$

and notice that $\sum_{k=1}^\infty \lambda_k^{-\sigma} |f_k||h_k| \le [f]_{H^s(\partial\Omega)} [h]_{H^{-s-\sigma}(\partial\Omega)}$. □

3.3 Pointwise formulas for $\partial_\nu^{\pm\sigma}$

Starting from the semigroup formulas given in Lemmas 3.2.1 and 3.2.2 we can obtain pointwise integro-differential formulas for ∂_ν^σ and $\partial_\nu^{-\sigma}$. The kernels are given in terms of the kernel $W_t(x,z)$ of $\{e^{-t\partial_\nu}\}$, which has Poisson estimates (see Lemma 3.1.1).

Lemma 3.3.1. *Suppose that $\partial\Omega$ is a smooth manifold. Define*

$$K_\sigma(x,z) := \frac{1}{2|\Gamma(-\sigma)|}\int_0^\infty W_t(x,z)\,\frac{dt}{t^{1+\sigma}}, \quad x,z\in\partial\Omega.$$

Then

$$0 \le K_\sigma(x,z) \le \frac{C}{d(x,z)^{n-1+\sigma}}, \quad \textit{for any } x,z\in\partial\Omega,$$

and, for the constant $R_0>0$ of Lemma 3.1.1, we have

$$K_\sigma(x,z) \ge \frac{c}{d(x,z)^{n-1+\sigma}}, \quad \textit{whenever } d(x,z)<R_0/2.$$

Moreover, for $g,h\in H^{\sigma/2}(\partial\Omega)$,

$$\langle \partial_\nu^\sigma g, h\rangle = \langle \partial_\nu^{\sigma/2} g, \partial_\nu^{\sigma/2} h\rangle = \iint_{\partial\Omega} \big(g(x)-g(z)\big)\big(h(x)-h(z)\big)K_\sigma(x,z)\,dS_x\,dS_z, \tag{3.3}$$

where the double integral is absolutely convergent.

Proof. If $\partial\Omega$ is a smooth manifold then we have estimates (1) and (2) of Lemma 3.1.1. As $W_t\ge 0$ we have $K_\sigma\ge 0$. By the change of variables $r=d(x,z)/t$ we obtain

$$K_\sigma(x,z) \le C\int_0^\infty \frac{t^{1-\sigma-n}}{(1+(d(x,z)/t)^2)^{n/2}}\,\frac{dt}{t} = \frac{C}{d(x,z)^{n-1+\sigma}}\int_0^\infty \frac{r^{n-1+\sigma}}{(1+r^2)^{n/2}}\,\frac{dr}{r},$$

and the last integral is convergent because $n\ge 2$ and $0<\sigma<1$. For the lower bound, suppose that $x,z\in\partial\Omega$ satisfy $d(x,z)<R_0/2$. Then for any $t<d(x,z)$ we have $t+d(x,z)<R_0$, so we can apply estimate (2) of Lemma 3.1.1 to get

$$K_\sigma(x,z) \ge c\int_0^{d(x,z)} \frac{t^{1-\sigma-n}}{(1+(d(x,z)/t)^2)^{n/2}}\,\frac{dt}{t} = \frac{c}{d(x,z)^{n-1+\sigma}}\int_1^\infty \frac{r^{n-1+\sigma}}{(1+r^2)^{n/2}}\,\frac{dr}{r},$$

and the last integral is finite because $\sigma<1$. Recall that in the smooth case the spaces $H^{\sigma/2}(\partial\Omega)$ coincide with the fractional trace spaces defined with the usual integral seminorm on $\partial\Omega$ (see [4]).

Then, from (3.1) and Lemma 3.1.1,

$$\begin{aligned}\langle \partial_\nu^\sigma g, h\rangle &= \frac{1}{\Gamma(-\sigma)}\int_0^\infty \iint_{\partial\Omega} W_t(x,z)g(z)\big(h(x)-h(z)\big)\,dS_z\,dS_x\,\frac{dt}{t^{1+\sigma}}\\ &= \frac{1}{\Gamma(-\sigma)}\int_0^\infty \iint_{\partial\Omega} W_t(x,z)g(x)\big(h(z)-h(x)\big)\,dS_z\,dS_x\,\frac{dt}{t^{1+\sigma}},\end{aligned}$$

where we used the symmetry of W_t. By adding both expressions and applying Fubini's Theorem (use the upper bound on K_σ) we obtain (3.3). □

Lemma 3.3.2. *Let $\partial\Omega$ be a smooth manifold. Define*

$$K_{-\sigma}(x,z) := \frac{1}{\Gamma(\sigma)}\int_0^\infty W_t(x,z)\,\frac{dt}{t^{1-\sigma}}, \quad x,z\in\partial\Omega.$$

Then

$$0 \le K_{-\sigma}(x,z) \le \frac{C}{d(x,z)^{n-1-\sigma}}, \quad \text{for any } x,z\in\partial\Omega,$$

and, for the constant $R_0>0$ of Lemma 3.1.1, we have

$$K_{-\sigma}(x,z) \ge \frac{c}{d(x,z)^{n-1-\sigma}}, \quad \text{whenever } d(x,z) < R_0/2.$$

For $f,h\in L^2(\partial\Omega)$ we have

$$\langle \partial_\nu^{-\sigma} f, h\rangle = \langle \partial_\nu^{-\sigma/2} f, \partial_\nu^{-\sigma/2} h\rangle = \iint_{\partial\Omega} K_{-\sigma}(x,z)f(z)h(x)\,dS_x\,dS_z, \tag{3.4}$$

where the double integral is absolutely convergent.

Proof. The estimates on $K_{-\sigma}(x,z)$ are obtained by applying the heat kernel estimates of Lemma 3.1.1 as we did in the proof of Lemma 3.3.1. On one hand, by Hölder's inequality,

$$\begin{aligned}\int_0^1\int_{\partial\Omega}\left[\int_{\partial\Omega} W_t(x,z)|f(z)|\,dS_z\right]|h(x)|\,dS_x\,\frac{dt}{t^{1-\sigma}} &\le \int_0^1 \|e^{-t\partial_\nu}|f|\|_{L^2(\partial\Omega)}\|h\|_{L^2(\partial\Omega)}\,\frac{dt}{t^{1-\sigma}}\\ &\le \|f\|_{L^2(\partial\Omega)}\|h\|_{L^2(\partial\Omega)}\int_0^1 t^{\sigma-1}\,dt < \infty.\end{aligned}$$

On the other hand, by using the heat kernel estimates of Lemma 3.1.1,

$$\begin{aligned}\int_1^\infty\int_{\partial\Omega}\left[\int_{\partial\Omega} W_t(x,z)|f(z)|\,dS_z\right]|h(x)|\,dS_x\,\frac{dt}{t^{1-\sigma}} &\le C\iint_{\partial\Omega}|f(z)||h(x)|\,dS_z\,dS_x\int_1^\infty t^{\sigma-n}\,dt\\ &\le C\|f\|_{L^2(\partial\Omega)}\|h\|_{L^2(\partial\Omega)} < \infty,\end{aligned}$$

because $\partial\Omega$ has finite measure, $\sigma<1$ and $n\ge 2$. Hence we can write down the formula with the heat kernel of Lemma 3.1.1 into (3.2) and apply Fubini's Theorem. This gives (3.4). □

3.3.1 Scaling

Let Ω be a C^1 bounded domain and take $u \in C^1(\overline{\Omega})$. Denote by $\frac{1}{\lambda}\Omega$, $\lambda > 0$, the set of points $\bar{x}$ such that $\bar{x} = x/\lambda$, for some $x \in \Omega$. For example, if $\Omega = B_\lambda$ then $\frac{1}{\lambda}\Omega = B_1$. Let us define the function

$$u_\lambda(\bar{x}) = u(\lambda \bar{x}), \quad \text{for } \bar{x} \in \tfrac{1}{\lambda}\Omega. \tag{3.5}$$

This is well defined because $\lambda\bar{x} \in \Omega$. Observe that the exterior unit normal $\nu_{\bar{x}}$ at a point $\bar{x} = x/\lambda$ at the boundary $\partial(\frac{1}{\lambda}\Omega)$ is exactly the exterior unit normal ν_x at the corresponding $x = \lambda\bar{x} \in \partial\Omega$. The operator ∂_ν depends on the boundary of the domain Ω. Let us call $\partial_{\nu,\lambda}$ the normal derivative operator for $\frac{1}{\lambda}\Omega$. Then, for $\bar{x} = x/\lambda$ at the boundary of $\frac{1}{\lambda}\Omega$,

$$\partial_{\nu,\lambda} u_\lambda(\bar{x}) = \nabla u_\lambda(\bar{x}) \cdot \nu_{\bar{x}} = \lambda(\nabla u)(\lambda\bar{x}) \cdot \nu_x = \lambda \partial_\nu u(\lambda\bar{x}).$$

In other words, the normal derivative of the scaled function u_λ at a boundary point $\bar{x}$ in the scaled domain $\frac{1}{\lambda}\Omega$ is λ times the normal derivative of the original u at the original boundary point $x = \lambda\bar{x} \in \partial\Omega$. We say that the normal derivative *scales like* λ.

Lemma 3.3.3. *Let $u : \Omega \to \mathbb{R}$ and let $u_\lambda : \frac{1}{\lambda}\Omega \to \mathbb{R}$ be defined as in (3.5). Then*

$$(\partial_{\nu,\lambda})^{\pm\sigma} u_\lambda(\bar{x}) = \lambda^{\pm\sigma}(\partial_\nu^\sigma u)(\lambda\bar{x}),$$

for every $\bar{x} \in \partial(\frac{1}{\lambda}\Omega)$. That is, the fractional operators $\partial_\nu^{\pm\sigma}$ scale like $\lambda^{\pm\sigma}$.

Proof. Let $v(x,t) = e^{-t\partial_\nu}u(x)$, for $x \in \partial\Omega$. Then the semigroup $v_\lambda(\bar{x},t)$ generated by $\partial_{\nu,\lambda}$ on $\partial(\frac{1}{\lambda}\Omega)$ is related to v via

$$v_\lambda(\bar{x},t) \equiv e^{-t\partial_{\nu,\lambda}}u_\lambda(\bar{x}) = e^{-(\lambda t)\partial_\nu}u(\lambda\bar{x}) = v(\lambda\bar{x},\lambda t), \quad \text{for } \bar{x} \in \tfrac{1}{\lambda}\Omega,\ t > 0.$$

Indeed, $v_\lambda(\bar{x},0) = u(\lambda\bar{x}) = u_\lambda(\bar{x})$ and

$$\partial_t v_\lambda(\bar{x},t) = \lambda(\partial_t v)(\lambda\bar{x},\lambda t) = -\lambda(\partial_\nu v)(\lambda\bar{x},\lambda t) = -\partial_{\nu,\lambda}v_\lambda(\bar{x},t).$$

Hence, by (3.1), the following identities hold in the weak sense:

$$(\partial_{\nu,\lambda})^\sigma u_\lambda(\bar{x}) = \frac{1}{\Gamma(-\sigma)} \int_0^\infty \big(v(\lambda\bar{x},\lambda t) - u_\lambda(\bar{x})\big)\,\frac{dt}{t^{1+\sigma}}$$

$$= \frac{\lambda^\sigma}{\Gamma(-\sigma)} \int_0^\infty \big(v(\lambda\bar{x}, r) - u_\lambda(\bar{x})\big) \frac{dr}{r^{1+\sigma}}$$
$$= \frac{\lambda^\sigma}{\Gamma(-\sigma)} \int_0^\infty \big(e^{-r\partial_\nu} u(\lambda\bar{x}) - u(\lambda\bar{x})\big) \frac{dr}{r^{1+\sigma}}$$
$$= \lambda^\sigma (\partial_\nu^\sigma u)(\lambda\bar{x}).$$

Analogously, the scaling $(\partial_{\nu,\lambda})^{-\sigma} u_\lambda(\bar{x}) = \lambda^{-\sigma}(\partial_\nu^{-\sigma} u)(\lambda\bar{x})$ follows by using (3.2). □

Lemma 3.3.1 shows that ∂_ν^σ is an integro-differential nonlocal operator of order σ on $\partial\Omega$. Moreover, by Lemma 3.3.3, this operator scales like λ^σ. Let us make explicit all this in the case of the upper half space.

Example 3.3.4. *Recall Examples 2.1.2 and 2.1.6. If a function g defined on $\mathbb{R}^{n-1}$ is written in terms of its Fourier–Steklov expansion (2.6) then, according to our definition of normal derivative ∂_ν, we get*

$$-\partial_{x_n} g(x') = \frac{1}{(2\pi)^{(n-1)/2}} \int_{\mathbb{R}^{n-1}} |\xi'| \widehat{g}(\xi') e^{ix'\cdot\xi'}\, d\xi' = (-\Delta_{\mathbb{R}^{n-1}})^{1/2} g(x'),$$

the fractional Laplacian of order $1/2$ on $\mathbb{R}^{n-1}$, which is an operator of order 1. Now, the semigroup generated by ∂_ν is nothing but the usual Poisson semigroup:

$$v(x', t) = e^{-t(-\Delta_{\mathbb{R}^{n-1}})^{1/2}} g(x) = \frac{\Gamma(n/2)}{\pi^{n/2}} \int_{\mathbb{R}^{n-1}} \frac{t}{(t^2 + |x' - z'|^2)^{n/2}}\, g(z')\, dz', \tag{3.6}$$

for $x' \in \mathbb{R}^{n-1}$ and $t > 0$. In particular, W_t in Lemma 3.1.1 is the usual Poisson kernel in the upper half space $\mathbb{R}^n_+$ and v solves the fractional heat equation

$$\partial_t v + (-\Delta_{\mathbb{R}^{n-1}})^{1/2} v = 0.$$

Next, for $0 < \sigma < 1$, by definition and by using Lemma 3.3.1,

$$(-\partial_{x_n})^\sigma g(x') = \frac{1}{(2\pi)^{(n-1)/2}} \int_{\mathbb{R}^{n-1}} |\xi'|^\sigma \widehat{g}(\xi') e^{ix'\cdot\xi'}\, d\xi'$$
$$= (-\Delta_{\mathbb{R}^{n-1}})^{\sigma/2} g(x') = c_{n,\sigma} \int_{\mathbb{R}^{n-1}} \frac{g(x') - g(z')}{|x' - z'|^{n-1+\sigma}}\, dz',$$

where $c_{n,\sigma} = \frac{2^\sigma \Gamma((n-1+\sigma)/2)}{\pi^{(n-1)/2}\Gamma(-\sigma/2)}$, see also [34]. Similarly,

$$(-\partial_{x_n})^{-\sigma} g(x') = (-\Delta_{\mathbb{R}^{n-1}})^{-\sigma/2} g(x') = c_{n,-\sigma} \int_{\mathbb{R}^{n-1}} \frac{g(z')}{|x' - z'|^{n-1-\sigma}}\, dz'.$$

Clearly the scaling of Lemma 3.3.3 is satisfied.

CHAPTER 4. EXTENSION PROBLEM FOR THE OPERATOR ∂_ν

We present the extension problem characterizations, which are particular cases of [33, 34]. We can look at the extension problem by working directly on a cylinder on top of $\partial\Omega$. This point of view is not completely local in nature. Indeed, it makes use of a Sobolev space given in terms of $\partial_\nu^{1/2}$. Equivalently, we can use the fact that the Steklov eigenfunctions are harmonic functions in Ω and write down the extension problem on a cylinder on top of $\overline{\Omega}$. The latter approach is completely local. From here on we let

$$a := 1 - 2\sigma \in (-1, 1).$$

4.1 Trace spaces

To understand the extension problem characterization in the case of the operator ∂_ν, we begin with some preliminary definitions and results. Let X be a Banach space. We define vector-valued spaces as in [15, Chapter 5]. Let $1 \leq p < \infty$. A strongly measurable function $u : (0, \infty) \to X$ is said to be in the space $L^p\left((0,\infty); X\right)$ if

$$\|u\|_{L^p((0,\infty);X)} = \left(\int_0^\infty \|u(y)\|_X^p \, dy \right)^{1/p} < \infty.$$

We can modify the measure with respect to y to obtain the space

$$L^p_a((0,\infty); X) = L^p\left((0,\infty), y^a \, dy; X\right)$$

with norm

$$\|u\|_{L^p_a((0,\infty),y^a\,dy;X)} = \left(\int_0^\infty \|u\|_X^p \, y^a \, dy \right)^{1/p}.$$

When $p = 2$, a direct computation permits us to write this weighted space as a weightless space via the equivalence

$$u \in L^2_a\left((0,\infty); X\right)$$

if and only if

$$y^{a/2}u \in L^2\left((0,\infty);X\right).$$

When $a=0$, we recover the weightless vector-valued L^p-spaces. We now define weak derivatives in this setting, which will be useful when considering the extension problem characterization. Given $u \in L^1\left((0,\infty);X\right)$, we say $v \in L^1\left((0,\infty);X\right)$ is the weak derivative of u if

$$\int_0^\infty \phi'(y)u(y)\,dy = -\int_0^\infty \phi(y)v(y)\,dy$$

for all real valued test functions $\phi \in C_c^\infty(0,\infty)$. In the case that u has a weak derivative v, we denote $v = u'$ or $v = u_y$.

We now define a weaker notion of vector-valued Sobolev spaces. Given Banach spaces X, Y with X continuously embedded in Y, the space $H_a\left(X,Y\right)$ is defined to be all strongly measurable functions $u:(0,\infty)\to X$ satisfying

$$u \in L_a^2\left((0,\infty);X\right)$$

and

$$u_y \in L_a^2\left((0,\infty);Y\right).$$

We follow closely the construction in [27] to characterize the trace spaces of $H_a(H^{1/2}(\partial\Omega), L^2(\partial\Omega))$. A priori, one knows $u(0) \in L^2(\partial\Omega)$ by classical theory, see for example [15, Chapter 5.8, Theorem 3], since $H_a(H^{1/2}(\partial\Omega), L^2(\partial\Omega)) \subseteq W^{1,2}((0,\infty), L^2(\partial\Omega))$. Let $G(y)f = e^{-y\partial_\nu^{1/2}}f := \sum_{k=0}^\infty e^{-y\lambda_k^{1/2}} f_k \hat{s}_k$ be the semigroup generated by $\partial_\nu^{1/2}$ defined on $L^2(\partial\Omega)$. Define the space $E(\alpha)$ by $u(0) \in L^2(\partial\Omega)$ such that

$$y^{\alpha-1}\left(e^{-y\partial_\nu^{1/2}}u(0) - u(0)\right) \in L^2\left((0,\infty);L^2(\partial\Omega)\right)$$

where $1/2 < \alpha < 1/2$. The norm on this space is given by

$$\|u\|_{L^2(\partial\Omega)} + \left(\int_0^\infty y^{(\alpha-1)/2}\|G(t)u-u\|^2_{L^2(\partial\Omega)}\,dy\right)^{1/2}.$$

Lemma 4.1.1. *If $u(0) \in E(\alpha)$ where $\alpha = (1-2\sigma)/2$, then $u(0) \in H^{\sigma/2}(\partial\Omega)$. The converse also holds.*

Proof. Suppose $f \in E(\alpha)$. Then $f = \sum_{k=0}^{\infty} f_k \hat{s}_k$ since $f \in L^2(\partial\Omega)$. Writing the second piece of the norm on $E(\alpha)$ explicitly, we obtain

$$\begin{aligned}\int_0^\infty y^{(\alpha-1)2}\|G(t)f - f\|^2_{L^2(\partial\Omega)}\, dy &= \int_0^\infty y^{(\alpha-1)2}\left(\sum_{k=0}^{\infty}\left(e^{-y\lambda_k^{1/2}} - 1\right)^2 f_k^2\right) dy \\ &= \sum_{k=0}^{\infty}\left(\int_0^\infty \frac{\left(e^{-y\lambda_k^{1/2}} - 1\right)^2}{y^{(\alpha-1)2}}\, dy\right) f_k^2 \\ &= \sum_{k=0}^{\infty}\lambda_k^\sigma\left(\int_0^\infty \frac{\left(e^{-v} - 1\right)^2}{v^{1-2\sigma}}\, dv\right) f_k^2.\end{aligned}$$

Notice

$$\int_0^\infty \frac{\left(e^{-u} - 1\right)^2}{u^{1-2\sigma}}\, du = C < \infty$$

since $-1 < 1 - 2\sigma < 1$. In particular, for $\varepsilon > 0$, we have

$$\int_0^\varepsilon \frac{\left(e^{-u} - 1\right)^2}{u^{1-2\sigma}}\, du \leq 4\int_0^\varepsilon \frac{1}{u^{1-2\sigma}}\, du = C\left(u^{a+1}\right)\Big|_0^\varepsilon < \infty.$$

We obtain a similar inequality for the tail. Hence

$$\|f\|_{E(\alpha)} = C\|f\|_{H^{\sigma/2}(\partial\Omega)}.$$

$\square$

We now state an analagous theorem as in [27, Theorem 1.1] to obtain the following trace theorem

Theorem 4.1.2. *The space $H_a\left(H^{1/2}(\partial\Omega), L^2(\partial\Omega)\right)$ surjects continuously onto $E(a/2)$ via the mapping $u \mapsto u(0)$, and consequently, surjects continuously onto $H^{\sigma/2}(\partial\Omega)$.*

The proof of this result relies on lemma 4.1.1.

We also have an analagous result for the space $H_{-a}\left(L^2(\partial\Omega), H^{-1/2}(\partial\Omega)\right)$ by using the same semigroup generated by $\partial_\nu^{1/2}$. In fact:

Theorem 4.1.3. *The space $H_{-a}\left(L^2(\partial\Omega), H^{-1/2}(\partial\Omega)\right)$ surjects continuously onto $E(a/2)$ via the mapping $w \mapsto w(0)$, and consequently, surjects continuously onto $H^{-\sigma/2}(\partial\Omega)$.*

We have now justified that the spaces $H^{-\sigma}(\partial\Omega)$ for $(-1,1)$ are trace spaces for the Sobolev spaces in which we will work from here on out.

4.1.1 Continuous functions are dense in $H_a(X,Y)$

We say that a weight $w : \mathbb{R} \to [0,\infty)$ belongs to A_2 if w is locally integrable and there is a constant C such that, for any ball $B \subseteq \mathbb{R}$, we have

$$\left(\frac{1}{|B|}\int_B w(y)\,dy\right)\left(\frac{1}{|B|}\int_B w(y)^{-1}\,dy\right) \leq C < \infty.$$

It is a well known fact that $w(y) = |y|^a$ is a weight in A_2. For a proof of this, see [21, Example 9.1.7].

We now extend the one-dimensional density results in [24] to Banach space valued functions. For the remainder of this section, fix $\phi \in C_c^\infty(0,\infty)$ nonnegative with $\int_{\mathbb{R}} \phi\,dy = 1$ and let Z be a Banach space. We define $\phi_\varepsilon(y) := \frac{1}{\varepsilon}\phi\left(\frac{y}{\varepsilon}\right)$, for $\varepsilon > 0$.

Lemma 4.1.4. *For $u \in L^2((0,\infty), y^a\,dy; Z)$ we have*

$$\sup_{\varepsilon>0} \|u * \phi_\varepsilon\|_Z \leq M\|u\|_Z$$

where M is the Hardy-Littlewood maximal function.

Proof. Let $t \in (0,\infty)$. Then

$$\|u * \phi_\varepsilon(t)\|_Z \leq \int_0^\infty \phi_\varepsilon(t-s)\|u(s)\|_Z\,ds.$$

Taking the supremum over ε implies the desired result. □

Using this lemma, we can show a pointwise convergence result.

Lemma 4.1.5. *For $u \in L^2((0,\infty), y^a\,dy; Z)$ and fixed t we have*

$$u * \phi_\varepsilon(t) \to u(t) \text{ in } Z.$$

Proof. Let $B \subseteq (0,\infty)$ be any ball. Let B' be any ball contained in the closure of B with $\delta = \mathrm{dist}(B, B') > 0$. Set

$$u_1(t) = \begin{cases} u(t) & t \in B' \\ 0 & \text{otherwise} \end{cases}$$

and $u_2 = u - u_1$. It follows that $u_1 \in L^1((0,\infty); Z)$ since

$$\int_0^\infty \|u_1\|_Z \, ds \le \left(\int_0^\infty y^a \|u_1\|^2 \, dy\right)^{1/2} \left(\int_0^\infty y^{-a} \, dy\right)^{1/2}$$

and u_1 is clearly in $L^2((0,\infty), y^a\,dy; Z)$ as it is a truncation of u. Now, an application of Lebesgue's differentiation theorem for vector-valued integrals implies that

$$\|u_1 * \phi_\varepsilon(t) - u_1(t)\|_Z \to 0$$

as $\varepsilon \to 0$ for a.e. $t \in B$.

Now, consider the following for $t \in B$

$$\begin{aligned}\|u_2 * \phi_\varepsilon(t)\|_Z &\le \int \phi_\varepsilon(t-s)\|u_2(s)\|_Z \, ds \\ &\le \left(\int_0^\infty y^a \|u_2(s)\|_Z^2 \, ds\right)^{1/2} \left(\int_{|t-s|>\delta} y^{-a} \phi_\varepsilon(t-s)\, ds\right)^{1/2}.\end{aligned}$$

The final term on the right is 0 for sufficiently small ε since ϕ is assumed to have compact support. □

Using this lemma, we prove that smooth functions are dense in $L^2((0,\infty), y^a\,dy; Z)$.

Theorem 4.1.6. *If $u \in L^2((0,\infty), y^a\,dy; Z)$, then*

$$u * \phi_\varepsilon \to u \text{ in } L^2((0,\infty), y^a\,dy; Z).$$

Proof. By 4.1.4,

$$\|u * \phi_\varepsilon - u\|_Z \le M\|u\|_Z + \|u\|_Z,$$

both of which are in $L^2((0,\infty), y^a\,dy)$. Consquently, Lebesgue's dominated convergence theorem along with 4.1.5 implies

$$\lim_{\varepsilon\to 0} \|u * \phi_\varepsilon - u\|_{L^2((0,\infty),y^a\,dy;Z)} = \lim_{\varepsilon\to 0}\left(\int_0^\infty \|u * \phi_\varepsilon - u\|_Z^2 y^a \, dy\right)^{1/2} = 0.$$

□

We are now in a position to prove various density results in $H_a(X,Y)$ for Banach spaces X and Y.

Theorem 4.1.7. *Given Banach spaces X and Y with X densely embedded in Y, the space $C^\infty((0,\infty);Y) \cap H_a(X,Y)$ is dense in $H_a(X,Y)$.*

Proof. By 4.1.6, $u * \phi_\varepsilon \to u$ in $L^2((0,\infty), y^a\, dy; X)$. So it suffices to show that $(u * \phi_\varepsilon)_y \to u_y$ in $L^2((0,\infty), y^a\, dy; Y)$. Notice that

$$(u * \phi_\varepsilon)_y = u_y * \phi_\varepsilon \in L^2((0,\infty), y^a\, dy; Y).$$

Another application of 4.1.6 implies

$$(u * \phi_\varepsilon)_y \to u_y \text{ in } L^2((0,\infty), y^a\, dy; Y) \text{ as } \varepsilon \to 0.$$

Hence $u * \phi_\varepsilon \to u$ in $H_a(X,Y)$. □

To show that compactly supported, smooth functions are dense in $H_a(X,Y)$, we must first prove some results about reflections. Namely, define

$$\overline{u}(y) = \begin{cases} u(y) & y > 0 \\ u(-y) & y < 0 \end{cases}$$

to be the even reflection of $u \in L^2((0,\infty), X)$.

Lemma 4.1.8. *Let $u \in H_a(X,Y)$. Then $\overline{u} \in L^2(\mathbb{R}, |y|^a\, dy; X)$.*

Proof. Observe

$$\begin{aligned} \int_{-\infty}^{\infty} \|\overline{u}(y)\|_X |y|^a\, dy &= \int_0^\infty \|u(y)\|_X |y|^a\, dy + \int_{-\infty}^0 \|u(-y)\|_X |y|^a\, dy \\ &= \int_0^\infty \|u(y)\|_X |y|^a\, dy + \int_0^\infty \|u(y)\|_X |y|^a\, dy \\ &\le 2\|u\|_{L^2_a((0,\infty);X)} < \infty. \end{aligned}$$

□

We now compute the weak derivative of the even reflection.

Lemma 4.1.9. *Let $u \in H_a(X,Y)$. Then*

$$\overline{u}'(y) = \begin{cases} u'(y) & y > 0 \\ -u'(-y) & y < 0 \end{cases}$$

is in $L^2\left(\mathbb{R}, |y|^a\, dy; Y\right)$

Proof. We first compute the derivative distributionally. Let $\psi \in C_c^\infty(\mathbb{R})$. Suppose $\operatorname{supp}(\psi)$ is contained in $(-\infty, 0)$. Then

$$\begin{aligned} \overline{u}'(\psi) &= -\int_{-\infty}^{\infty} \overline{u}(y)\psi'(y)\, dy \\ &= -\int_{-\infty}^{0} u(-y)\psi'(y)\, dy \\ &= \int_{0}^{\infty} u(y)(\psi(-y))'\, dy \\ &= -\int_{0}^{\infty} u'(y)\psi(-y)\, dy \\ &= -\int_{-\infty}^{0} u'(-y)\psi(y)\, dy. \end{aligned}$$

If $\operatorname{supp}(\psi) \subseteq (0, \infty)$, then

$$\begin{aligned} \overline{u}'(\psi) &= -\int_{-\infty}^{\infty} \overline{u}(y)\psi'(y)\, dy \\ &= -\int_{0}^{\infty} u(y)\psi'(y)\, dy \\ &= \int_{0}^{\infty} u'(y)\psi(y)\, dy. \end{aligned}$$

Now, if $0 \in \operatorname{supp}(\psi)$, say $\operatorname{supp}(\psi) = (-1, 1)$, we need to modify about 0. Let $\varepsilon > 0$ and define η_ε to be constantly 1 on $(-\infty, -2\varepsilon) \cup (2\varepsilon, \infty)$, 0 on $(-\varepsilon, \varepsilon)$, and smoothly interpolating on $(-2\varepsilon, -\varepsilon)$ and $(\varepsilon, 2\varepsilon)$ with bounded derivative $|\eta_\varepsilon'| \leq \frac{2}{\varepsilon}$. Using this, we can 'delete' 0 in the following computations. Consider

$$\begin{aligned} \overline{u}'(\psi) &= -\int_{-\infty}^{\infty} \overline{u}(y)\psi(y)\, dy \\ &= -\lim_{\varepsilon \to 0} \int_{-1}^{1} \overline{u}(y)\left(\eta_\varepsilon \psi'\right)(y) \end{aligned}$$

$$
\begin{aligned}
&= -\lim_{\varepsilon\to 0}\int_{-1}^{1} \overline{u}(y)\left(\eta_\varepsilon\psi\right)'(y)\,dy + \lim_{\varepsilon\to 0}\int_{-1}^{1}\overline{u}(y)\eta_\varepsilon'(y)\psi(y)\,dy \\
&=: \lim_{\varepsilon\to 0}(I+II).
\end{aligned}
$$

Let us compute carefully the term I. Observe

$$
\begin{aligned}
I &= -\int_0^1 u(y)\left(\eta_\varepsilon\psi\right)'(y)\,dy - \int_{-1}^0 u(-y)\left(\eta_\varepsilon\psi\right)'(y)\,dy \\
&= \int_0^1 u'(y)\eta_\varepsilon(y)\psi(y)\,dy - \int_{-1}^0 u'(-y)\eta_\varepsilon(y)\psi(y)\,dy \\
&= \int_{-1}^1 \overline{u}'(y)\eta_\varepsilon(y)\psi(y)\,dy.
\end{aligned}
$$

Therefore

$$
\lim_{\varepsilon\to 0} I = \int_{-1}^1 \overline{u}'(y)\psi(y)\,dy.
$$

To compute II we utilize the definition of η_ε. Notice

$$
\begin{aligned}
II &= \int_\varepsilon^{2\varepsilon} u(y)\eta_\varepsilon'(y)\psi(y)\,dy + \int_{-2\varepsilon}^{-\varepsilon} u(-y)\eta_\varepsilon'(y)\psi(y)\,dy \\
&= \int_\varepsilon^{2\varepsilon} u(y)\eta_\varepsilon'(y)\psi(y)\,dy - \int_{2\varepsilon}^{\varepsilon} u(y)(-\eta_\varepsilon'(y))\psi(-y)\,dy \\
&= \int_\varepsilon^{2\varepsilon} u(y)\eta_\varepsilon'(y)\left(\psi(y)-\psi(-y)\right)\,dy \\
&\le \frac{2}{\varepsilon}\int_\varepsilon^{2\varepsilon} u(y)\left(\psi(y)-\psi(-y)\right)\,dy \\
&= 2\left(\frac{2}{2\varepsilon}\int_0^{2\varepsilon} u(y)(\psi(y)-\psi(-y))\,dy - \frac{1}{\varepsilon}\int_0^\varepsilon u(y)(\psi(y)-\psi(-y))\right) \\
&\le 2\left(\frac{2}{2\varepsilon}\int_0^{2\varepsilon} \|u(y)\|_Y|\psi(y)-\psi(-y)|\,dy - \frac{1}{\varepsilon}\int_0^\varepsilon \|u(y)\|_Y|\psi(y)-\psi(-y)|\right).
\end{aligned}
$$

Hence, by Lebesgue's differentiation theorem,

$$
\lim_{\varepsilon\to 0} II = 0.
$$

Therefore

$$
\overline{u}'(\psi) = \int_{-1}^1 \overline{u}'(y)\psi(y)\,dy
$$

where $\overline{u}'$ is defined as in the statement. Finally,

$$
\int_{-\infty}^{\infty}\|\overline{u}'(y)\|_Y^2|y|^a\,dy = \int_0^\infty \|u'(y)\|_Y^2\,|y|^a\;dy + \int_{-\infty}^0 \|-u'(-y)\|_Y^2\,|y|^a\;dy
$$

$$= 2\int_0^\infty \|u'(y)\|_Y^2 y^a \, dy$$
$$< \infty.$$

□

For the following results, let $H_{|y|^a}(X,Y)$ be the space of functions u satisfying

$$u \in L^2\left((-\infty,\infty), |y|^a \ dy; X\right)$$

and

$$u' \in L^2\left((-\infty,\infty), |y|^a \ dy; Y\right).$$

Lemma 4.1.10. *If $u \in H_a(X,Y)$, then $\overline{u} \in H_{|y|^a}(X,Y)$ and $\|\overline{u}\|_{H_{|y|^a}(X,Y)} \leq 2\|u\|_{H_a(X,Y)}$.*

Proof. This follows immediately from 4.1.8 and 4.1.9. □

Theorem 4.1.11. *Let $u \in H_a(X,Y)$. Let $\psi \in C_c^\infty(\mathbb{R})$ be nonnegative with $\int_{\mathbb{R}} \phi \, dy = 1$. Then $\overline{u} * \psi_\varepsilon \in C_c^\infty(\mathbb{R}; X)$ and*

$$\overline{u} * \psi_\varepsilon \to \overline{u} \text{ in } H_{|y|^a(X,Y)}.$$

Proof. This follows the same as in 4.1.7 since $|y|^a$ is of class A_2. □

As a consequence, we can prove compactly supported smooth functions are dense in $H_a(X,Y)$.

Corollary 4.1.12. *The space $C_c^\infty([0,\infty); X)$ is dense in $H_a(X,Y)$.*

Proof. Let $u \in H_a(X,Y)$. Consider the restriction $\overline{u} * \psi_\varepsilon \Big|_{[0,\infty)} \in C_c^\infty([0,\infty); X)$. Notice

$$\|u - \left(\overline{u} * \psi_\varepsilon\Big|_{[0,\infty)}\right)\|_{H_a(X,Y)} \leq \|\overline{u} - (\overline{u} * \psi_\varepsilon)\|_{H_{|y|^a}(X,Y)} \to 0 \text{ as } \varepsilon \to 0$$

by 4.1.11 □

4.2 Weak solutions to the extension problems

4.2.1 The extension problem on top of $\partial\Omega$

Let $u = \sum_{k=0}^{\infty} u_k \hat{s}_k \in L^2(\partial\Omega)$. Consider the extension problem

$$\begin{cases} y^a \partial_\nu w = \partial_y (y^a \partial_y w), & \text{for } x \in \partial\Omega,\ y > 0, \\ w(x,0) = u(x), & \text{for } x \in \partial\Omega. \end{cases} \tag{4.1}$$

The natural Sobolev space for weak solutions to (4.1) is the boundary mixed norm space $H^{1/2,1}_{\partial,a}$, which is defined as the closure of $C^\infty(\partial\Omega \times [0,\infty))$ under the norm

$$\|w\|_a^2 := \int_0^\infty \int_{\partial\Omega} y^a \big(|w|^2 + |\partial_\nu^{1/2} w|^2 + |\partial_y w|^2 \big)\, dS_x\, dy.$$

A function $w \in H^{1/2,1}_{\partial,a}$ is a weak solution to the boundary extension problem (4.1) if

$$\int_0^\infty \int_{\partial\Omega} y^a \big(\partial_\nu^{1/2} w \partial_\nu^{1/2} \psi + w_y \psi_y \big)\, dS_x\, dy = 0,$$

for every $\psi \in H^{1/2,1}_{\partial,a}$ such that $\psi(x,0) = 0$ on $\partial\Omega$, and

$$\lim_{y \to 0^+} w(x,y) = u(x), \quad \text{in } L^2(\partial\Omega).$$

It was shown in [33, 34] that the function

$$\begin{aligned} w(x,y) &:= \frac{2^{1-\sigma}}{\Gamma(\sigma)} \sum_{k=0}^{\infty} (y\lambda_k^{1/2})^\sigma \mathcal{K}_\sigma(y\lambda_k^{1/2}) u_k \hat{s}_k(x) \\ &= \frac{y^{2\sigma}}{4^\sigma \Gamma(\sigma)} \int_0^\infty e^{-y^2/(4t)} e^{-t\partial_\nu} u(x)\, \frac{dt}{t^{1+\sigma}}, \end{aligned} \tag{4.2}$$

is a weak solution. Here $\mathcal{K}_\sigma(z)$ the modified Bessel function of the third kind or Macdonald's function, see [25]. Moreover, w as above is the unique minimizer of the energy functional

$$\mathcal{J}_\partial(w) := \int_0^\infty \int_{\partial\Omega} y^a \big(|\partial_\nu^{1/2} w|^2 + |w_y|^2 \big)\, dS_x\, dy,$$

among the set of functions $w \in H^{1/2,1}_{\partial,a}$ such that $w(x,0) = u(x)$ in $L^2(\partial\Omega)$.

As a consequence of [33, 34], if $u \in H^{\sigma/2}(\partial\Omega)$ then

$$-\lim_{y \to 0^+} y^a w_y = c_\sigma \partial_\nu^\sigma u, \quad \text{in } H^{-\sigma/2}(\partial\Omega),$$

where $c_\sigma = \frac{\Gamma(1-\sigma)}{4^{\sigma-1/2}\Gamma(\sigma)} > 0$. Furthermore, the following energy identity holds:

$$\int_0^\infty \int_{\partial\Omega} y^a \big(|\partial_\nu^{1/2} w|^2 + |w_y|^2\big)\, dS_x\, dy = c_\sigma \int_{\partial\Omega} |\partial_\nu^{\sigma/2} u|^2\, dS_x.$$

Therefore, if $u : \partial\Omega \to \mathbb{R}$ is a solution to $\partial_\nu^\sigma u = f$ on $\partial\Omega$, with $f = \sum_{k=1}^\infty f_k \hat{s}_k \in H_0^{1/2-\sigma}(\partial\Omega)$ (see Section 2.3), then $u(x) = w(x,0) \in H^{1/2}(\partial\Omega) \subset H^{\sigma/2}(\partial\Omega)$, where w solves the extension problem with Neumann boundary condition

$$\begin{cases} y^a \partial_\nu w = \partial_y (y^a w_y), & \text{in } \partial\Omega,\ y > 0, \\ -\lim_{y\to 0^+} y^a w_y = c_\sigma f, & \text{on } \partial\Omega. \end{cases} \tag{4.3}$$

From [33, 34], a solution w (unique up to an additive constant) is given by

$$\begin{aligned} w(x,y) &= \frac{2^{1-\sigma}}{\Gamma(\sigma)} \sum_{k=1}^\infty (y\lambda_k^{1/2})^\sigma \mathcal{K}_\sigma(y\lambda_k^{1/2}) \frac{f_k}{\lambda_k^\sigma} \hat{s}_k(x) \\ &= \frac{1}{\Gamma(\sigma)} \int_0^\infty e^{-y^2/(4t)} e^{-t\partial_\nu} f(x)\, \frac{dt}{t^{1-\sigma}}. \end{aligned} \tag{4.4}$$

Of course, we can always normalize w in such a way that $\int_{\partial\Omega} w\, dS_x = 0$, for each $y \geq 0$.

By Lemma 3.3.3 we see that if w solves (4.1) in $\partial\Omega \times (0,\infty)$ then

$$w_\lambda(\bar{x},\bar{y}) = w(\lambda\bar{x}, \lambda^{1/2}\bar{y}), \quad \text{for } \bar{x} \in \partial(\tfrac{1}{\lambda}\Omega),\ \bar{y} > 0,$$

is a solution to

$$\begin{cases} \bar{y}^a \partial_{\nu,\lambda} w_\lambda = \partial_{\bar{y}}(\bar{y}^a \partial_{\bar{y}} w_\lambda), & \text{for } \bar{x} \in \partial(\tfrac{1}{\lambda}\Omega),\ \bar{y} > 0, \\ w_\lambda(\bar{x},0) = u_\lambda(\bar{x}), & \text{for } \bar{x} \in \partial(\tfrac{1}{\lambda}\Omega). \end{cases}$$

In addition, if $f_\lambda(\bar{x}) = f(\lambda\bar{x})$, for $\bar{x} \in \frac{1}{\lambda}\Omega$, then from (4.3) we get the Neumann condition

$$\bar{y}^a \partial_{\bar{y}} w_\lambda(\bar{x},\bar{y})\big|_{\bar{y}=0} = c_\sigma \lambda^\sigma (\partial_\nu^\sigma u)(\lambda\bar{x}) = c_\sigma (\partial_{\nu,\lambda})^\sigma u_\lambda(\bar{x}) = c_\sigma \lambda^\sigma f_\lambda(\bar{x}).$$

Example 4.2.1. *Consider the case of the boundary $\mathbb{R}^{n-1}$ of the upper half space $\mathbb{R}^n_+$. Then the boundary extension problem for $w = w(x',y)$ reads*

$$\begin{cases} y^a (-\Delta_{\mathbb{R}^{n-1}})^{1/2} w = \partial_y (y^a \partial_y w), & \textit{for } x' \in \mathbb{R}^{n-1},\ y > 0, \\ w(x',0) = u(x'), & \textit{for } x' \in \mathbb{R}^{n-1}, \end{cases}$$

and the Sobolev space $H^{1/2,1}_{\partial,a}$ *is the closure of* $C^\infty(\mathbb{R}^{n-1}\times[0,\infty))$ *under the norm*

$$\int_0^\infty\int_{\mathbb{R}^{n-1}} y^a\big(|w|^2+|(-\Delta_{\mathbb{R}^{n-1}})^{1/4}w|^2+|w_y|^2\big)\,dx'\,dy.$$

Since we have the kernel of $e^{-t(-\Delta_{\mathbb{R}^{n-1}})^{1/2}}$*, see* (3.6)*, the solution* w *can be written as*

$$\begin{aligned} w(x',y) &= \frac{y^{2\sigma}}{4^\sigma\Gamma(\sigma)}\int_0^\infty e^{-y^2/(4t)}e^{-t(-\Delta_{\mathbb{R}^{n-1}})^{1/2}}u(x')\,\frac{dt}{t^{1+\sigma}} \\ &= \int_{\mathbb{R}^{n-1}} P_y^\sigma(x'-z')u(z')\,dz', \end{aligned}$$

where

$$P_y^\sigma(x')=\frac{\Gamma(n/2)}{4^\sigma\Gamma(\sigma)\pi^{n/2}}\cdot y^{2\sigma}\int_0^\infty \frac{e^{-y^2/(4t)}}{(t^2+|x'|^2)^{n/2}}\,\frac{dt}{t^\sigma}=\frac{1}{(y^2)^{n-1}}P_1^\sigma\Big(\frac{x'}{y^2}\Big).$$

When $\sigma=1/2$*,* $w(x',y)$ *is the subordinated Poisson semigroup of* $e^{-t(-\Delta_{\mathbb{R}^{n-1}})^{1/2}}$*, namely,*

$$w(x',y)=e^{-y(-\Delta_{\mathbb{R}^{n-1}})^{1/4}}u(x')=\frac{y}{2\sqrt{\pi}}\int_0^\infty e^{-y^2/(4t)}e^{-t(-\Delta_{\mathbb{R}^{n-1}})^{1/2}}u(x')\,\frac{dt}{t^{3/2}},$$

which solves $\partial_y w+(-\Delta_{\mathbb{R}^{n-1}})^{1/4}w=0$*. We also have the energy identity*

$$\int_0^\infty\int_{\mathbb{R}^{n-1}} y^a\big(|(-\Delta_{\mathbb{R}^{n-1}})^{1/4}w|^2+|w_y|^2\big)\,dx'\,dy=c_\sigma\int_{\mathbb{R}^{n-1}}|(-\Delta_{\mathbb{R}^{n-1}})^{\sigma/4}u|^2\,dx'.$$

Finally, we can also write the solution as (see (4.4)*)*

$$\begin{aligned} w(x,y) &= \frac{1}{\Gamma(\sigma)}\int_0^\infty e^{-y^2/(4t)}e^{-t(-\Delta_{\mathbb{R}^{n-1}})^{1/2}}\big((-\Delta_{\mathbb{R}^{n-1}})^\sigma u\big)(x')\,\frac{dt}{t^{1-\sigma}} \\ &= \int_{\mathbb{R}^{n-1}} Q_y^\sigma(x'-z')(-\Delta_{\mathbb{R}^{n-1}})^\sigma u(z')\,dz', \end{aligned}$$

where

$$Q_y^\sigma(x')=\frac{\Gamma(n/2)}{\pi^{n/2}\Gamma(\sigma)}\int_0^\infty \frac{t^\sigma e^{-y^2/(4t)}}{(t^2+|x'|^2)^{n/2}}\,dt=\frac{1}{(y^2)^{n-1-\sigma}}Q_1^\sigma\Big(\frac{x'}{y^2}\Big).$$

Observe that this scaling corresponds to an approximation of the identity in fractional dimension $n-1-\sigma$*.*

Remark 4.2.2. (Scaling in the flat case) Suppose that w solves

$$\partial_{x_n}w+\frac{a}{y}w_y+w_{yy}=0,\quad \text{for } x\in T_1=B_1\cap\{x_n=0\},\ 0<y<1.$$

Then

$$w_\lambda(x,y) = w\left(\frac{x}{\lambda}, \frac{y}{\sqrt{\lambda}}\right), \quad \text{for } x \in T_\lambda = B_\lambda \cap \{x_n = 0\},\ 0 < y < \sqrt{\lambda},$$

solves the same equation in $T_\lambda \times (0, \sqrt{\lambda})$. The other way around, if w solves the equation

$$\partial_{x_n} w + \frac{a}{y} w_y + w_{yy} = 0, \quad \text{for } x \in T_r,\ 0 < y < \sqrt{r},$$

for some $r > 0$, then

$$w_r(x,y) = w(rx, \sqrt{r}y), \quad \text{for } x \in T_1,\ 0 < y < 1,$$

solves the equation in $T_1 \times (0,1)$.

4.2.2 The extension problem on top of $\overline{\Omega}$

Next we consider the extension problem on $\overline{\Omega} \times [0,\infty)$. Let $u = \sum_{k=0}^{\infty} u_k \hat{s}_k \in L^2(\partial\Omega)$. If w solves (4.1) then w extends inside Ω as a harmonic function for each $y \geq 0$ just by noticing that the Steklov eigenfunctions in (4.2) are harmonic in Ω. This observation leads us to consider the following extension problem for a function $U = U(x,y)$:

$$\begin{cases} \Delta_x U(x,y) = 0, & \text{for } x \in \Omega,\ \text{for every } y \geq 0, \\ y^a \partial_\nu U = \partial_y (y^a U_y), & \text{on } \partial\Omega,\ \text{for } y > 0, \\ U(x,0) = u(x), & \text{for } x \in \partial\Omega. \end{cases} \tag{4.5}$$

It is clear that the restriction $U\big|_{\partial\Omega \times (0,\infty)}$ solves (4.2), so it coincides with w as in (4.2). Thus, if $u \in H^{\sigma/2}(\partial\Omega)$ then

$$-\lim_{y \to 0^+} y^a \partial_y \big(U\big|_{\partial\Omega \times (0,\infty)}\big) = c_\sigma \partial_\nu^\sigma u,$$

in $H_0^{-\sigma/2}(\partial\Omega)$. Let us next analyze (4.5).

In order to define the notion of weak solution we multiply the first equation in (4.5) by a test function $\Psi \in C_c^\infty(\overline{\Omega} \times [0,\infty))$, integrate over $\Omega \times (0,\infty)$ with respect to the measure $y^a dxdy$ and then integrate by parts in x. We obtain

$$\int_0^\infty \int_\Omega y^a \nabla_x U \nabla_x \Psi \, dx \, dy = \int_0^\infty \int_{\partial\Omega} \Psi (y^a \partial_\nu U) \, dS_x \, dy.$$

Next in the integral in the right hand side we use the second equation of (4.5), interchange the order of integration and integrate by parts in y to get

$$\int_0^\infty y^a\Big[\int_\Omega \nabla_x U \nabla_x \Psi\,dx + \int_{\partial\Omega} U_y \Psi_y\,dS_x\Big]\,dy = \int_{\partial\Omega} \Psi(x,0)\Big[-\lim_{y\to 0^+} y^a U_y(x,y)\Big]\,dS_x.$$

Let $u \in L^2(\partial\Omega)$. We say that U is a weak solution to (4.5) if $\nabla_x U \in L^2(\Omega\times(0,\infty), y^a dxdy)$, $\partial_y U \in L^2(\partial\Omega\times(0,\infty), y^a dS_x dy)$ and

$$\int_0^\infty y^a\Big[\int_\Omega \nabla_x U \nabla_x \Psi\,dx + \int_{\partial\Omega} U_y \Psi_y\,dS_x\Big]\,dy = 0,$$

for every test function Ψ such that $\Psi(x,0) = 0$ on $\partial\Omega$, with

$$\lim_{y\to 0^+} \operatorname{tr}_{\partial\Omega} U(x,y) = u(x), \quad \text{in } L^2(\partial\Omega).$$

The weak solution U is the unique minimizer of the energy functional

$$\mathcal{J}_\Omega(U) = \int_0^\infty y^a\Big[\int_\Omega |\nabla_x U|^2\,dx + \int_{\partial\Omega} |U_y|^2\,dS_x\Big]\,dy,$$

among all $V(x,y)$ such that $V(x,0) = u(x)$ in $L^2(\partial\Omega)$. As in (4.2), we can write

$$\begin{aligned} U(x,y) &= \frac{2^{1-\sigma}}{\Gamma(\sigma)}\sum_{k=0}^\infty (y\lambda_k^{1/2})^\sigma \mathcal{K}_\sigma(y\lambda_k^{1/2}) u_k s_k(x) \\ &= \frac{y^{2\sigma}}{4^\sigma\Gamma(\sigma)}\int_0^\infty e^{-y^2/(4t)}\mathcal{E}(e^{-t\partial_\nu}u)(x)\,\frac{dt}{t^{1+\sigma}}, \end{aligned}$$

where $\mathcal{E}$ denotes the harmonic extension operator, see Lemma 2.1.3. Notice that $U(\cdot,y) \in \mathcal{H}(\Omega)$, for every $y \geq 0$. Moreover, if $\int_{\partial\Omega} u\,dS_x = 0$ then we have $\int_{\partial\Omega} U\,dS_x = 0$, for each $y \geq 0$.

Let $f = \sum_{k=1}^\infty f_k \hat{s}_k \in H_0^{1/2-\sigma}(\partial\Omega)$ and suppose that $u \in \mathcal{H}(\Omega)$ is the unique weak solution to

$$\begin{cases} \Delta u = 0, & \text{in } \Omega, \\ \partial_\nu^\sigma u = f, & \text{on } \partial\Omega, \end{cases} \tag{4.6}$$

having zero mean on $\partial\Omega$. Then the weak solution U to the problem above satisfies

$$\begin{cases} \Delta_x U(x,y) = 0, & \text{in } \Omega, \text{ for every } y \geq 0, \\ y^a \partial_\nu U = \partial_y(y^a U_y), & \text{on } \partial\Omega, \text{ for } y > 0, \\ -\lim_{y\to 0^+} y^a \partial_y U = c_\sigma f, & \text{in } H^{1/2-\sigma}(\partial\Omega), \end{cases} \tag{4.7}$$

and has zero mean on $\partial\Omega$, for every $y \geq 0$. Observe that U satisfies

$$\int_0^\infty y^a \left[\int_\Omega \nabla_x U \nabla_x \Psi \, dx + \int_{\partial\Omega} U_y \Psi_y \, dS_x \right] dy = c_\sigma \langle f, \Psi(\cdot, 0) \rangle,$$

for every test function Ψ. Moreover, the following energy identity holds:

$$\begin{aligned}\int_0^\infty y^a \left[\int_\Omega |\nabla_x U|^2 \, dx + \int_{\partial\Omega} |U_y|^2 \, dS_x \right] dy &= c_\sigma \langle f, u \rangle = c_\sigma \int_{\partial\Omega} |\partial_\nu^{\sigma/2} u|^2 \, dS \\ &= [u]^2_{H^{\sigma/2}(\partial\Omega)} = [f]^2_{H^{-\sigma/2}(\partial\Omega)}.\end{aligned}$$

The weak solution to (4.7) having zero mean on $\partial\Omega$ can be written as

$$\begin{aligned}U(x,y) &= \frac{2^{1-\sigma}}{\Gamma(\sigma)} \sum_{k=1}^\infty (y\lambda_k^{1/2})^\sigma \mathcal{K}_\sigma(y\lambda_k^{1/2}) \frac{f_k}{\lambda_k^\sigma} s_k(x) \\ &= \frac{1}{\Gamma(\sigma)} \int_0^\infty e^{-y^2/(4t)} \mathcal{E}(e^{-t\partial_\nu} f)(x) \, \frac{dt}{t^{1-\sigma}}.\end{aligned}$$

Conversely, if U is a weak solution to (4.7) then $u(x) = U(x,0)$ is a weak solution to (4.6).

Let us summarize the relation between U and w.

Corollary 4.2.3. *Let U be a solution to*

$$\int_0^\infty y^a \left[\int_\Omega \nabla_x U \nabla_x \Psi \, dx + \int_{\partial\Omega} U_y \Psi_y \, dS_x \right] dy = c_\sigma \int_{\partial\Omega} \Psi(x,0) f(x) \, dS_x.$$

Then $w := \mathrm{tr}_{\partial\Omega} U$ is a solution to

$$\int_0^\infty \int_{\partial\Omega} y^a \left(\partial_\nu^{1/2} w \partial_\nu^{1/2} \psi + w_y \psi_y \right) dS_x \, dy = c_\sigma \int_{\partial\Omega} \psi(x,0) f(x) \, dS_x.$$

Conversely, if w is a solution to the latter equation, then $U(x,y) := \mathcal{E}_x\big(w(\cdot,y)\big)$ is a solution to the former equation. In both cases, $u(x) = U(x,0)$ and $u(x) = (\mathcal{E}w)(x,0)$ are weak solutions to (4.6) *if and only if U and w are solutions to the equations above.*

Example 4.2.4. *In the case of the upper half space, the extension problem for $U = U(x', x_n, y)$ reads*

$$\begin{cases} \Delta U(x', x_n, y) = 0, & \text{in } \mathbb{R}^n_+, \text{ for every } y \geq 0, \\ -y^a \partial_{x_n} U(x', 0, y) = \partial_y(y^a U_y(x', 0, y)), & \text{on } \mathbb{R}^{n-1}, \text{ for } y > 0, \\ \lim_{y \to 0^+} U(x', 0, y) = u(x'), & \text{in } L^2(\mathbb{R}^{n-1}), \\ -\lim_{y \to 0^+} y^a \partial_y U(x', 0, y) = c_\sigma (-\Delta_{\mathbb{R}^{n-1}})^{\sigma/2} u(x'), & \text{in } H^{1/2-\sigma}(\partial\Omega), \end{cases}$$

where Δ above is the Laplacian in the variables $(x', x_n) \in \mathbb{R}^n_+$. The energy identity becomes

$$\int_0^\infty y^a \left[\int_{\mathbb{R}^n_+} |\nabla U|^2 \, dx' \, dx_n + \int_{\mathbb{R}^{n-1}} |U_y|^2 \, dx' \right] dy = c_\sigma \int_{\mathbb{R}^{n-1}} |(-\Delta_{\mathbb{R}^{n-1}})^{\sigma/2} u|^2 \, dx'.$$

Given a function $g = g(x') : \mathbb{R}^{n-1} \to \mathbb{R}$, the harmonic extension operator $\mathcal{E}$ on g is

$$\mathcal{E} g(x', x_n) = e^{-x_n(-\Delta_{\mathbb{R}^{n-1}})^{1/2}} g(x'), \quad (x', x_n) \in \mathbb{R}^n_+.$$

Hence, the solution U above is given by

$$\begin{aligned} U(x', x_n, y) &= \frac{y^{2\sigma}}{4^\sigma \Gamma(\sigma)} \int_0^\infty e^{-y^2/(4t)} e^{-(x_n+t)(-\Delta_{\mathbb{R}^{n-1}})^{1/2}} u(x') \, \frac{dt}{t^{1+\sigma}} \\ &= \frac{1}{\Gamma(\sigma)} \int_0^\infty e^{-y^2/(4t)} e^{-(x_n+t)(-\Delta_{\mathbb{R}^{n-1}})^{1/2}} \big((-\Delta_{\mathbb{R}^{n-1}})^{\sigma/2} u \big)(x') \, \frac{dt}{t^{1-\sigma}}. \end{aligned}$$

From here kernel expressions for U can be easily derived.

4.3 Flattening the boundary

We now flatten the extension on top of $\overline{\Omega}$. Let U be a solution to

$$\int_0^\infty y^a \left[\int_\Omega \nabla_x U \nabla_x \Psi \, dx + \int_{\partial\Omega} U_y \Psi_y \, dS_x \right] dy = \int_{\partial\Omega} \Psi(x, 0) f(x) \, dS_x, \tag{4.8}$$

for every test function Ψ. We take a point x_0 at the boundary of Ω and we flatten the equation near $\partial\Omega$ around that point, for each $y \geq 0$, by using a usual flattening map which will be made precise in the following chapter. Notice that all our ingredients in the equation (4.8) are local in nature. We need to fix some notation. For $r > 0$,

$$\begin{aligned} B_r^+ &:= B_r \cap \{x_n > 0\} = \{(x', x_n) \in B_r : x_n > 0\}, \\ T_r &:= B_r \cap \{x_n = 0\}. \end{aligned}$$

Now, after flattening the boundary around x_0, which follows the same as in the proof of 5.1.1, the transformed solution and right hand side, that we still call U and f, satisfy

$$\int_0^1 y^a \left[\int_{B_1^+} A(x) \nabla_x U \nabla_x \Psi \, dx + \int_{T_1} U_y \Psi_y \Phi \, dx' \right] dy = \int_{T_1} \Psi(x', 0) f(x') \Phi \, dx', \tag{4.9}$$

for every test function Ψ. The function Φ is a Lipschitz map, depending on the boundary, coming from the flattening process. In our setting, Φ is, at worst, Lipschitz. The coefficients $A(x)$ are symmetric, uniformly elliptic and

- bounded and measurable if $\partial\Omega$ is Lipschitz;
- continuous if $\partial\Omega$ is C^1, or;
- C^α Hölder continuous if $\partial\Omega$ is $C^{1,\alpha}$ for some $\alpha \in (0,1)$.

By an orthogonal change of variables we will always assume that

$$A(0) = I,$$

the identity matrix.

Remark 4.3.1. In the weak formulation (4.9), the trace $u(x) = U(x,0)$ would correspond to a solution to a fractional Neumann problem for a divergence form elliptic operator with flat boundary:

$$\begin{cases} \operatorname{div}(A(x)\nabla u) = 0, & \text{in } \Omega, \\ \partial_A^\sigma u = f, & \text{on } \partial\Omega. \end{cases}$$

Here ∂_A^σ denotes the fractional power of the conormal derivative $\partial_A u = \nu^T A(x)\nabla u$. On the flat part of $\partial\Omega$, $\nu = -e_n$, so that $\nu^T A(x)\nabla u = -\sum_{j=1}^n A_{nj}(x)u_{x_j}$. The assumption $A(0) = I$ then means that $\partial_A u(0) = -\partial_{x_n} u(0)$.

CHAPTER 5. GLOBAL L^2 TO L^∞ ESTIMATE

The main result of this chapter a nonlocal 'L^2 to L^∞' type estimate for solutions to (2.1). We do this by modifying the theory of De Giorgi and utilizing facts about harmonic extensions. We state the result here:

Theorem 5.0.1. *Let $\Omega \subseteq \mathbb{R}^n$ be flat or a bounded domain with Lipschitz boundary. Let u be a solution to*

$$\begin{cases} \Delta u = 0 & \text{in } \Omega \\ \partial_\nu^\sigma u = f & \text{on } \partial\Omega \end{cases}$$

with $f \in L^q(\partial\Omega)$, $q > (n-1)/\sigma$. Then $u \in L^\infty(\overline{\Omega})$.

5.1 Fractional Sobolev Inequality

We need a few lemmas before proving 5.0.1. First, we show that flattening the boundary gives us a way to relate energy in $\overline{\Omega}$ to that in the flat setting. Moreover, we prove a similar relationship in the extended domain. We begin with some preliminaries about flattening the boundary of a Lipschitz domain.

A bounded Lipschitz domain in $\mathbb{R}^n$ is a bounded connected domain Ω such that the boundary $\partial\Omega$ can be covered by finitely many open balls K_j in $\mathbb{R}^n$, $j = 1, \ldots, J$, centered at $\partial\Omega$ such that

$$K_j \cap \Omega = K_j \cap \Omega_j, \quad j = 1, \ldots, J,$$

where Ω_j are rotations of suitable special Lipschitz domains in $\mathbb{R}^n$ given by Lipschitz functions φ_j. One may then assume that $\partial\Omega \cap K_j$ can be represented in local coordinates by $x_n = \varphi_j(x')$, where φ_j is a Lipschitz function on $\mathbb{R}^{n-1}$ with $\varphi_j(0) = 0$. Remember also that if φ is a Lipschitz function defined on an set $A \subset \mathbb{R}^{n-1}$ with Lipschitz constant M, then there exists an extension $\overline{\varphi} : \mathbb{R}^{n-1} \to \mathbb{R}$ of φ such that $\overline{\varphi} = \varphi$ on A and the Lipschitz constant of $\overline{\varphi}$ does not exceed M. One can also find an extension having the same Lipschitz constant that the original function φ. For this see [16, Chapter 3]. Let Ω be a bounded Lipschitz domain. Consider the balls K_j, $j = 1, \ldots, n$ that cover $\partial\Omega$ as above. Then there are bi-Lipschitz transformations

$$y = \psi^{(j)}(x) : K_j \Longleftrightarrow V_j = \psi^{(j)}(K_j),$$

such that

$$\psi^{(j)}(K_j \cap \Omega) \subset \mathbb{R}^n_+, \quad \psi^{(j)}(K_j \cap \partial\Omega) \subset \mathbb{R}^{n-1} = \mathbb{R}^n \cap \{y_n = 0\}.$$

We can assume that V_j is simply connected and that the upper boundary of $\psi^{(j)}(K_j \cap \Omega)$ can be described by $y_n = \tau^{(j)}(y')$, where $\tau^{(j)}$ is a Lipschitz function. In fact (upon relabeling and reorienting the coordinate axes if necessary), we can take

$$\psi^{(j)} : x = (x_1, \ldots, x_n) \in K_j \to y = (y_1, \ldots, y_n) = \psi^{(j)}(x) = (\psi^{(j)}_1(x), \ldots, \psi^{(j)}_n(x)) \in V_j \subset \mathbb{R}^n$$

of the form (see also [15, Appendix C.1])

$$y_i = \psi^{(j)}_i(x) := x_i,\ i = 1, \ldots, n-1, \quad \text{and} \quad y_n = \psi^{(j)}_n(x) := x_n - \varphi_j(x').$$

This is the flattening map. In addition, the normal directions $\nu_z = \nu(z)$ with $z \in \partial\Omega \cap K_j$ are mapped in normal y_n-directions with the foot-points $\psi^{(j)}(z) \in \mathbb{R}^n \cap \{y_n = 0\}$.

The inverse flattening map

$$(\psi^{(j)})^{-1} : y \in V_j \subset \mathbb{R}^n \to x = (x_1, \ldots, x_n) = (\psi^{(j)})^{-1}(y) = ((\psi^{(j)})^{-1}_1(y), \ldots, (\psi^{(j)})^{-1}_n(y)) \in K_j$$

is given by (see also [15, Appendix C.1])

$$x_i = (\psi^{(j)})^{-1}_i(y) := y_i,\ i = 1, \ldots, n-1, \quad \text{and} \quad x_n = (\psi^{(j)})^{-1}_n(y) := y_n + \varphi_j(y').$$

Let K_j, $j = 1, \ldots, J$, be the balls as above, so that $\partial\Omega \subset \bigcup_{j=1}^J K_j$. Let Ω_0 be an open subset of Ω such that $\overline{\Omega_0} \subset \Omega$. A partition of unity $\{\xi_j\}_{j=0}^J$ subordinated to $\{\Omega_0, K_1, \ldots, K_J\}$ is a family of nonnegative smooth functions ξ_j on $\mathbb{R}^n$ such that

$$\xi_0 \in C^\infty_c(\Omega_0), \quad \xi_j \in C^\infty_c(K_j),\ j = 1, \ldots, J, \quad \text{and} \quad \sum_{j=0}^J \xi_j(x) = 1 \text{ for every } x \in \overline{\Omega}.$$

From the last sum condition, it follows that

$$0 \leq \xi_0, \xi_j \leq 1, \quad j = 1, \dots, J.$$

Obviously the family $\{\xi_j\}_{j=1}^J$ is a partition of unity subordinated to the open cover $\{K_1, \dots, K_J\}$ of $\partial\Omega$ and $\sum_{j=1}^J \xi_j(x) = 1$ for every $x \in \partial\Omega$.

Lemma 5.1.1. *Let Ω be a bounded domain in $\mathbb{R}^n$. Let $\{\xi_j\}_{j=1}^J$ be the partition of unity subordinated to the open balls $\{K_1, \dots, K_J\}$ that cover $\partial\Omega$. Consider the flattening maps $\psi^{(j)} = \psi^j(x) : K_j \to V_j$ and their inverses $\left(\psi^{(j)}\right)^{-1} = \left(\psi^{(j)}\right)^{-1}(z) : V_j \to K_j$. For a measurable function $u = u(x)$ on $\partial\Omega$, define functions $u_j : \mathbb{R}^{n-1} \equiv \mathbb{R}^{n-1} \cap \{z_n = 0\} \to \mathbb{R}$ as*

$$u_j = u_j(z) = \begin{cases} (\xi_j u) \circ \left(\psi^{(j)}\right)^{-1}(z), & \text{for } z \in V_j \cap \{z_n = 0\}, \\ 0, & \text{for } z \in \mathbb{R}^{n-1} \setminus (V_j \cap \{z_n = 0\}) \end{cases}$$

for each $j = 1, \dots, J$. Then for any $1 \leq p \leq \infty$ and $0 < \sigma < 1$, the following are equivalent:

1. *$u \in L^p(\partial\Omega)$*
2. *$u_j \in L^p\left(\mathbb{R}^{n-1}\right)$ for each $j = 1, \dots, J$.*

Moreover, if any of the above conditions hold, then

$$\|u\|_{L^p(\partial\Omega)} \sim \sum_{j=1}^J \|u_j\|_{L^p(\mathbb{R}^{n-1})}.$$

Furthermore, if we consider flattening in x for each $y \geq 0$ in the extended domain $\overline{\Omega} \times (0, \infty)$, we have

$$\int_0^\infty y^a \left(\int_\Omega |\nabla U|^2 \, dx + \int_{\partial\Omega} U_y^2 \, dx' \right) dy \sim C \sum_{j=1}^J \int_0^\infty y^a \left(\int_{\mathbb{R}^n_+} |\nabla U_j|^2 \, dx + \int_{\mathbb{R}^{n-1}} (U_j)_y^2 \, dx' \right) dy.$$

Proof. Consider the flattening maps $\psi^{(j)} = (\psi_1^{(j)}, \dots, \psi_n^{(j)}) : K_j \to V_j \subseteq \overline{\mathbb{R}^n_+}$ given by

$$z_i = \psi_i^{(j)}(x) = x_i, \quad i = 1, \dots, n-1, \qquad z_n = \varphi_n^{(j)}(x) = x_n - \varphi^{(j)}(x')$$

and its inverse $(\psi^{(j)})^{-1} : V_j \to K_j$ given by

$$x_i = (\psi_i^{(j)})^{-1}(z) = z_i, \ i = 1, \dots, n-1, \qquad x_n = (\psi_n^{(j)})^{-1}(z) = z_n + \varphi^{(j)}(z').$$

Notice that

$$\psi^{(j)}(x) = (z', 0) = (x', 0) \quad \text{for every } x = (x', \varphi^{(j)}(x')) \in \partial\Omega \cap K_j$$

and

$$(\psi^{(j)})^{-1}(z', 0) = (x', \varphi^{(j)}(x')) = (z', \varphi^{(j)}(z')) \in \partial\Omega \cap K_j \quad \text{for every } (z', 0) \in \partial\mathbb{R}^n_+ \cap V_j \equiv \mathbb{R}^{n-1} \cap V_j.$$

Let $1 < p < \infty$. Since $u_j(y)$ is defined in $\mathbb{R}^{n-1} \equiv \mathbb{R}^n \cap \{y_n = 0\}$, we can identify it with a function of $y' \in \mathbb{R}^{n-1}$ in the obvious way, and we still call this new function $u_j(y')$. Similarly, $(\psi^{(j)})^{-1}(y') \equiv (\psi^{(j)})^{-1}(y)$ when $y \in V_j \cap \{y_n = 0\}$ is identified with $y' \in V_j \cap \mathbb{R}^{n-1}$.

The area formula says that if $f : \mathbb{R}^{n-1} \to \mathbb{R}^n$ is a Lipschitz injective map and $g \in L^1(\mathbb{R}^n, dH^{n-1}|_{f(\mathbb{R}^{n-1})})$ then

$$\int_{f(\mathbb{R}^{n-1})} g\, dH^{n-1} = \int_{\mathbb{R}^{n-1}} (g \circ f)(y') Jf(y')\, dy',$$

where $Jf(y') = \sqrt{\det(\nabla f(y')^* \nabla f(y'))}$ is the Jacobian of f, with $\nabla f(y')^*$ the adjoint of the gradient map $\nabla f(y') : \mathbb{R}^{n-1} \to \mathbb{R}^n$, see [29, Remark 8.3]. In the case when $\varphi : \mathbb{R}^{n-1} \to \mathbb{R}$ is a Lipschitz function and $\Gamma(\varphi) = \{(y', \varphi(y')) \in \mathbb{R}^n : y' \in \mathbb{R}^{n-1}\}$ is the graph of φ (representing the boundary of a special Lipschitz domain), the area formula with $f(y') = (y', \varphi(y'))$ and $f(\mathbb{R}^{n-1}) = \Gamma(\varphi)$ implies that

$$\int_{\Gamma(\varphi)} g\, dH^{n-1} = \int_{\mathbb{R}^{n-1}} g(y', \varphi(y')) \sqrt{1 + |\nabla\varphi(y')|^2}\, dy',$$

see [29, Theorem 9.1]. This is called the area formula for codimension one (Lipschitz) graphs. Notice that

$$\left(\left[\frac{\partial \psi_j}{\partial x_i} \right]^T \left[\frac{\partial \psi_\ell}{\partial x_i} \right] \right)_{j,\ell=1}^n = \begin{bmatrix} 1 + |\partial_1\varphi|^2 & \partial_1\varphi\partial_2\varphi & \cdots & \partial_1\varphi\partial_{n-1}\varphi & -\partial_1\varphi \\ \partial_2\varphi\partial_1\varphi & 1 + |\partial_2\varphi|^2 & \cdots & \partial_2\varphi\partial_{n-1}\varphi & -\partial_2\varphi \\ \vdots & \cdots & \ddots & \vdots & \vdots \\ \partial_{n-1}\varphi\partial_1\varphi & \cdots & \cdots & 1 + |\partial_{n-1}\varphi|^2 & -\partial_{n-1}\varphi \\ -\partial_1\varphi & -\partial_2\varphi & \cdots & -\partial_{n-1}\varphi & 1 \end{bmatrix}.$$

Applying this formula to $g = |\xi_j u|^p$, $f(y') = (\psi^{(j)})^{-1}(y') = (y', \varphi_j(y'))$ and $\varphi(y') = \varphi_j(y')$,

$$\int_{\mathbb{R}^{n-1}} |u_j|^p = \int_{V_j \cap \{y_n = 0\}} |u_j(y')|^p\, dy'$$

$$
\begin{aligned}
&= \int_{V_j\cap\{y_n=0\}} |(\xi_j u)\circ(\psi^{(j)})^{-1}(y')|^p\,dy' \\
&\le \int_{V_j\cap\{y_n=0\}} |(\xi_j u)(y',\varphi_j(y'))|^p\sqrt{1+|\nabla\varphi_j(y')|^2}\,dy' \\
&= \int_{\{(\psi^{(j)})^{-1}(V_j\cap\{y_n=0\})\}} |(\xi_j u)(x)|^p\,dH_x^{n-1} \\
&= \int_{K_j\cap\partial\Omega} |(\xi_j u)(x)|^p\,dH_x^{n-1} \le \int_{\partial\Omega} |u(x)|^p\,dS_x.
\end{aligned}
$$

In a similar way, by the $L^p(\partial\Omega)$ Minkowski inequality and using that $K_j\cap\partial\Omega = f(V_j\cap\{y_n=0\}) = (\psi^{(j)})^{-1}(V_j\cap\{y_n=0\})$ in the area formula for codimension one graphs and $|\nabla\varphi_j|\le C$,

$$
\begin{aligned}
\left(\int_{\partial\Omega}|u(x)|^p\,dS_x\right)^{1/p} &\le \sum_{j=1}^J\left(\int_{\partial\Omega}|(\xi_j u)(x)|^p\,dS_x\right)^{1/p} \\
&= \sum_{j=1}^J\left(\int_{K_j\cap\partial\Omega}|(\xi_j u)(x)|^p\,dH_x^{n-1}\right)^{1/p} \\
&= \sum_{j=1}^J\left(\int_{V_j\cap\{y_n=0\}}|(\xi_j u)(y',\varphi_j(y'))|^p\sqrt{1+|\nabla\varphi_j(y')|^2}\,dy'\right)^{1/p} \\
&\le C\sum_{j=1}^J\left(\int_{\mathbb{R}^{n-1}}|u_j|^p\right)^{1/p}.
\end{aligned}
$$

Therefore, $u\in L^p(\partial\Omega)$ if and only if $u_j\in L^p(\mathbb{R}^{n-1})$ for every $j=1,\dots,J$, $1\le p<\infty$. We have shown that

$$\|u\|_{L^p(\partial\Omega)} \sim \sum_{j=1}^J \|u_j\|_{L^p(\mathbb{R}^{n-1})}.$$

We now look at the extended case. Consider the change of variables $x=(\psi^{(j)})^{-1}(z)$ and for $U(x,y)\in H^1(\Omega)$ define

$$
U_j = U_j(z,y) = \begin{cases} (\xi_j U)\left(\left(\psi^{(j)}\right)^{-1}(z), y\right) & \text{for } z\in V_j\cap\mathbb{R}^n_+, \\ 0, & \text{for } z\in\mathbb{R}^n_+\setminus\left(V_j\cap\mathbb{R}^n_+\right) \end{cases}
$$

We have

$$\nabla\psi^{(j)}(x) = \left[\frac{\partial \psi_i^{(j)}(x)}{\partial x_k}\right]_{i,k=1}^n = \begin{bmatrix} 1 & 0 & \cdots & \cdots & 0 \\ 0 & 1 & \cdots & \cdots & 0 \\ \vdots & \cdots & \ddots & \vdots & \vdots \\ 0 & \cdots & \cdots & 1 & 0 \\ -\frac{\partial \varphi^{(j)}}{\partial x_1} & -\frac{\partial \varphi^{(j)}}{\partial x_2} & \cdots & -\frac{\partial \varphi^{(j)}}{\partial x_{n-1}} & 1 \end{bmatrix}$$

and

$$\nabla(\psi^{(j)})^{-1}(z) = \left[\frac{\partial (\psi_i^{(j)})^{-1}(z)}{\partial z_k}\right]_{i,k=1}^n = \begin{bmatrix} 1 & 0 & \cdots & \cdots & 0 \\ 0 & 1 & \cdots & \cdots & 0 \\ \vdots & \cdots & \ddots & \vdots & \vdots \\ 0 & \cdots & \cdots & 1 & 0 \\ \frac{\partial \varphi^{(j)}}{\partial z_1} & \frac{\partial \varphi^{(j)}}{\partial z_2} & \cdots & \frac{\partial \varphi^{(j)}}{\partial z_{n-1}} & 1 \end{bmatrix}.$$

Then $\det \nabla\psi^{(j)}(x) = \det \nabla(\psi^{(j)})^{-1}(z) = 1$, for every $x \in K_j$ and $z \in V_j$.

Observe that

$$U_j(z', 0, y) = U_j(z', \varphi^{(j)}(z'), y) \quad \text{and} \quad U(x, y) = U_j(\psi^{(j)}(x), y).$$

For $x \in K_j$ and $y \geq 0$, the chain rule gives

$$\partial_{x_i}(\xi_j U)(x, y) = \sum_{\ell=1}^n \partial_{z_\ell} U_j(\psi^{(j)}(x), y) \partial_i \psi_\ell^{(j)}(x).$$

Then, by using the change of variables $x = \psi^{-1}(z)$ with $dx = |\det \nabla\psi^{-1}(z)|\, dz = dz$,

$$\begin{aligned} \int_{K_j} \nabla(\xi_j U)\nabla(\xi_j U)\, dx &= \sum_{i=1}^n \int_{K_j} \partial_{x_i}(\xi_j U)(x, y)\partial_{x_i}(\xi_j U)(x, y)\, dx \\ &= \sum_{i=1}^n \int_{K_j} \left(\sum_{\ell=1}^n \partial_{z_\ell} U_j\left(\psi^{(j)}(x), y\right) \partial_i \psi_\ell^{(j)}(x)\right)\left(\sum_{k=1}^n \partial_{z_k} U_j(\psi^{(j)}(x), y)\partial_i \psi_k^{(j)}(x)\right) dx \\ &= \sum_{k,\ell}^n \int_{V_j} \left(\sum_{i=1}^n \partial_i \psi_k^{(j)}\left((\psi^{(j)})^{-1}(z)\right) \partial_i \psi_\ell\left(\psi^{(j)-1}(z)\right)\right) \partial_{z_k} U_j(z, y)\partial_{z_\ell} U_j(z, y)\, dz \\ &= \int_{V_j} A_j(z)\nabla U_j \nabla U_j\, dz \end{aligned}$$

where

$$A_j(z) = \left(a_j^{k\ell}(z)\right) = \left(\sum_{i=1}^{n} \partial_i \psi_k \left((\psi^{(j)})^{-1}(z)\right) \partial_i \psi_\ell^{(j)} \left((\psi^{(j)})^{-1}(z)\right)\right)$$

is bounded and uniformly elliptic provided that $\partial\Omega$ is Lipschitz. Hence we obtain the following inequalities

$$\int_\Omega |\nabla U|^2 \, dx \le \sum_{j=1}^{J} \left(\int_{\mathbb{R}^n_+} A_j(z) \left|\nabla U_j\right|^2 \, dz\right) \le C \sum_{j=1}^{J} \left(\int_{\mathbb{R}^n_+} \left|\nabla U_j\right|^2 \, dz\right).$$

and

$$\begin{aligned} \int_{\mathbb{R}^n_+} \left|\nabla U_j\right|^2 \, dz &= \int_{V_j} \left|\nabla U_j\right|^2 \, dz \\ &\le c \int_{V_j} A_j(z) \left|\nabla \left(U_j\right)\right|^2 \, dz \\ &= c \int_{K_j} \left|\nabla \left(\xi_j U\right)\right|^2 \, dx \\ &\le c \int_\Omega |\nabla U|^2 \, dx. \end{aligned}$$

Integrating the inequalities against $y^a \, dy$ gives the equivalence

$$\int_0^\infty y^a \int_\Omega |\nabla U|^2 \, dx \, dy \sim \sum_{j=1}^{J} y^a \int_0^\infty \int_{\mathbb{R}^n_+} \left|\nabla U_j\right|^2 \, dz \, dy.$$

We use a similar argument to deal with the term including U_y. However, this term is easier to work with since the flattening is independent of y. In particular, the derivative with respect to y will not change after flattening. We first fix $y \in (0, \infty)$ to get

$$\|U_y\|_{L^2(\partial\Omega)} \sim \sum_{k=1}^{J} \|(U_j)_y\|_{L^2(\mathbb{R}^{n-1})}$$

by using the first part of this theorem. Integrating against $y^a \, dy$ gives

$$\int_0^\infty y^a \int_{\partial\Omega} U_y^2 \, dS \, dy \sim \sum_{j=1}^{J} \int_0^\infty y^a \int_{\mathbb{R}^{n-1}_+} (U_j)_y^2 \, dS \, dy.$$

$\square$

We now prove that solutions to (4.5) minimize a particular energy functional associated to the extension problem on top of $\overline{\Omega}$.

Lemma 5.1.2. *The solution U to the extension problem (4.5) is the unique minimizer of the energy functional*

$$\mathcal{J}(V) = \int_0^\infty y^a \left(\int_\Omega |\nabla_x V|^2 \, dx + \int_{\partial\Omega} V_y^2 \, dx' \right) dy$$

among all functions V with satisfying $V(x,0) = u(x)$.

Proof. Suppose $v(x,0) = u(x)$. Then we obtain the following energy identity by definition of U being a solution

$$\int_0^\infty \int_{\partial\Omega} y^a (\partial_\nu^{1/2} w \partial_\nu^{1/2} v + w_y v_y) \, dS \, dy = c_\sigma \int_{\partial\Omega} \partial_\nu^{\sigma/2} w(x,0) \partial_\nu^{\sigma/2} v(x,0) \, dS = c_\sigma \int_{\partial\Omega} \left| \partial_\nu^{\sigma/2} u \right|^2 \, dS.$$

Now, recalling that w extends harmonically to Ω as U, we have

$$\int_0^\infty \int_{\partial\Omega} y^a \partial_\nu w v \, dS \, dy = \int_0^\infty \int_\Omega y^a \nabla U \nabla V \, dx \, dy.$$

Hence

$$\int_0^\infty \int_\Omega y^a \nabla U \nabla V \, dx + \int_{\partial\Omega} y^a U_y V_y \, dS \, dy = c_\sigma \int_{\partial\Omega} \left| \partial_\nu^{\sigma/2} u \right|^2 \, dS.$$

Furthermore, a direct computation reveals

$$\begin{aligned} J(V - U + U) &= \int_0^\infty y^a \left(\int_\Omega |\nabla_x (V - U + U)|^2 \, dx + \int_{\partial\Omega} (V - U + U)_y^2 \, dx' \right) dy \\ &= \int_0^\infty y^a \left(\int_\Omega |\nabla (V - U)|^2 \, dx + \int_{\partial\Omega} (V - U)_y^2 \, dS \right) dy \\ &\quad + 2 \int_0^\infty y^a \left(\int_\Omega \nabla (V - U) \nabla U \, dx + \int_{\partial\Omega} (V - U)_y U_y \, dS \right) dy \\ &\quad + \int_0^\infty y^a \left(\int_\Omega |\nabla U|^2 \, dx + \int_{\partial\Omega} U_y^2 \, dS \right) dy \\ &\geq 2 \int_0^\infty y^a \left(\int_\Omega \nabla (V - U) \nabla U \, dx + \int_{\partial\Omega} (V - U)_y U_y \, dS \right) dy + J(U). \end{aligned}$$

By our initial remarks, the first term in the final line is 0. Hence

$$J(V) \geq J(U).$$

∎

We are now in a position to prove a 'fractional' Sobolev inequality.

Lemma 5.1.3 (Fractional Sobolev embedding)**.** *There exists a constant $C > 0$ depending only on $\partial\Omega$ and σ such that for any $u \in H^{\sigma/2}(\partial\Omega)$ satisfying $\mathcal{H}^{n-1}(\{u = 0\}) > 0$ or $\int_{\partial\Omega} u\, dS = 0$ we have*

$$\|u\|_{L^{2^*_\sigma}(\partial\Omega)} \leq C\|\partial_\nu^{\sigma/2} u\|_{L^2(\partial\Omega)}$$

where $2^*_\sigma = \frac{2(n-1)}{n-1-\sigma} > 2$.

Proof. Given u as in the statement, let U be the corresponding solution to the extension problem (4.5). We have the following energy identity

$$\int_0^\infty y^a \left(\int_\Omega |\nabla U|^2 \, dx + \int_{\partial\Omega} U_y^2 \, dS \right) dy = c_\sigma \int_{\partial\Omega} \left| \partial_\nu^{\sigma/2} u \right|^2 \, dS.$$

Therefore, it is enough to show that

$$\left(\int_{\partial\Omega} u^{2^*_\sigma} \, dS \right)^{2/2^*_\sigma} \leq C \int_0^\infty y^a \left(\int_\Omega |\nabla U|^2 \, dx + \int_{\partial\Omega} U_y^2 \, dS \right) dy.$$

Let $\{\xi_j\}_{j=1}^J$ be a partition of unity subordinated to a finite family of open balls $\{K_1, \ldots, K_J\}$ that cover $\partial\Omega$. Consider the flattening maps on the sets $K_i \cap \overline{\Omega}$ and let u_j and U_j be the flattened functions as in Lemma 5.1.1. Then, by Lemma 5.1.1,

$$\|u\|^2_{L^{2^*_\sigma}(\partial\Omega)} \leq C \sum_{j=1}^J \|u_j\|^2_{L^{2^*_\sigma}(\mathbb{R}^{n-1})}.$$

For each of the flattened functions u_j, let U_j^* denote the solution to the flat extension problem as given in Example 4.2.4. By the classical fractional Sobolev embedding in $\mathbb{R}^{n-1}$ and the fact that U_j^* minimize the corresponding energy,

$$\begin{aligned}
\|u_j\|^2_{L^{2^*_\sigma}(\mathbb{R}^{n-1})} &\leq C\| \left(-\Delta\right)^{\sigma/4} u_j\|^2_{L^2(\mathbb{R}^{n-1})} \\
&= C \int_0^\infty y^a \left(\int_{\mathbb{R}^n_+} \left|\nabla U_j^*\right|^2 \, dx + \int_{\mathbb{R}^{n-1}} (U_j^*)_y^2 \, dx' \right) dy \\
&\leq C \int_0^\infty y^a \left(\int_{\mathbb{R}^n_+} |\nabla U_j|^2 \, dx + \int_{\mathbb{R}^{n-1}} (U_j)_y^2 \, dx' \right) dy.
\end{aligned}$$

Hence, by Lemma 5.1.1 again

$$\|u\|^2_{L^{2^*_\sigma}(\partial\Omega)} \leq C \sum_{j=1}^J \|u_j\|^2_{L^{2^*_\sigma}(\mathbb{R}^{n-1})}$$

$$\leq C\sum_{j=1}^{J}\int_0^\infty y^a\left(\int_{\mathbb{R}^n_+}|\nabla U_j|^2\,dx+\int_{\mathbb{R}^{n-1}}(U_j)_y^2\,dx'\right)dy$$
$$\leq C\int_0^\infty y^a\left(\int_\Omega|\nabla U|^2\,dx+\int_{\partial\Omega}U_y^2\,dS\right)dy$$

which is the desired inequality. □

5.2 Solutions to (2.1) are bounded

We prove the main result of this chapter which follows closely the ideas presented in [22, Chapter 3] accounting for our fractional setting.

Proof of theorem 5.0.1. Define $u_k=(u-C_k)^+$ where $C_k=2^{-1}\left(1-2^{-k}\right)$, $k\geq 0$. We multiply the equation on the boundary by u_k and integrate:

$$\int_{\partial\Omega}u_k\partial_\nu^\sigma u\,dS=\int_{\partial\Omega}u_kf\,dS. \tag{5.1}$$

Denote by U^* the solution to the extension problem for u on top of $\overline{\Omega}$ and let $(U^*)_k=(U^*-C_k)^+$ be its truncation. Let $(U_k)^*$ be the solution to the extension problem on top of $\overline{\Omega}$ with datum u_k. On one hand, by using the extension characterization of ∂_ν^σ we can estimate the left hand side as

$$\begin{aligned}\int_{\partial\Omega}u_k\partial_\nu^\sigma u\,dS&=-c_\sigma\lim_{y\to 0}\int_{\partial\Omega}(U^*)_kU_y^*y^a\,dS\\&=\int_0^\infty y^a\left(\int_\Omega\nabla(U^*)_k\nabla U^*\,dx+\int_{\partial\Omega}((U^*)_k)_yU_y^*\,dS\right)dy\\&=\int_0^\infty y^a\left(\int_\Omega|\nabla(U^*)_k|^2\,dx+\int_{\partial\Omega}((U^*)_k)_y^2\,dS\right)dy\\&\geq\int_0^\infty y^a\left(\int_\Omega|\nabla(U_k)^*|^2\,dx+\int_{\partial\Omega}((U_k)^*)_y^2\,dS\right)dy\\&=c_\sigma\int_{\partial\Omega}\left|\partial_\nu^{\sigma/2}u_k\right|^2\,dS\end{aligned}$$

where in the inequality above we used the minimization property of $(U_k)^*$, see Lemma 5.1.2. We now estimate the right hand side of (5.1). Let 2^*_σ be the critical exponent in the fractional Sobolev embedding (Lemma 5.1.3). Observe that

$$\int_{\partial\Omega}u_kf\,dS\leq\|f\|_{L^q(\partial\Omega)}\|u_k\|_{L^{2^*_\sigma}(\partial\Omega)}\left|\{u_k>0\}\right|^{1-\frac{1}{2^*_\sigma}-\frac{1}{q}}$$

$$\leq C\|f\|_{L^q(\partial\Omega)}\|\partial_\nu^{\sigma/2}u_k\|_{L^2(\partial\Omega)}\left|\{u_k>0\}\right|^{1-\frac{1}{2^*_\sigma}-\frac{1}{q}}$$
$$\leq \frac{c_\sigma}{2}\|\partial_\nu^{\sigma/2}u_k\|^2_{L^2(\partial\Omega)}+CF^2\left|\{u_k>0\}\right|^{2(1-\frac{1}{2^*_\sigma}-\frac{1}{q})}$$

where $F=\|f\|_{L^q(\partial\Omega)}$ and C depends only on $\partial\Omega$ and σ. Also, explicitly computing the exponent on the right-most term gives

$$2\left(1-\frac{1}{2^*_\sigma}-\frac{1}{q}\right)=1+\frac{\sigma}{n-1}-\frac{2}{q}.$$

Therefore, by plugging the last two estimates into (5.1),

$$\int_{\partial\Omega}\left|\partial_\nu^{\sigma/2}u_k\right|^2\,dS\leq CF^2\left|\{u_k>0\}\right|^{1+\frac{\sigma}{n-1}-\frac{2}{q}}.$$

Note that if $u_k>0$, then $u_{k-1}>2^{-1}2^{-k}=2^{-k-1}$. Applying Chebyshev's inequality to the right hand side, we see

$$\left|\{u_k>0\}\right|=\left|\{u_{k-1}>2^{-k-1}\}\right|=\left|\{u_{k-1}^2>2^{-2(k+1)}\}\right|\leq 2^{2(k+1)}\int_{\partial\Omega}u_{k-1}^2\,dx'.$$

Hence

$$\|\partial_\nu^{\sigma/2}u_k\|^2_{L^2(\partial\Omega)}\leq 2^{2k}C\left(\left|\partial\Omega\right|,F,\sigma\right)\left(\int_{\partial\Omega}u_{k-1}^2\,dx'\right)^{1+\frac{\sigma}{n-1}-\frac{2}{q}}.$$

We now proceed with the De Giorgi iteration. Denote the energy levels by

$$v_k=\int_{\partial\Omega}u_k^2\,dx'.$$

By Hölder's inequality, we get

$$\begin{aligned}v_k&\leq\|u_k\|^2_{L^{2^*_\sigma}(\partial\Omega)}\left|\{u_k>0\}\right|^{1/(2^*_\sigma/2)'}\\&\leq\|\partial_\nu^{\sigma/2}u_k\|^2_{L^2(\partial\Omega)}C^kv_{k-1}^{\sigma/(n-1)}\\&\leq C^kv_{k-1}^{1+\varepsilon}v_{k-1}^{\sigma/(n-1)}\\&=C^kv_{k-1}^{1+\gamma}.\end{aligned}$$

Therefore, if v_0 were small enough (which is always the case under appropriate scaling), then $\|v_k\|_{L^2(\partial\Omega)}\to 0$ as $k\to\infty$. In other words,

$$\int_{\partial\Omega}(u-1/2)_+^2\,dx'=0$$

which implies $|u| \leq 1/2$ in $\partial\Omega$. $\square$

CHAPTER 6. REGULARITY FOR SOLUTIONS TO A HARMONIC-LIKE EXTENSION PROBLEM

Let $B_1^+ = B_1 \cap \{x_n > 0\}$ be the half ball in $\mathbb{R}^n$ and $T_1 = B_1 \cap \{x_n = 0\}$ the face in $\mathbb{R}^{n-1}$. The goal of this chapter is to localize the L^2 to L^∞ estimate of Chapter 5 for solutions to

$$\begin{cases} \Delta_x U = 0 & \text{in } (-1,1) \times (B_1^+ \cup T_1) \\ \left(|y|^a U_y\right)_y = |y|^a \partial_{x_n} U & \text{on } (-1,1) \times T_1. \end{cases} \tag{6.1}$$

We then show solutions to (6.1) are Hölder continuous via a critical density and oscillation decay argument. Consequently, we show these harmonic-like solutions are smooth in x and enjoy estimates in y.

6.1 Localized L^2 to L^∞ estimate

In the following analysis, we obtain a localized L^2 to L^∞ estimate like the one in [12] with respect to the weighted measure $|y|^a\,dy$.

Lemma 6.1.1 (Interpolation)**.** *Let $\Omega \subseteq \mathbb{R}^n$. Suppose $f \in L^{q_1}(I, |y|^a\,dy; L^{p_1}(\Omega)) \cap L^{q_2}(I, |y|^a\,dy; L^{p_2}(\Omega))$. Define*

$$\frac{1}{p} = \frac{\theta}{p_2} + \frac{1-\theta}{p_1}$$

and

$$\frac{1}{q} = \frac{1-\theta}{q_1} + \frac{\theta}{q_2}.$$

Then $f \in L^q(I, |y|^a\,dy; L^p(\Omega))$.

Proof. Define $|f|_p := \|f\|_{L^p(\Omega)}$. Observe

$$\|f\|_{L^q(I,|y|^a\,dy;L^p(\Omega))} = \|\,|f|_p\,\|_q$$

$$\leq \| \, |f|_{p_1}^{1-\theta} \, |f|_{p_2}^{\theta} \, \|_q .$$

by Hölder's inequality. To see this, the usual interpolation argument using the fact that $1 = \frac{p\theta}{p_2} + \frac{(1-\theta)p}{p_1}$ yields

$$\begin{aligned} |f|_p^p &= \int_\Omega f^p \\ &= \int_\Omega f^{p(1-\theta)} f^{p\theta} \\ &\leq \left(\int_\Omega f^{p(1-\theta)\left(\frac{p_1}{(1-\theta)p}\right)} \right)^{\frac{(1-\theta)p}{p_1}} \left(\int_\Omega f^{p\theta\left(\frac{p_2}{p\theta}\right)} \right)^{\frac{p\theta}{p_2}} . \end{aligned}$$

Therefore

$$|f|_p \leq |f|_{p_1}^{1-\theta} \, |f|_{p_2}^{\theta} .$$

Now, applying essentially the same argument, we get

$$\| \, |f|_{p_1}^{1-\theta} \, |f|_{p_2}^{\theta} \, \|_q \leq \| \, |f|_{p_1} \, \|_{q1}^{1-\theta} \| \, |f|_{p_2} \, \|_{q_2}^{\theta} = \|f\|_{L^{q_1}(I,|y|^a \, dy; L^{p_1}(\Omega))}^{1-\theta} \|f\|_{L^{q_2}(I,|y|^a \, dy; L^{p_2}(\Omega))}^{\theta} .$$

□

Now, a quick application of interplolation gives the following inequality.

Lemma 6.1.2. *Suppose $q > 2$. There exists an $r > 2$ for which*

$$\|U\|_{L^r(T_1 \times (-1,1), |y|^a \, dy \, dx')}^2 \leq \|U\|_{L^q(I,|y|^a \, dy; L^2(\Omega))}^{\theta} \|U\|_{L^2(I,|y|^a \, dy; L^q(\Omega))}^{1-\theta} .$$

Proof. Let $2 < r < q$. We verify that there is a $0 < \theta < 1$ for which the hypothesis of the previous lemma is satisfied. We have

$$\frac{1}{r} = \frac{\theta}{q} + \frac{1-\theta}{2} \text{ and } \frac{1}{r} = \frac{\theta}{2} + \frac{1-\theta}{q} .$$

Therefore

$$\theta = \frac{\left(\frac{1}{r} - \frac{1}{2}\right)}{\frac{1}{q} - \frac{1}{2}} \in (0,1)$$

since $2 < r < q$.

□

Now, we have the following inequality:

Theorem 6.1.3. *There is an $r > 2$ and $0 < \theta < 1$ for which*

$$\left(\int_{-1}^{1} |y|^a \int_{T_1} U^r \, dx' \, dy\right)^{2/r} \leq C \left(\int_{-1}^{1} y^a \int_{\mathbb{R}^{n-1}} \left|(-\Delta)^{1/4} U\right|^2 \, dx' \, dy\right)^{\theta} \left(\int_{-1}^{1} |y|^a \int_{T_1} U_y^2 \, dx' \, dy\right)^{1-\theta}.$$

Proof. By Minkowski's integral inequality, the classical fractional Sobolev inequality, and q satisfying the hypothesis of Lemma 6.1.2, we have

$$\begin{aligned}
\left(\int_{-1}^{1} |y|^a \int_{T_1} U^r \, dx' \, dy\right)^{2/r} &\leq \left(\int_{-1}^{1} |y|^a \left(\int_{T_1} U^2 \, dx'\right)^{q/2} dy\right)^{2\theta/q} \left(\int_{-1}^{1} |y|^a \left(\int_{T_1} U^q \, dx'\right)^{2/q} dy\right)^{1-\theta} \\
&\leq \left(\int_{T_1} \left(\int_{-1}^{1} U^q \, |y|^a \, dy\right)^{2/q} dx'\right)^{\theta} \left(\int_{-1}^{1} |y|^a \left(\int_{T_1} U^q \, dx'\right)^{2/q} dy\right)^{1-\theta} \\
&\leq C \left(\int_{-1}^{1} |y|^a \int_{T_1} U_y^2 \, dx' \, dy\right)^{\theta} \left(\int_{-1}^{1} |y|^a \int_{\mathbb{R}^{n-1}} \left|(-\Delta)^{1/4} U\right|^2 \, dx' \, dy\right)^{1-\theta}.
\end{aligned}$$

□

We immediately obtain the following corollary:

Corollary 6.1.4. *There is an $r > 2$ so that*

$$\left(\int_{-1}^{1} |y|^a \int_{T_1} U^r \, dS \, dy\right)^{2/r} \leq C \left(\int_{-1}^{1} |y|^a \left(\int_{B_1^+} |\nabla U|^2 \, dx + \int_{T_1} U_y^2 \, dx'\right) dy\right).$$

Proof. Let U^* be the solution to the flat extension with trace term U on T_1. We have the energy identity

$$\int_{-1}^{1} |y|^a \int_{\mathbb{R}^{n-1}} \left|(-\Delta)^{1/4} U\right|^2 \, dS \, dy = \int_{-1}^{1} |y|^a \int_{\mathbb{R}^n_+} |\nabla U^*|^2 \, dx \, dy.$$

Note that $\mathrm{tr}(U^*)|_{B_1^+} = \mathrm{tr}(U)|_{B_1^+}$ where U is the solution to (6.1). Then

$$\int_{-1}^{1} |y|^a \int_{\mathbb{R}^{n-1}} \left|(-\Delta)^{1/4} U\right|^2 \, dS \, dy = \int_{-1}^{1} |y|^a \int_{\mathbb{R}^n_+} |\nabla U^*|^2 \, dx \, dy \leq \int_{-1}^{1} |y|^a \int_{B_1^+} |\nabla U|^2 \, dx \, dy$$

since U^* minimizes the Dirichlet energy in the middle term. Then Theorem 6.1.3 along with this inequality implies the desired conclusion. □

Lemma 6.1.5. *Suppose U is a subsolution to*

$$\begin{cases} \Delta U = 0 & \text{in } B_1^+ \times (-1,1) \\ (|y|^a U_y)_y = |y|^a \partial_{x_n} U & \text{on } T_1 \times (-1,1). \end{cases}$$

That is,

$$\int_{-1}^{1} |y|^a \left(\int_{B_1^+} \nabla U \nabla \phi \, dx + \int_{T_1} U_y \phi_y \, dx' \right) dy \leq 0$$

for every test function ϕ. Then for any cutoff function $\eta \subseteq C_0^\infty\left((B_1^+ \cup T_1) \times (-1,1)\right)$ we have the following energy inequality:

$$\int_{-1}^{1} |y|^a \left(\int_{B_1^+} |\nabla (U_+ \eta)|^2 \, dx + \int_{T_1} (U_+\eta)_y^2 \, dx' \right) dy \leq \int_{-1}^{1} |y|^a \left(\int_{B_1^+} |\nabla \eta|^2 U_+^2 \, dx + \int_{T_1} \eta_y^2 U_+^2 \, dx' \right) dy.$$

Proof. Testing against $\eta^2 U_+$ and using the fact that U is a subsolution, we get

$$0 \geq \int_{-1}^{1} |y|^a \left(\int_{B_1^+} \nabla\left(\eta^2 U_+\right) \nabla U \, dx + \int_{T_1} \left(\eta^2 U_+\right)_y U_y \, dx' \right) dy = I + II.$$

Now, $\nabla\left(\eta^2 U_+\right) = \nabla\eta\,(\eta U_+) + \eta \nabla(\eta U_+)$ and $\eta \nabla U = \nabla(\eta U) - \nabla \eta U$. Therefore

$$\begin{aligned} I &= \int_{-1}^{1} |y|^a \int_{B_1^+} \nabla \eta\,(\eta U_+) \nabla U + \eta \nabla(\eta U_+) \nabla U \, dx \, dy \\ &= \int_{-1}^{1} |y|^a \int_{B_1^+} \nabla \eta U_+ \left(\nabla(\eta U) - \nabla \eta U\right) + \nabla(\eta U_+)\left(\nabla(\eta U) - \nabla\eta U\right) \, dx \, dy \\ &= \int_{-1}^{1} |y|^a \int_{B_1^+} \nabla\eta U_+ \nabla(\eta U) - \nabla \eta U_+ \nabla \eta U + \nabla(\eta U_+)\nabla(\eta U) - \nabla(\eta U_+)\nabla\eta U \, dx \, dy. \end{aligned}$$

Notice that $\nabla \eta U_+ \nabla(\eta U) = \nabla\eta U_+ \nabla(\eta U_+) = \nabla \eta U \nabla(\eta U_+)$. Indeed, we can expand the expression to obtain

$$\nabla \eta U_+ \nabla(\eta U) = \nabla \eta U_+ \nabla \eta U + \nabla \eta U_+ \eta \nabla U.$$

Now, $U_+ U = (U_+)^2$ by definition of U_+. Similarly, $U_+ \nabla U = U_+ \nabla U_+ = U \nabla U_+$. Hence

$$I = \int_{-1}^{1} |y|^a \int_{B_1^+} |\nabla(\eta U_+)|^2 - |\nabla \eta|^2 U_+^2 \, dx \, dy.$$

The exact same computation with the boundary term gives

$$II = \int_{-1}^{1} |y|^a \int_{T_1} (\eta U_+)_y^2 - \eta_y^2 U_+^2 \, dx' \, dy.$$

Since $I + II \leq 0$, we obtain the desired energy inequality. $\square$

We now follow an argument similar to that in [12] to obtain a localized L^2 to L^∞ estimate on solutions U to (6.1). The main idea is to utilize various barriers which will allow us to localize statements about U by comparison. We modify the idea to account for the weighted measure $|y|^a \, dx\, dy$ and the distinct energy associated to our problem.

Theorem 6.1.6. *There exists $\varepsilon_0 > 0$ and $\lambda > 0$, depending only on N, such that for every subsolution U to (6.1), the following property holds true: If $U \le 2$ in $\left(B_2^+ \cup T_2\right) \times (-1,1)$, and*

$$\int_{-1}^{1} |y|^a \left(\int_{B_2^+} U^2 \, dx + \int_{T_2} U^2 \, dx' \right) dy < \varepsilon_0$$

then

$$U \le 2 - \lambda \text{ on } T_{1/2} \times (-1/2, 1/2)\,.$$

Before proving this theorem, we redefine B_r^+ to be the cube $T_r \times [0,r]$, where T_r is a cube of radius r, in $\mathbb{R}^{N-1}$, centered at the origin. This local cube perspective is equivalent to the ball setting we discuss prior. We also denote $P(x_n)$ to be the Poisson kernel.

Proof. We begin by defining barrier functions. Consider b_1 which solves

$$\begin{cases} \Delta b_1 = 0 & \text{in } B_2^+ \\ b_1 = 2 & \text{on the sides of the cube } B_2^+ \text{ except for } x_n = 0 \,\cdot \\ b_1 = 0 & \text{for } x_n = 0 \end{cases}$$

Then, by the maximum principle, we can find $\lambda > 0$ for which

$$b_1(x', x_n) \le 2 - 4\lambda \text{ on } B_{1^+}.$$

We also define the elliptic barrier

$$b_2(x,y) := 2\cos(y) e^{-x}, \qquad x, y \in \mathbb{R}$$

which is harmonic in $\{x > 0, 0 < y < 1\}$ and bounded by $2e^{-x}$ in this set.

Lemma 6.1.7. *There exist $0 < \delta < 1$ and $M > 1$ such that for every $k > 0$:*

$$2(n-1)\overline{C}e^{-\frac{2^{-k}}{4(\sqrt{2}+1)\delta^k}} \leq \lambda 2^{-k-2}$$

$$\frac{M^{-k/2}}{\delta^{\frac{(n-1)(k+1)}{2}}}\|P(1)\|_{L^2} \leq \lambda 2^{-k-2}$$

$$M^{-k} \geq C_0^k M^{-(1+1/(n-1))(k-3)} \qquad \text{for } k \geq 12$$

where $\overline{C}$ is defined previously for the barrier b_2 and C_0 a constant to be defined later.

The proof of this lemma can be found in [12]. We now set up the induction step which will be used to prove a decay of energy levels on shrinking rectangles. Define

$$u_k = (u - C_k)_+ \text{ and } U_k = (U - C_k)_+$$

where $u = \mathrm{tr}(U)|_{T_1}$ and $C_k = 2 - \lambda(1 + 2^{-k})$. Notice that $C_k \to 2 - \lambda$ as $k \to \infty$. Define

$$T_k' = T_{\frac{1}{2}(1+2^{-k})} \times \left(-\frac{1}{2}\left(1+2^{-k}\right), \frac{1}{2}\left(1+2^{-k}\right)\right)$$

and notice that $T_k' \to T_{1/2} \times (-1/2, 1/2)$ as $k \to \infty$. We define the cut-off functions η_k on T_0' where

$$\mathbf{1}_{T_{k+1}'} \leq \eta_k \leq \mathbf{1}_{T_k'} \text{ and } |\nabla_{x'}\eta_k|, |(\eta_k)_y| \leq C2^k.$$

Define the energy levels

$$V_k = \int_{-1}^{1} |y|^a \int_0^{\delta^k} \int_{\mathbb{R}^{N-1}} |\nabla(\eta_k U_k)|^2 \, dx' \, dx_n \, dy + \int_{-1}^{1} |y|^a \int_{\mathbb{R}^N} (\eta_k u_k)_y^2 \, dx' \, dy.$$

We will now show that for all $k \geq 0$,

$$V_k \leq M^{-k}$$

$$\eta_k U_k = 0 \text{ for } x_n = \delta_k.$$

For $0 \leq k \leq 12(n-1)$, plugging $\eta_k(x', y)\psi(x_n)$ into the energy inequality, we obtain

$$\begin{aligned}\int_{-1}^{1} |y|^a \left(\int_{B_1^+} |\nabla(U_k\eta_k\psi)|^2 \, dx + \int_{T_1} (U_k\eta_k\psi)_y^2 \, dx'\right) dy &\leq \int_{-1}^{1} |y|^a \left(\int_{B_1^+} |\nabla\eta_k\psi|^2 U_+^2 \, dx + \int_{T_1} (\eta_k\psi)_y^2 U_k^2 \, dx'\right) dy \\ &\leq C2^{24N}\left(\int_{-1}^{1} |y|^a \left(\int_{B_1^+} U_k^2 \, dx + \int_{T_1} U_k^2 \, dx'\right) dy\right)\end{aligned}$$

$$\leq C2^{24N}\varepsilon_0$$
$$\leq M^{-12N}$$

for ε_0 small enough.

Now, by the maximum principle,

$$U \leq \left(u_+ \mathbf{1}_{T_1}\left(x', y\right)\right) * P(x_n) + b_1(x', x_n)$$

since the right hand side is positive, harmonic, and the trace on the boundary is bigger than that of U ($U \leq 2$ by assumption). Recall that $b_1(x', x_n) \leq 2 - 4\lambda$ on B_1^+ and

$$\|u_+ \mathbf{1}_{T_1} * P(x_n)\|_{L^\infty(x_n \geq 1)} \leq C\|P(1)\|_{L^2}\sqrt{\varepsilon_0} \leq C\sqrt{\varepsilon_0}.$$

Choosing ε_0 small enough so that the right hand side is smaller than 2λ gives

$$U \leq 2 - 2\lambda \text{ for } x_n = 1, x' \in T_1.$$

Hence

$$U_0 = \left(U - (2 - 2\lambda)\right)_+ \leq 0 \text{ for } 1 = x_n, x' \in T_1.$$

So $\eta_0 U_0 = 0$ for $1 = \delta^0$.

Next we claim that if the induction hypotheses are satisfied up until k, then

$$\eta_{k+1} U_{k+1} \leq 0 \text{ for } \delta^{k+1} = x_n$$

and that

$$\eta_{k+1} U_{k+1} \leq \left((\eta_k u_k) * P(z)\right) \eta_{k+1} \text{ on } T_{1+2^{-k}} \times [0, \delta^k].$$

To this end, we utilize the barrier b_2 and try to control U_k by harmonic functions on the boundary. Consider the smaller set $T_{1+2^{-k-1/2}} \times [0, \delta^k]$. When $z = \delta^k$, we get that $U_k = 0$ by the induction hypothesis. When $z = 0$, we can control U_k by $\eta_k u_k * P(x_n)$ since their traces are the same on the set we consider. Now, to control U_k on the remaining sides of the cube. To do this, we use the barrier b_2. In particular, given a point $(x_1, \ldots, x_{n-1}) \in T_{1+2^{-k-1/2}}$, we define

$$b_2\left((x_i - x^+)/\delta^k, x_n/\delta^k\right) + b_2\left((-x_i + x^-)/\delta^k, x_n/\delta^k\right)$$

where $x^+ = (1 + 2^{-k-1/2})$ and $x^- = -x^+$. Here, the division by δ^k normalizes the second argument so that $0 \le x_n/\delta^k \le 1$ which aligns with the defintion of b_2. Also, if $x_i = x^+$ or $x_i = x^-$, that is, $x' = (x_1, \ldots, x_i, \ldots, x_{n-1})$ is a point on the boundary of $T_{1+2^{-k-1/2}}$, then the sum is larger than 2. By the maximum principle, we get

$$U_k \le \sum_{k=1}^{n-1} \left(b_2 \left((x_i - x^+)/\delta^k, x_n/\delta^k \right) + b_2 \left((-x_i + x^-)/\delta^k, x_n/\delta^k \right) \right) + (\eta_k u_k) * P\left(x_n\right).$$

By the initial bound we had on b_2, we see

$$\sum_{k=1}^{n-1} \left(b_2 \left((x_i - x^+)/\delta^k, x_n/\delta^k \right) + b_2 \left((-x_i + x^-)/\delta^k, x_n/\delta^k \right) \right) \le 2\left(n-1\right) C e^{-\frac{2^{-k}}{4(\sqrt{2}+1)\delta^k}}.$$

Then, by 6.1.7, we see

$$\sum_{k=1}^{n-1} \left(b_2 \left((x_i - x^+)/\delta^k, x_n/\delta^k \right) + b_2 \left((-x_i + x^-)/\delta^k, x_n/\delta^k \right) \right) \le \lambda 2^{-k-2}.$$

Therefore

$$U_{k+1} \le \left(U_k - \lambda 2^{-k-1} \right)_+,$$

and, in particular,

$$U_{k+1} \le \left((\eta_k u_k) * P_{x_n} - \lambda 2^{-k-2} \right)_+.$$

Multiplying the cuttoff η_{k+1} on the left gives

$$\eta_{k+1} U_{k+1} \le \left((\eta_k u_k) * P_{x_n} - \lambda 2^{-k-2} \right)_+.$$

Now, let $\delta^{k+1} \le x_n \le \delta^k$. Then

$$|(\eta_k u_k) * P_{x_n}| \le \sqrt{V_k} \|P_{x_n}\|_{L^2(T_{1+2^{-k-1/2}})} \le \frac{M^{-k/2}}{\delta^{(k+1)(n-1)/2}} \|P_1\|_{L^2\left(T_{1+2^{-k-1/2}}\right)} \le \lambda 2^{-k-2}$$

where we have again used Lemma 6.1.7. So,

$$\eta_{k+1} U_{k+1} \le 0 \text{ for } \delta^{k+1} \le z \le \delta^k.$$

Now, we will show that $V_k \le C^k \left(V_{k-3}\right)^{1+1/(n-1)}$ for $k \ge 12(n-1)+1$. Doing so will imply that $V_k \to 0$ for $k \to \infty$. Indeed, taking the initial function to be $\omega \frac{U_0}{V_0}$ for small ω allows us to run an induction argument as in the typical De Giorgi proof.

Now,

$$V_{k-3} \geq C\|\eta_{k-3}u_{k-3}\|^2_{L^{2r}(\mathbb{R}^{n-1}\times(-1,1))}$$

for some r such that $2r > 2$ (see Corollary 6.1.4). Notice that

$$\frac{1}{r} + \frac{1}{1} = \frac{1}{r} + 1,$$

so by the Young convolution inequality we have

$$\|\eta_{k-2}U_{k-2}\|_{L^{2r}\left(\mathbb{R}^n_+\times(-1,1)\right)} \leq \|P(1)\|_{L^1}\|\eta_{k-3}u_{k-3}\|_{L^{2r}_a} = \|\eta_{k-3}u_{k-3}\|_{L^{2r}(\mathbb{R}^{n-1}\times(-1,1))}.$$

Therefore

$$V_{k-3} \geq C\|\eta_{k-2}U_{k-2}\|^2_{L^{2r}} + C\|\eta_{k-3}u_{k-3}\|^2_{L^{2r}} \geq \|\eta_{k-2}U_{k-1}\|^2_{L^{2r}} + C\|\eta_{k-2}u_{k-1}\|^2_{L^{2r}}$$

since, by definition, $U_k \leq U_{k-1}$ and $\eta_k \leq \eta_{k-1}$ for all k. We now bridge the gap between V_k and V_{k-3}. By the energy inequality, we have

$$V_k \leq 2^{2k}\left(\int_{-1}^{1}\int_0^{\delta^k}\int_{\mathbb{R}^{N-1}} \eta_{k-1}^2 U_k^2 \, dx\, dy + \int_{-1}^{1}\int_{\mathbb{R}^{N-1}} \eta_{k-1}^2 u_k^2 \, dx'\, dy\right).$$

By Hölder's, Jensen's inequality, and the fact that

$$\{\eta_{k-1}U_k > 0\} = \{\eta_{k-2}U_{k-1} > 2^{-k}\lambda\},$$

we have

$$\begin{aligned}
\int \eta_{k-1}^2 U_k^2 &\leq \left(\int (\eta_{k-1}U_k)^{2r}\right)^{1/r} |\{\eta_{k-1}U_k > 0\}|^{1/r'} \\
&= \left(\int (\eta_{k-1}U_k)^{2r}\right)^{1/r} \left|\{(\eta_{k-2}U_{k-1})^{2r} > (2^{-k}\lambda)^{2r}\}\right|^{1/r'} \\
&\leq \left(\int (\eta_{k-2}U_{k-1})^{2r}\right)^{1/r} \left(\frac{2^{2kr}}{\lambda^{2r}}\left(\int (\eta_{k-2}U_{k-1})^{2r}\right)\right)^{1/r'} \\
&= \left(\frac{2^{2k}}{\lambda^2}\right)^{r/r'} \int (\eta_{k-2}U_{k-1})^{2r}.
\end{aligned}$$

The exact same computation applied to the trace term gives

$$\int \eta_{k-1}^2 u_k^2 \leq \left(\frac{2^{2k}}{\lambda^2}\right)^{r/r'} \int (\eta_{k-2}u_{k-1})^{2r}.$$

Hence

$$V_k \le 2^{2k}\left(\frac{2^{2k}}{\lambda^2}\right)^{r/r'}\left(\int (\eta_{k-2}U_{k-1})^{2r} + \int (\eta_{k-2}u_{k-1})^{2r}\right).$$

Combining the previous inequalities with the above estimate, we get

$$V_k \le C^k V_{k-3}^r$$

where $r > 1$ and $C_0^k = C2^{2k}\left(\frac{2^{2k}}{\lambda^2}\right)^{r/r'}$.

We now show that

$$\int_{-1}^{1} |y|^a \left(\int_{T_1} (\eta_k U_k)^p \, dx'\right)^{2/p} dy \le CV_k$$

for any k, with C independent of k, and $p > 2$ chosen for a particular embedding. This would imply, by the above iteration, that

$$\int_{-1}^{1} |y|^a \left(\int_{B_1^+} (\eta_\infty U_\infty)^p \, dx'\right)^{2/p} dy = 0.$$

In other words,

$$U \le 2 - \lambda$$

in the smaller, limiting set. To do this, fix $y \in (-1, 1)$ and note that if U is compactly supported in T_1, we can use the fractional Sobolev embedding to obtain

$$\left(\int_{T_1} U^p \, dx'\right)^{1/p} \le C\left(\int_{\mathbb{R}^{n-1}} \left|(-\Delta)^{1/4} U^*\right|^2 dx'\right)^{1/2} = C\left(\int_{\mathbb{R}^n} |\nabla U^*|^2 \, dx\right)^{1/2}$$

where U^* is the harmonic extension of $\mathrm{tr}|_{T_1}(U)$. Since the harmonic extension U^* minimizes the Dirichlet energy and $\mathrm{tr}|_{T_1} U^* = \mathrm{tr}|_{T_1}(\chi_{B_1^+} U)$, we have

$$\int_{\mathbb{R}^n} |\nabla U^*|^2 \, dx \le \int_{B_1^+} |\nabla U|^2 \, dx.$$

Integrating with respect to the measure $|y|^a \, dy$, we get

$$\begin{aligned}\int_{-1}^{1} |y|^a \left(\int_{T_1} U^p \, dx'\right)^{2/p} dy &\le C \int_{-1}^{1} |y|^a \int_{B_1^+} |\nabla U|^2 \, dx \, dy \\ &\le C\left(\int_{-1}^{1} |y|^a \left(\int_{B_1^+} |\nabla U|^2 \, dx + \int_{T_1} U_y^2 \, dx'\right) dy\right)\end{aligned}$$

where C is universal with respect to the cutoffs we consider. $\square$

6.2 Critical Density Argument

Let us fix some notation. Define $Q_r = (B_r^+ \cup T_r) \times (-r, r)$, that is, Q_r is the half cylinder including a face. We first prove a compact embedding result which we will use to prove a Fabes' lemma. As a reminder, the space $\mathcal{H}_a^1$, in the context of the flat problem, is all $U : \left(B_1^+ \times T_1\right) \times (-1,1) \to \mathbb{R}$ for which $U, \nabla_x U \in L^2\left(B_1^+ \times (-1,1)\right)$ and $U_y \in L^2\left(T_1 \times (-1,1)\right)$ with norm

$$\|U\|_{\mathcal{H}_a^1} = \int_{-1}^{1} |y|^a \left(\int_{B_1^+} |\nabla U|^2 + U^2 \, dx + \int_{T_1} U_y^2 \, dx' \right) dy.$$

Lemma 6.2.1 (Compact Embedding)**.** *The space $\mathcal{H}_a^1$ embeds compactly into $L^2\left((B_1^+ \cup T_1) \times (-1,1), |y|^a \, dx \, dy\right)$.*

Proof. The integrability conditions in the space $\mathcal{H}_a^1$ can be retwritten in a vector-valued setting as

$$U \in L_a^2\left((-1,1); H^1(B_1^+)\right)$$

and

$$U_y \in L_a^2\left((-1,1); L^2(T_1)\right)$$

with norm

$$\|U\|_{\mathcal{H}_a^1}^2 = \int_{-1}^{1} |y|^a \left(\int_{B_1^+} |\nabla U|^2 + U^2 \, dx + \int_{T_1} U_y^2 \, dx' \right) dy.$$

We now setup to use the Aubin-Lion lemma to prove the compact embedding. Namely, we show that $H^1(B_1^+) \subset\subset L^2(B_1^+ \cup T_1) \hookrightarrow L^2(T_1)$. Then, the Aubin-Lion lemma modified to account for the weighted measure implies the spaces defining $\mathcal{H}_a^1$:

$$U \in L_a^2\left((-1,1); H^1(B_1^+)\right)$$

and

$$U_y \in L_a^2\left((-1,1); L^2(T_1)\right),$$

embed compactly in

$$U \in L_a^2\left((-1,1); L^2\left(B_1^+ \cup T_1\right)\right)$$

provided that

$$H^1(B_1^+) \subset\subset L^2(B_1^+ \cup T_1) \hookrightarrow L^2(T_1).$$

We know by the Rellich-Kondrachov theorem that $H^1(B_1^+) \subset\subset L^2(B_1^+)$. Using the trace operator, we can identify $H^1\left(B_1^+\right)$ with $H^{1/2}(T_1)$ by [31]. Further, $H^{1/2}(T_1) \subset\subset L^2(T_1)$.

We will now show that $H^1(B_1^+) \subset\subset L^2\left(B_1^+ \cup T_1\right)$ via a sequential argument. Let $\{U_k\}$ be a bounded sequence in $H^1(B_1^+)$. Then there exists a weakly convergent subsequence, call it $\{U_k\}$ again, converging to some $U \in H^1(\Omega)$. By the compact embedding mentioned above, $U_k \to U$ in $L^2(B_1^+)$ strongly. Also, $\{\mathrm{tr}(U_k)\}$ is a bounded sequence in $H^{1/2}(T_1)$. So there exists a further subsequence, call it $\{\mathrm{tr}(U_k)\}$ again, converging weakly to some $W \in H^{1/2}(T_1)$. Then $\mathrm{tr}(U_k) \to W$ strongly in $L^2(T_1)$ by the compact embedding mentioned above. Finally, we conclude that $W = \mathrm{tr}(U)$ since the trace operator is continuous. Therefore

$$H^1(B_1^+) \subset\subset L^2(B_1^+ \cup T_1) \hookrightarrow L^2(T_1).$$

□

Let us first define a measure which will be used in the following statements. For $r > 0$, define

$$\mu_r(E) = \int_{-r}^{r} |y|^a \left(\int_{B_r^+} \chi_E \, dx + \int_{T_r} \chi_E \, dx' \right) dy$$

for measurable subsets $E \subseteq \left(B_1^+ \times T_1\right) \times (-1,1)$. If $r = 1$, we denote $\mu_1 = \mu$ for convenience. We now prove a variation of Fabes' lemma which will be used to prove a critical density result. The idea is, if $V = 0$ with positive density anywhere in Q_1 (including the flat face), then V satisfies the following energy inequality:

Lemma 6.2.2 (Fabes' Lemma)**.** *For all $\varepsilon > 0$ there exists $C > 0$ depending only on dimension and ε such that if $V \in \mathcal{H}_a^1$ satisfies*

$$\mu(\{V = 0\}) \geq \varepsilon \, |Q_1|$$

then

$$\int_{-1}^{1} |y|^a \left(\int_{B_1^+} V^2 \, dx + \int_{T_1} V^2 \, dx' \right) dy \leq C \left(\int_{-1}^{1} |y|^a \left(\int_{B_1^+} |\nabla V|^2 \, dx + \int_{T_1} V_y^2 \, dx' \right) dy \right).$$

Proof. We proceed by contradiction. Suppose there exists a sequence $\{V_k\} \subseteq \mathcal{H}_a^1$ for which

$$\mu\left(\{V_k = 0\}\right) \geq \varepsilon \left|Q_1\right|$$

and

$$\int_{-1}^{1} |y|^a \left(\int_{B_1^+} V_k^2 \, dx + \int_{T_1} V_k^2 \, dx' \right) dy = 1$$

for all $k \geq 1$ but

$$\int_{-1}^{1} |y|^a \left(\int_{B_1^+} \left|\nabla V_k\right|^2 \, dx + \int_{T_1} (V_k)_y^2 \, dx' \right) dy \to 0 \text{ as } k \to \infty.$$

Then $\{V_k\}$ is a bounded sequence in $\mathcal{H}_a^1$. Therefore, there is a subsequence that, up to reindexing, we denote by $\{V_k\}$ again, and $V_\infty \in \mathcal{H}^1$ such that $V_k \to V_\infty$ weakly in $\mathcal{H}_a^1$ as $k \to \infty$. Since $\mathcal{H}_a^1$ is compactly embedded in $L_a^2\left((-1,1), L^2(B_1^+ \cup T_1)\right)$ by 6.2.1, there exists a further subsequence, that we also denote by $\{V_k\}$, for which $V_k \to V_\infty$ strongly in $L_a^2\left((-1,1), L^2\left(B_1^+ \cup T_1\right)\right)$. In particular, $\|V_\infty\|_{L_a^2\left((-1,1),L^2\left(B_1^+\cup T_1\right)\right)} = 1$ and, by Fatou's lemma,

$$\|V_\infty\|^2_{L_a^2\left((-1,1),L^2\left(B_1^+\cup T_1\right)\right)} \leq \liminf_{k\to\infty} \|V_k\|^2_{L_a^2\left((-1,1),L^2\left(B_1^+\cup T_1\right)\right)} = 0.$$

Hence, V_∞ is constant in (x', x_n), constant in y, and, consequently, constant everywhere. Thus,

$$\begin{aligned}
0 &= \lim_{k\to\infty} \int_{-1}^{1} |y|^a \int_{B_1^+} \left|V_k - V_\infty\right|^2 \, dx \, dy + \int_{-1}^{1} |y|^a \int_{T_1} \left|V_k - V_\infty\right|^2 \, dx' \, dy \\
&\geq \lim_{k\to\infty} \left(\int_{\{(x,y)\in B_1^+\times(-1,1):V_k=0\}} |y|^a \left|V_k - V_\infty\right|^2 \, dx \, dy + \int_{\{(x',y)\in T_1\times(-1,1):V_k=0\}} |y|^a \left|V_k - V_\infty\right|^2 \, dx' \, dy \right) \\
&\geq (V_\infty)^2 \inf_{k\geq 1} \mu\left(\{V_k = 0\}\right) \\
&\geq (V_\infty)^2 \varepsilon \left|Q_1\right| \\
&> 0
\end{aligned}$$

which is a contradiction. $\square$

We are now able to prove a critical density estimate.

Theorem 6.2.3 (Critical density estimate)**.** *For any $\varepsilon > 0$ there exists $0 < c < 1$ depending only on n and ε such that if U is a solution in Q_2 and*

$$|\{x \in Q_1 : U \geq 1\}| \geq \varepsilon |Q_1|$$

then

$$\inf_{Q_{1/2}} U \geq c.$$

Proof. We may assume $U \geq \delta$ in Q_2 for some small $\delta > 0$. For the general case we consider the solution $U + \delta > 0$ in Q_2. Then

$$\left|\{(x', x_n, y) \in Q_1 : (U + \delta) \geq 1\}\right| \geq \left|\{(x', x_n, y) \in Q_1 : U \geq 1\}\right| \geq \varepsilon |Q_1|.$$

So

$$\inf_{Q_{1/2}} (U + \delta) = \inf_{Q_{1/2}} U + \delta \geq c.$$

Since $\delta > 0$ is arbitrary, the result follows for U.

Let $V = -\min(0, \log(U)) = (\log U)^-$ which is a composition of U with $\Phi(t) = (\log(t))^-$. We will prove that V is a subsolution in Q_2. By assumption, $0 \leq V \leq \log(\delta^{-1})$ so that $V \in L^2(Q_2)$. We see that Φ is convex, nonincreasing, and Lipschitz on $[\delta, \infty)$, and, in particular, supported on $[\delta, 1]$. We also know that Φ is differentiable in $[\delta, \infty)$ except at the cutoff point $t = 1$. Since Φ is Lipschitz, we have $\nabla_x(V(U)) = V'(U)\nabla_x U$ and $\partial_y V(U) = V'(U) U_y$ are both in $L^2_a\left((-2,2) \times B_2^+\right)$ and $L^2_a((-2,2) \times T_2)$, respectively. That is to say, $V \in \mathcal{H}^1_a$. To verify that V is a subsolution, we use a mollification. Let $\zeta \in C_c^\infty(-1,1)$ be nonegative and even with $\int \zeta = 1$. Define $\zeta_\varepsilon = \frac{1}{\varepsilon}\zeta(z/\varepsilon)$ and $\Phi_\varepsilon = (\zeta_\varepsilon * \Phi)$. Then $\Phi_\varepsilon \in C_c^\infty(\delta - \varepsilon, \delta + \varepsilon)$. Since Φ is nonincreasing and Lipschitz in $[\delta, \infty)$, we have $\Phi'_\varepsilon = \eta_\varepsilon * \Phi' \leq 0$. By convexity, $\Phi''_\varepsilon = \zeta_\varepsilon * \Phi'' \geq 0$. Let $\phi \in C_c^1(Q_2)$. Then Φ'_ε is an admissible nonpositive test function. Using this along with the fact that U is a solution in Q_2, we get the following

$$\begin{aligned}
\int_{-2}^{2} |y|^a \int_{B_2^+} \nabla\left(\Phi_\varepsilon(U)\right) \nabla\phi \, dx \, dy &= \int_{-2}^{2} |y|^a \int_{B_2^+} \Phi'_\varepsilon \nabla U \nabla \phi \, dx \, dy \\
&= \int_{-2}^{2} |y|^a \left(\int_{B_1^+} \nabla U \nabla \left(\Phi'_\varepsilon(U)\phi\right) - |\nabla U|^2 \phi \Phi''_\varepsilon \, dx \right) dy
\end{aligned}$$

$$\leq 0.$$

The same exact argument applied to the y derivative gives

$$\int_{-2}^{2} |y|^a \int_{T_2} \left(\Phi_\varepsilon(U)\right)_y \phi_y \, dx' \, dy \leq 0.$$

So, $\Phi_\varepsilon(U)$ is a subsolution converging pointwise a.e. to $\Phi(U)$ in $\mathcal{H}_a^1$ by Corollary 4.1.12, and, by the dominated convergence theorem,

$$\int_{-2}^{2} |y|^a \, (\int_{B_2^+} \nabla \left(\Phi(U)\right) \nabla \phi \, dx + \int_{T_2} \left(\Phi(U)\right)_y \phi_y \, dx' \, dy =$$
$$\lim_{\varepsilon \to 0^+} \int_{-2}^{2} |y|^a \left(\int_{B_2^+} \nabla \left(\Phi_\varepsilon(U)\right) \nabla \phi \, dx + \int_{T_2} \left(\Phi_\varepsilon(U)\right)_y \phi_y \, dx' \right) dy \leq 0.$$

Hence V is a subsolution.

Now, observe

$$\mu_2\left(\{V = 0\}\right) = \mu_2\left(\{U = 1\}\right)$$
$$\geq \varepsilon \left|Q_2\right|.$$

By the L^2 to L^∞ estimate and Fabe's lemma applied to V, we get

$$\sup_{Q_1} V \leq C \|V\|_{L_a^2\left((B_1^+ \cup T_1) \times (-1,1)\right)} \leq C \left(\int_{-1}^{1} y^a \left(\int_{B_1^+} |\nabla V|^2 \, dx + \int_{T_1} V_y^2 \, dx' \right) dy \right).$$

Furthermore, if we can prove the right hand side is universally bounded, we get

$$\sup_{Q_1} V = \sup_{Q_1} \left(\log(U)\right)^- \leq C$$

which implies $\log(U) \geq \min\left(0, \log(U)\right) \geq -C$ in Q_1. Therefore

$$\inf_{Q_1} U \geq e^{-C} = c > 0.$$

Let us proceed with the required bounding. Observe $\nabla V = -\chi_{U<1} \left(\frac{1}{U}\right) \nabla U$. Let $\eta \in C_c^1\left(Q_2\right)$ and

$$\Phi = \frac{1}{U} \eta^2 > 0.$$

Since $U \geq \delta$, Φ is a valid test function which is nonnegative. Since U is a solution

$$0 = \int_{-1}^{1} \Bigg(\int_{B_1^+} \frac{\nabla_x U}{U} 2\eta \nabla_x \eta \, dx - \int_{B_1^+} \frac{\nabla_x U}{U} \frac{\nabla_x U}{U} \eta^2 \, dx + \int_{T_1} \frac{U_y}{U} 2\eta \eta_y \, dx' - \int_{T_1} \frac{U_y}{U} \frac{U_y}{U} \eta^2 \, dx' \Bigg) dy.$$

In particular, rearranging and applying Cauchy's ε-inequality, we get

$$\begin{aligned}
&\int_{-1}^{1} \left(\int_{B_1^+} \eta^2 \left(\frac{\nabla_x U}{U} \right)^2 dx + \int_{T_1} \eta^2 \left(\frac{U_y}{U} \right)^2 dx' \right) dy \\
&\qquad = \int_{-1}^{1} \left(\int_{B_1^+} \left(\frac{\nabla_x U}{U} \eta \right) (2\nabla_x \eta) \, dx + \int_{T_1} \left(\frac{U_y}{U} \eta \right) (2\eta_y) \, dx' \right) dy \\
&\qquad \leq \int_{-1}^{1} \Bigg(\varepsilon \int_{B_1^+} \left| \frac{\nabla_x U}{U} \right|^2 \eta^2 \, dx + \frac{1}{4\varepsilon^2} \int_{B_1^+} (2\nabla_x \eta)^2 \, dx' \\
&\qquad\quad + \varepsilon \int_{T_1} \left| \frac{U_y}{U} \right|^2 \eta^2 \, dx' + \frac{1}{4\varepsilon^2} \int_{T_1} (2\eta_y)^2 \, dx' \Bigg) dy.
\end{aligned}$$

Choose $\varepsilon = 1/2$. Then

$$\int_{-1}^{1} \int_{B_1^+} \left| \frac{\nabla_x U}{U} \right|^2 \eta^2 \, dx + \varepsilon \int_{T_1} \left| \frac{U_y}{U} \right|^2 \eta^2 \, dx' \leq C \int_{-1}^{1} \left(\int_{B_1^+} |\nabla_x \eta|^2 \, dx + \int_{T_1} \eta_y^2 \, dx' \right) dy \leq C_0$$

for a particular choice of η where $\eta = 1$ in Q_1. □

We now prove an oscillation decay result using the previous theorem. Notice now that we have the critical density theorem, we can prove the oscillation decay as usual.

Corollary 6.2.4. *There exists $0 < \omega < 1$ depending only on n such that if U is a bounded solution in Q_2, then*

$$\operatorname{osc}_{Q_1} U \leq \omega \operatorname{osc}_{Q_2} U.$$

In particular, U is locally Hölder continuous in Q_2.

Proof. For $r > 0$, let

$$M(r) = \sup_{Q_r} U \text{ and } m(r) = \inf_{Q_r} U.$$

Consider the two nonnegative solutions

$$\frac{U - m(2)}{M(2) - m(2)} \text{ and } \frac{M(2) - U}{M(2) - m(2)}$$

in Q_2. Notice

$$U \geq \frac{1}{2}\left(M(2) - m(2)\right)$$

if and only if

$$\frac{U - m(2)}{M(2) - m(2)} \geq \frac{1}{2}$$

and

$$u \leq \frac{1}{2}(M(2) + m(2))$$

if and only if

$$\frac{M(2) - U}{M(2) - m(2)} \geq \frac{1}{2}.$$

We break into two cases. First, suppose

$$\mu\left(\left\{\frac{2\left(U - m(2)\right)}{M(2) - m(2)} \geq 1\right\}\right) \geq \frac{1}{2}\left|B_1\right|.$$

We now apply the critical density result 6.2.3 to the nonnegative solution $\frac{2(U-m(2))}{M(2)-m(2)}$ with $\varepsilon = 1/2$ to get that, for some $0 < c < 1$,

$$\inf_{Q_{1/2}} \frac{(U - m(2))}{M(2) - m(2)} \geq c.$$

That is,

$$m(1/2) = \inf_{Q_{1/2}} U \geq m(2) + c\left(M(2) - m(2)\right).$$

Therefore

$$\operatorname{osc}_{Q_{1/2}} U = M(1/2) - m(1/2) \leq M(2) - m(1/2) \leq (1-c)\left(M(2) - m(2)\right) = \omega \operatorname{osc}_{Q_2} U.$$

For the second case, assume

$$\mu\left(\left\{\frac{2\left(M(2) - U\right)}{M(2) - m(2)} \geq 1\right\}\right) \geq \frac{1}{2}\left|Q_1\right|.$$

Applying Theorem 6.2.3 to $\frac{2(M(2)-U)}{M(2)-m(2)}$,

$$\inf_{Q_{1/2}} \frac{M(2) - U}{M(2) - m(2)} \geq c.$$

Rearranging

$$M\left(1/2\right) = \sup_{Q_{1/2}} U \leq M(2) - c\left(M(2) - m(2)\right).$$

Therefore,

$$\mathrm{osc}_{Q_{1/2}}U = M\,(1/2) - m\,(1/2) \le M(1/2) - m(2) \le (1-c)(M(2) - m(2)) = \omega\,\mathrm{osc}_{Q_2}(U).$$

□

Penultimately, we prove corollary about local Hölder continuity, from which we can derive further estimates on solutions to (6.1).

Corollary 6.2.5. *There exist $0 < \alpha < 1$ and $C > 0$ depending on n such that if U is a solution in Q_2 then $U \in C^\alpha$ at the origin with*

$$\left|U(x', x_n, y) - U(0,0,0)\right| \le C\|U\|_{L^2(Q_2)} \left|(x', x_n, y)\right|^\alpha$$

for every $(x', x_n, y) \in Q_1$.

Proof. We begin by zooming out U from $Q_{1/4^k}$ to Q_2 to apply the oscillation decay 6.2.4. Let $k \ge 0$ and consider

$$\overline{U}(x', x_n, y) = U\left(x'/4^{k+1/2}, x_n/4^{k+1/2}, y/4^{\sqrt{k+1/2}}\right)$$

in Q_2. Then $\overline{U}$ is a solution in Q_2, so, by Corollary 6.2.4,

$$\mathrm{osc}_{B_{1/4^{k+1}}}U = \mathrm{osc}_{B_{1/2}}\overline{U} \le \omega \mathrm{osc}_{B_2}\overline{U} = \omega\,\mathrm{osc}_{B_{1/4^k}}U.$$

Iterating this process along with local boundedness implies

$$\mathrm{osc}_{B_{1/4^{k+1}}}U \le \omega^k\,\mathrm{osc}_{Q_1}U \le 2\omega^k\|U\|_{L^\infty(Q_1)} \le C\omega^k\|U\|_{L^2(Q_2)}.$$

where C depends only n. Let us assume that $\omega > 1/4$. Choose $0 < \alpha < 1$ such that

$$\omega = \left(\frac{1}{4}\right)^\alpha.$$

Then α depends only on ω. Now, for any $(x', x_n, y) \in Q_1$, there is a k for which $(x', x_n, y) \in Q_{\frac{1}{4^k}} \setminus Q_{\frac{1}{4^{k+1}}}$, so that

$$\left|U(x', x_n, y) - U(0,0,0)\right| \le \mathrm{osc}_{Q_{1/4^k}}U$$

$$
\begin{aligned}
&\leq C\omega^k \|U\|_{L^2(Q_2)} \\
&= C\left(\frac{1}{4^k}\right)\alpha \|U\|_{L^2(Q_2)} \\
&= 4^\alpha C\left(\frac{1}{4^{k+1}}\right)^\alpha \|u\|_{L^2(Q_2)} \\
&\leq C\left|(x', x_n, y)\right|^\alpha \|U\|_{L^2(Q_2)}
\end{aligned}
$$

where C depends only on n. □

Finally, we prove a lemma regarding the smoothness properties of U. The ideas for the proof presented resemble those in [8, Chapter 5] and [9].

Lemma 6.2.6 (Estimates on harmonic functions)**.** *Let $U \in \mathcal{H}_a^1$ be harmonic in the sense of* (6.1)*.*

(1) *For each integer $k \geq 0$,*

$$\sup_{(B^+_{1/4}\cup T_{1/4})\times(-1/2,1/2)} |D_x^k U| \leq C_{n,a,k}\|U\|_{L^2_a}.$$

(2) *There exists a constant $C = C_{n,a} > 0$ such that for any $0 \leq y < 1/2$,*

$$\sup_{x\in B^+_{1/4}\cup T_{1/4}} |U_y(x, y)| \leq Cy.$$

Proof. Proof of (1). We first prove that (1) holds for fixed y. To that end, fix $y \in (-1/2, 1/2)$ and a unit tangential vector e in $(B_1 \cup T_1) \times \{y\}$. Define the incremental quotient

$$U_e = \frac{U(x + e, y) - U(x, y)}{|h|^\alpha}.$$

Then U_e is bounded in $\left(B_{1/2} \cup T_{1/2}\right) \times \{y\}$ independent of h. By linearity and translation invariance of the equation in x, we have that U_e is a solution to (6.1). So, Corollary 6.2.5 implies that U_e is C^α. Therefore, U is $C^{2\alpha}$ (see, for example, [8, Chapter 5]). If we iterate this argument, we can conclude that U is Lipschitz in this slice in x. Applying the incremental quotient argument once more, we obtain that U is $C^{1,\alpha}$ in the tangential directions in $\left(B_{1/2} \cup T_{1/2}\right) \times \{y\}$. Differentiating the solution in x and repeating this argument, we conclude (1).

Proof of (2). Recall that $y^a U_y|_{y=0^+}$ since U is harmonic-like. By using the equation, we see

$$\begin{aligned} y^a U_y(x,y) &= \int_0^y \partial_t \left(t^a U_t\left(x,t\right)\right)\,dt = -\int_0^y t^a \partial_{x_n} U(x,t)\,dt \\ &\leq \|\partial_{x_n} U\|_{L^\infty\left(B^+_{1/4}\times(0,1/2)\right)} \int_0^y t^a\,dt \leq C y^{a+1}. \end{aligned}$$

$\square$

CHAPTER 7. SCHAUDER ESTIMATES

We first prove that solutions to (4.9) can be well approximated by harmonic-like solutions described in the previous chapter. We then use Schauder estimates to transfer regularity to solutions of (4.9) up to the boundary. Once we have estimates up to the boundary, we can conclude that the solutions to the fractional Neumann problem, $U(x',0,0)=u(x')$, must satisfy these estimates.

7.1 Approximation by a harmonic-like solution

Define $L^2_a := L^2\left(B_1^+ \times (0,1), y^a dxdy\right)$. The next result is similar to the one in Nekvinda [31].

Lemma 7.1.1 (Trace inequality)**.** *Let $U \in L^2_a$ be a function such that ∇U belongs to L^2_a and $U_y \in L^2(T_1 \times (0,1), y^a dx'dy)$. There exists a constant C depending only on n and a such that*

$$\int_{T_1} |U|^2\, dx' \le C \int_0^1 y^a \left[\int_{B_1^+} \left(|\nabla U|^2 + U^2\right) dx + \int_{T_1} |U_y|^2\, dx' \right] dy.$$

This inequality rescales in the following way:

$$\int_{T_\lambda} |U|^2\, dx' \le \frac{C}{\lambda^{1+\frac{a+1}{2}}} \int_0^{\sqrt{\lambda}} y^a \left[\int_{B_\lambda^+} \left(\lambda^2|\nabla U|^2 + U^2\right) dx + \int_{T_\lambda} \lambda^2 U_y^2\, dx' \right] dy. \tag{7.1}$$

Proof. For simplicity, suppose that U is smooth enough. Let η be a smooth cutoff such that $\eta\big|_{y=1} = \eta\big|_{x_n=1} = 0$, and $\eta\big|_{T_1} = 1$. Let us first take $a=0$. We can estimate

$$\begin{aligned}
\int_{T_1} U(x',0,0)^2\, dx' &= \int_{T_1} \left(U(x',0,0)\eta(x',0,0)\right)^2 dx' \\
&= -\int_{T_1}\int_0^1 \partial_y \left[\left(U(x',0,y)\eta(x',0,y)\right)^2\right] dy\, dx' \\
&= -2\int_{T_1}\int_0^1 U U_y \eta^2\, dy\, dx' - 2\int_{T_1}\int_0^1 \eta\eta_y U^2\, dy\, dx' \\
&\le C\int_{T_1}\int_0^1 |U||U_y|\, dy\, dx' + C\int_{T_1}\int_0^1 U^2\, dy\, dx'
\end{aligned}$$

$$\le C\int_{T_1}\int_0^1 U(x',0,y)^2\,dy\,dx' + C\int_{T_1}\int_0^1 (U_y(x',0,y))^2\,dy\,dx'.$$

Next we estimate the first term in the last line. Choose another smooth cutoff ζ such that $\zeta\big|_{x_n=1}=0$, and $\zeta\big|_{T_1\times[0,1]}=1$. Then

$$\begin{aligned}\int_{T_1}\int_0^1 U(x',0,y)^2\,dy\,dx' &= -\int_{T_1}\int_0^1\int_0^1 \partial_{x_n}\left[(U(x',x_n,y)\zeta(x',x_n,y))^2\right]dx_n\,dy\,dx'\\ &= -2\int_{T_1}\int_0^1\int_0^1 U\partial_{x_n}U\zeta^2\,dx_n\,dy\,dx' - 2\int_{T_1}\int_0^1\int_0^1 \zeta\partial_{x_n}\zeta U^2\,dx_n\,dy\,dx'\\ &\le C\int_{T_1}\int_0^1\int_0^1 |U||\partial_{x_n}U|\,dx_n\,dy\,dx' + C\int_{T_1}\int_0^1\int_0^1 U^2\,dx_n\,dy\,dx'\\ &\le C\int_{T_1}\int_0^1\int_0^1 U^2\,dx_n\,dy\,dx' + C\int_{T_1}\int_0^1\int_0^1 |\nabla U|^2\,dx_n\,dy\,dx'.\end{aligned} \tag{7.2}$$

With this the estimate follows.

Let us do the general case $a\neq 0$. Let $u=u(y)$ be a smooth function of $y\ge 0$. We can write

$$u(y)-u(0)=\int_0^1 \partial_t u(ty)\,dt=\int_0^1 yu_y(ty)\,dt.$$

From here,

$$y^{a/2}|u(0)|\le y^{a/2}|u(y)|+\int_0^1 y^{a/2+1}|u_y(ty)|\,dt,$$

so that, after integration in y from 0 to 1, we get

$$|u(0)|\frac{1}{a/2+1}\le\int_0^1 y^{a/2}|u(y)|\,dy+\int_0^1\int_0^1 y^{a/2}|u_y(ty)|\,dt\,dy.$$

Notice that $a\in(-1,1)$, so $a/2+1>0$ and also $y^{a/2+1}\le y^{a/2}$ because $y\le 1$. We estimate now the two terms in the last inequality. On one hand, by Hölder's inequality,

$$\int_0^1 y^{a/2}|u(y)|\,dy\le\left(\int_0^1 y^a|u(y)|^2\,dy\right)^{1/2}.$$

On the other hand, by the change of variables $ty=r$ in the integral in y and letting $\alpha>0$,

$$\begin{aligned}\int_0^1\int_0^1 y^{a/2}|u_y(ty)|\,dy\,dt &= \int_0^1\int_0^t r^{a/2}|u_y(r)|\,dr\,t^{-a/2-1}\,dt\\ &= \int_0^1\left(\int_0^t r^{a/2}|u_y(r)|t^{-a/2-1+\alpha}\,dr\right)t^{-\alpha}\,dt\end{aligned}$$

$$
\begin{aligned}
&\le \left(\int_0^1 t^{-2\alpha}\,dt\right)^{1/2}\left(\int_0^1\left[\int_0^t r^{a/2}|u_y(r)|t^{-a/2-1+\alpha}\,dr\right]^2 dt\right)^{1/2}\\
&\le \frac{1}{(1-2\alpha)^{1/2}}\left[\int_0^1\left(\int_0^t r^a|u_y(r)|^2\,dr\right)\left(\int_0^t t^{-a-2+2\alpha}\,dr\right)dt\right]^{1/2}\\
&\le \frac{1}{(1-2\alpha)^{1/2}}\left(\int_0^1 r^a|u_y(r)|^2\,dr\right)^{1/2}\left(\int_0^1 t^{-a-1+2\alpha}\,dt\right)^{1/2}\\
&= \frac{1}{(1-2\alpha)^{1/2}(2\alpha-a)^{1/2}}\left(\int_0^1 r^a|u_y(r)|^2\,dr\right)^{1/2}.
\end{aligned}
$$

This last expression will be finite as soon as we choose $\alpha > 0$ such that $1-2\alpha > 0$ and $2\alpha - a > 0$, that is, $a < 2\alpha < 1$. This is always possible because $a < 1$. As a conclusion, there exists a constant C depending only on a such that

$$|u(0)|^2 \le C\int_0^1 y^a\big(u^2+u_y^2\big)\,dy.$$

Now let U be as in the statement. By using the inequality we just proved, for each fixed $x' \in T_1$ we have

$$|U(x',0,0)|^2 \le C\int_0^1 y^a\big(U(x',0,y)^2+U_y(x',0,y)^2\big)\,dy.$$

Integrating in x' over T_1 we get

$$\int_{T_1}|U(x',0,0)|^2\,dx' \le C\int_{T_1}\int_0^1 y^a\big(U(x',0,y)^2+U_y(x',0,y)^2\big)\,dy\,dx'.$$

By a computation completely parallel to (7.2) we end up with

$$\int_{T_1}|U|^2\,dx' \le C\int_{T_1}\int_0^1\int_0^1 y^a\big(U^2+|\nabla U|^2\big)\,dx_n\,dy\,dx' + C\int_{T_1}\int_0^1 y^a|U_y|^2\,dy\,dx',$$

as desired.

Let us verify the scaling. Let $U(x,y)$ be defined in $B_\lambda^+\times[0,\sqrt{\lambda})$. Take $\bar U(\bar x,\bar y) := U(\lambda\bar x,\sqrt{\lambda}\bar y)$, for $\bar x \in B_1^+$ and $\bar y \in [0,1)$. Then, by the trace inequality,

$$
\begin{aligned}
\int_{T_\lambda} U^2\,dx' &= \lambda^{n-1}\int_{T_1}|\bar U|^2\,d\bar x'\\
&\le C\lambda^{n-1}\int_0^1 \bar y^a\left[\int_{B_1^+}\big(|\nabla\bar U|^2+\bar U^2\big)\,d\bar x + \int_{T_1}\bar U_y^2\,d\bar x'\right]d\bar y\\
&= C\lambda^{n-1}\int_0^1 \bar y^a\left[\int_{B_1^+}\big(\lambda^2|(\nabla U)(\lambda\bar x,\sqrt{\lambda}\bar y)|^2 + U(\lambda\bar x,\sqrt{\lambda}\bar y)^2\big)\,d\bar x\right.
\end{aligned}
$$

$$
\begin{aligned}
&\qquad + \int_{T_1} \lambda U_y^2(\lambda \bar{x}, \sqrt{\lambda}\bar{y})\, d\bar{x}' \Big] \, d\bar{y} \\
&= C\lambda^{n-1} \int_0^{\sqrt{\lambda}} \frac{1}{\lambda^{\frac{a+1}{2}}}\, y^a \Big[\int_{B_\lambda^+} \lambda^{-n} \big(\lambda^2 |\nabla U(x,y)|^2 + U(x,y)^2\big)\, dx \\
&\qquad + \int_{T_\lambda} \lambda^{-(n-1)} \lambda U_y^2(x,y)\, dx' \Big] \, dy \\
&= \frac{C}{\lambda^{1+\frac{a+1}{2}}} \int_0^{\sqrt{\lambda}} y^a \Big[\int_{B_\lambda^+} \big(\lambda^2 |\nabla U|^2 + U^2\big)\, dx + \int_{T_\lambda} \lambda^2 U_y^2\, dx' \Big] \, dy
\end{aligned}
$$

□

We now prove an energy estimate, which, when paired with a compact embedding result, will be used to prove the approximation of harmonic-like solutions.

Lemma 7.1.2 (Caccioppoli estimate)**.** *If* $U \in \mathcal{H}_a^1$ *satisfies* (4.9) *then*

$$
\begin{aligned}
&\int_0^1 y^a \Big[\int_{B_1^+} |\nabla U| \eta^2\, dx + \int_{T_1} U_y^2 \eta^2\, dx' \Big] \, dy \\
&\qquad \le C \int_0^1 y^a \Big[\int_{B_1^+} |\nabla \eta|^2 U^2\, dx + \int_{T_1} \eta_y^2 U^2\, dx' \Big] \, dy + \int_{T_1} \eta(x',0)^2 |U(x',0)||f(x')|\, dx'.
\end{aligned}
$$

In particular,

$$
\int_0^{1/2} y^a \Big[\int_{B_{1/4}^+} |\nabla U|^2\, dx + \int_{T_{1/4}} U_y^2\, dx' \Big] \, dy \le C \int_0^1 y^a \Big[\int_{B_1^+} U^2\, dx + \int_{T_1} U^2\, dx' \Big] \, dy + \int_{T_1} |U||f|\, dx'.
$$

This inequality scales as follows:

$$
\begin{aligned}
&\frac{\lambda}{\lambda^{\frac{a+1}{2}}} \int_0^{\sqrt{\lambda/4}} y^a \Big[\int_{B_{\lambda/4}^+} |\nabla U|^2\, dx + \int_{T_{\lambda/4}} U_y^2\, dx' \Big] \, dy \\
&\qquad \le \frac{C}{\lambda^{\frac{a+1}{2}}} \int_0^{\sqrt{\lambda}} y^a \Big[\int_{B_\lambda^+} \frac{U^2}{\lambda}\, dx + \int_{T_\lambda} U^2\, dx' \Big] \, dy + \int_{T_\lambda} |U||f|\, dx'. \quad (7.3)
\end{aligned}
$$

Proof. Take $\Psi = U\eta^2$ as test function in (4.9). Then

$$
\begin{aligned}
&\int_0^1 y^a \Big[\int_{B_1^+} A(x) \nabla U \nabla U \eta^2\, dx + \int_{T_1} U_y^2 \eta^2\, dx' \Big] \, dy \\
&\qquad = -\int_0^1 \Big[\int_{B_1^+} A(x) \nabla U 2\eta \nabla \eta U\, dx + \int_{T_1} U_y U 2\eta \eta_y\, dx' \Big] \, dy + \int_{T_1} \eta^2 U f\, dx'.
\end{aligned}
$$

The inequality follows by using the ellipticity and boundedness of A together with Cauchy's inequality with ε as usual. The second estimate follows by choosing η such that $\eta = 1$ in $(B^+_{1/4} \cup T_{1/4}) \times [0, 1/2)$, $0 \le \eta \le 1$ and supported in $B^+_1 \times [0, 1)$.

Let us verify the scaling. Let $U(x, y)$ be defined in $B^+_{\lambda/4} \times [0, \sqrt{\lambda/4})$, and let $f(x')$ be defined in $T_{\lambda/4}$. Consider $\bar{U}(\bar{x}, \bar{y}) = U(\lambda\bar{x}, \sqrt{\lambda}\bar{y})$, for $\bar{x} \in B^+_{1/4}$ and $0 \le \bar{y} < 1/2$, and $\bar{f}(\bar{x}') = f(\lambda\bar{x}')$, for $\bar{x}' \in T_{1/4}$. Then

$$
\begin{aligned}
&\int_0^{\sqrt{\lambda/4}} y^a \left[\int_{B^+_{\lambda/4}} |\nabla U|^2 \, dx + \int_{T_{\lambda/4}} U_y^2 \, dx' \right] dy \\
&\qquad = \lambda^{\frac{a+1}{2}} \int_0^{1/2} \bar{y}^a \left[\int_{B^+_{1/4}} \lambda^n \frac{|\nabla \bar{U}|^2}{\lambda^2} \, d\bar{x} + \int_{T_{1/4}} \lambda^{n-1} \frac{\bar{U}_y^2}{\lambda} \, d\bar{x}' \right] d\bar{y} \\
&\qquad = \lambda^{\frac{a+1}{2}+n-2} \int_0^{1/2} \bar{y}^a \left[\int_{B^+_{1/4}} |\nabla \bar{U}|^2 \, d\bar{x} + \int_{T_{1/4}} \bar{U}_y^2 \, d\bar{x}' \right] d\bar{y} \\
&\qquad \le \lambda^{\frac{a+1}{2}+n-2} \left\{ C \int_0^1 \bar{y}^a \left[\int_{B^+_1} \bar{U}^2 \, d\bar{x} + \int_{T_1} \bar{U}^2 \, d\bar{x}' \right] d\bar{y} + \int_{T_1} |\bar{U}||\bar{f}| \, d\bar{x}' \right\} \\
&\qquad = \lambda^{\frac{a+1}{2}+n-2} \left\{ C \int_0^{\sqrt{\lambda}} \lambda^{-\frac{a+1}{2}} y^a \left[\int_{B^+_\lambda} \lambda^{-n} U^2 \, dx + \int_{T_\lambda} \lambda^{-(n-1)} U^2 \, dx' \right] dy \right. \\
&\qquad\qquad \left. + \int_{T_\lambda} \lambda^{-(n-1)} |U||f| \, dx' \right\} \\
&\qquad = \lambda^{\frac{a+1}{2}-1} \left\{ C\lambda^{-\frac{a+1}{2}} \int_0^{\sqrt{\lambda}} y^a \left[\int_{B^+_\lambda} \frac{U^2}{\lambda} \, dx + \int_{T_\lambda} U^2 \, dx' \right] dy + \int_{T_\lambda} |U||f| \, dx' \right\}
\end{aligned}
$$

□

Corollary 7.1.3 (Approximation by harmonic functions)**.** *Fix any $\varepsilon_0 > 0$. There exists $\delta_0 > 0$ such that for any solution $U \in \mathcal{H}^1_a$ to (4.9) with*

$$
\int_0^1 y^a \left[\int_{B^+_1} U^2 \, dx + \int_{T_1} U^2 \, dx' \right] dy + \int_{T_1} U^2 \, dx' \le 1,
$$

the following assertion holds. If

$$
\int_{T_1} f^2 \, dx' + \int_{B^+_1} |A(x) - I|^2 \, dx < \delta_0^2,
$$

then there exists a harmonic function W, that is, a weak solution W to

$$\begin{cases} \Delta_x W(x,y) = 0, & \text{in } B^+_{1/2}, \text{ for every } 0 \le y < 1/2, \\ -y^a \partial_{x_n} W = \partial_y (y^a W_y), & \text{on } T_{1/4}, \text{ for } 0 \le y < 1/2, \\ -y^a W_y\big|_{y=0} = 0, & \text{for } x' \in T_{1/4}, \end{cases}$$

or, which is the same, a function W that satisfies

$$\int_0^{1/2} y^a \left[\int_{B^+_{1/4}} \nabla_x W \nabla_x \Psi \, dx + \int_{T_{1/4}} W_y \Psi_y \, dx' \right] dy = 0,$$

for every test function Ψ, such that

$$\int_0^{1/2} y^a \left[\int_{B^+_{1/4}} |U - W|^2 \, dx + \int_{T_{1/4}} |U - W|^2 \, dx' \right] dy < \varepsilon_0^2.$$

Proof. By contradiction. Suppose that there exist $\varepsilon_0 > 0$, solutions U_k, right hand sides f_k and coefficients $A_k(x)$ such that

$$\int_0^1 y^a \left[\int_{B^+_1} U_k^2 \, dx + \int_{T_1} U_k^2 \, dx' \right] dy + \int_{T_1} U_k^2 \, dx' \le 1,$$

and

$$\int_{T_1} f_k^2 \, dx' + \int_{B^+_1} |A_k(x) - I|^2 \, dx < \frac{1}{k^2},$$

but

$$\int_0^{1/2} y^a \left[\int_{B^+_{1/4}} |U_k - W|^2 \, dx + \int_{T_{1/4}} |U_k - W|^2 \, dx' \right] dy \ge \varepsilon_0^2,$$

for any $k \ge 1$ and for any harmonic function W. By the Caccioppoli estimate in Lemma 7.1.2,

$$\int_0^{1/2} y^a \left[\int_{B^+_{1/4}} |\nabla U_k|^2 \, dx + \int_{T_{1/4}} (U_k)_y^2 \, dx' \right] dy \le C,$$

that is, U_k is a bounded sequence in the Sobolev space H^1_a. By the compact embedding $\mathcal{H}^1_a \subset L^2_a$ (see Lemma 6.2.1) , up to a subsequence, we have

$$\begin{cases} U_k \to U_\infty, & \text{strongly in } L^2_a, \\ U_k \to U_\infty, & \text{weakly in } \mathcal{H}^1_a. \end{cases}$$

We show that U_∞ is a harmonic function, which will give us a contradiction. Indeed, for any test function ψ,

$$\int_0^{1/2} y^a \left[\int_{B^+_{1/4}} A_k(x) \nabla_x U_k \nabla_x \psi \, dx + \int_{T_{1/4}} (U_k)_y \psi_y \, dx' \right] dy = \int_{T_{1/4}} \psi(x',0) f_k(x') \, dx'.$$

By taking the limit $k \to \infty$ along the subsequence above, we get

$$\int_0^{1/2} y^a \left[\int_{B^+_{1/4}} \nabla_x U_\infty \nabla_x \psi \, dx + \int_{T_{1/4}} (U_\infty)_y \psi_y \, dx' \right] dy = 0.$$

$\square$

Note that the harmonic-like approximation is equivalent to the harmonic-like problem obtained by even reflection across $y = 0$:

$$\begin{cases} \Delta_x W = 0 & \text{in } B_1^+ \times (-1,1) \\ -|y|^a \, \partial_{x_n} W = \partial_y \left(|y|^a \, W_y \right) & \text{on } T_1 \times (-1,1) \end{cases} \tag{7.4}$$

Since the reflection is even, we get the condition $-y^a W_y|_{y=0} = 0$ on T_1 when restricting to $T_1 \times (0,1)$. Therefore, all of the localized results of the previous chapter can be applied to solutions in the half cylinder $(0,1) \times (B_1^+ \cup T_1)$.

7.2 Schauder Estimates

Lemma 7.2.1. *Let $0 < \alpha < 1$ and $0 < \sigma < 1$ be such that $0 < \alpha + \sigma < 1$. There exist universal constants $C_0 > 0$, $0 < \lambda < 1$ and $\delta_0 > 0$ such that for every f and every solution U to*

$$\int_0^1 y^a \left[\int_{B_1^+} A(x) \nabla U \nabla \Psi \, dx + \int_{T_1} U_y \Psi_y \, dx' \right] dy = \int_{T_1} \Psi f \, dx',$$

such that

$$\int_0^1 y^a \left[\int_{B_1^+} U^2 \, dx + \int_{T_1} U^2 \, dx' \right] dy + \int_{T_1} U^2 \, dx' \le 1,$$

if

$$\int_{T_1} f^2 \, dx' + \int_{B_1^+} |A(x) - I|^2 \, dx < \delta_0^2,$$

then there exists a constant $\mathcal{A}$ with $|\mathcal{A}| \leq C_0$ and such that

$$\frac{1}{\lambda^{n-1}} \int_{T_\lambda} |U - \mathcal{A}|^2 \, dx' < \lambda^{2(\alpha+\sigma)},$$

and

$$\frac{1}{\lambda^{\frac{a+1}{2}}} \int_0^{\sqrt{\lambda}} y^a \left[\frac{1}{\lambda^n} \int_{B_\lambda^+} |U - \mathcal{A}|^2 \, dx \, dy + \frac{1}{\lambda^{n-1}} \int_{T_\lambda} |U - \mathcal{A}|^2 \, dx' \right] dy < \lambda^{2(\alpha+\sigma)}.$$

Proof. Let $0 < \varepsilon_0 < 1$ and let W be the harmonic approximation given by Corollary 7.1.3.

Step 1: Estimates on W. Notice first that, by adding and subtracting U,

$$\begin{aligned} &\int_0^{1/2} y^a \left[\int_{B_{1/4}^+} |W|^2 \, dx + \int_{T_{1/4}} |W|^2 \, dx' \right] dy \\ &\qquad\qquad \leq 2\varepsilon_0^2 + 2 \int_0^{1/2} y^a \left[\int_{B_{1/4}^+} U^2 \, dx + \int_{T_{1/4}} U^2 \, dx' \right] dy \leq 4. \end{aligned}$$

Let us define $\mathcal{A} := W(0,0)$. Then, by using this estimate and Lemma 6.2.6, $|\mathcal{A}| \leq C_0$ and

$$\begin{aligned} |W - \mathcal{A}| = |W(x,y) - W(0,0)| &\leq |W(x,y) - W(x,0)| + |W(x,0) - W(0,0)| \\ &= |W_y(x,\xi)| y + |\nabla W(\eta, 0)||x| \\ &\leq C_0(y^2 + |x|), \end{aligned}$$

where we used the mean value theorem for some $0 \leq \xi \leq y$ and some η in the segment that joins x with 0.

Step 2: Estimate of $U - \mathcal{A}$ on T_λ. Let $0 < \lambda < 1/4$. By the rescaled trace inequality (7.1) and the rescaled Caccioppoli inequality (7.3) (observe that if U satisfies (4.9) then $U - \mathcal{A}$ also does),

$$\begin{aligned} \int_{T_{\lambda/4}} |U - \mathcal{A}|^2 \, dx' &\leq \frac{C}{\lambda^{1+\frac{a+1}{2}}} \int_0^{\sqrt{\lambda/4}} y^a \left[\int_{B_{\lambda/4}^+} \left(\lambda^2 |\nabla U|^2 + |U - \mathcal{A}|^2 \right) dx + \int_{T_{\lambda/4}} \lambda^2 U_y^2 \, dx' \right] dy \\ &= C \frac{\lambda}{\lambda^{\frac{a+1}{2}}} \int_0^{\sqrt{\lambda/4}} y^a \left[\int_{B_{\lambda/4}^+} |\nabla U|^2 \, dx + \int_{T_{\lambda/4}} U_y^2 \, dx' \right] dy \\ &\quad + \frac{C}{\lambda^{\frac{a+1}{2}}} \int_0^{\sqrt{\lambda/4}} y^a \int_{B_{\lambda/4}^+} \frac{|U - \mathcal{A}|^2}{\lambda} \, dx \, dy \\ &\leq \frac{C}{\lambda^{\frac{a+1}{2}}} \int_0^{\sqrt{\lambda}} y^a \left[\int_{B_\lambda^+} \frac{|U - \mathcal{A}|^2}{\lambda} \, dx + \int_{T_\lambda} |U - \mathcal{A}|^2 \, dx' \right] dy + \int_{T_\lambda} |U - \mathcal{A}||f| \, dx' \end{aligned}$$

$$\leq \frac{C}{\lambda^{\frac{a+1}{2}}} \int_0^{\sqrt{\lambda}} y^a \left[\int_{B_\lambda^+} \frac{|U-W|^2}{\lambda} \, dx + \int_{T_\lambda} |U-W|^2 \, dx' \right] dy + \|U-\mathcal{A}\|_{L^2(T_1)} \|f\|_{L^2(T_1)}$$
$$+ \frac{C}{\lambda^{\frac{a+1}{2}}} \int_0^{\sqrt{\lambda}} y^a \left[\int_{B_\lambda^+} \frac{|W-\mathcal{A}|^2}{\lambda} \, dx + \int_{T_\lambda} |W-\mathcal{A}|^2 \, dx' \right] dy$$
$$\leq \frac{C\varepsilon_0^2}{\lambda^{1+\frac{a+1}{2}}} + C\delta_0 + \frac{C}{\lambda^{\frac{a+1}{2}}} \int_0^{\sqrt{\lambda}} y^a \left[\int_{B_\lambda^+} \frac{(y^2+|x|)^2}{\lambda} \, dx + \int_{T_\lambda} (y^2+|x|)^2 \, dx' \right] dy$$
$$\leq \frac{C}{\lambda^{1+\frac{a+1}{2}}} \varepsilon_0^2 + C\delta_0 + C\lambda^{n-1}\lambda^2.$$

Thus, after relabeling $\lambda/4$ by λ,

$$\frac{1}{\lambda^{n-1}} \int_{T_\lambda} |U-\mathcal{A}|^2 \, dx' \leq C\left(\frac{\varepsilon_0^2}{\lambda^{n+\frac{a+1}{2}}} + \frac{\delta_0}{\lambda^{n-1}} + \lambda^2 \right),$$

where $C>0$ depends on n, a and the ellipticity constants of $A(x)$, but not on U, W or f. Observe that on the way (see the fourth line in the chain of inequalities above) we have also estimated

$$\frac{1}{\lambda^{\frac{a+1}{2}}} \int_0^{\sqrt{\lambda}} y^a \left[\frac{1}{\lambda^n} \int_{B_\lambda^+} |U-\mathcal{A}|^2 \, dx + \frac{1}{\lambda^{n-1}} \int_{T_\lambda} |U-\mathcal{A}|^2 \, dx' \right] dy \leq C\left(\frac{\varepsilon_0^2}{\lambda^{n+\frac{a+1}{2}}} + \frac{\delta_0}{\lambda^{n-1}} + \lambda^2 \right).$$

Step 3: Conclusion. We first choose $0<\lambda<1/4$ such that $C\lambda^2 < \frac{1}{3}\lambda^{2(\alpha+\sigma)}$ (remember we are in the case $\alpha+\sigma<1$). Then we fix $\varepsilon_0>0$ small enough such that $\frac{C\varepsilon_0^2}{\lambda^{n+\frac{a+1}{2}}} < \frac{1}{3}\lambda^{2(\alpha+\sigma)}$. With this choice of ε_0 we then choose δ_0 in the approximation lemma small enough so that $\frac{C\delta_0}{\lambda^{n-1}} < \frac{1}{3}\lambda^{2(\alpha+\sigma)}$. Hence,

$$\frac{1}{\lambda^{n-1}} \int_{T_\lambda} |U-\mathcal{A}|^2 \, dx' < \lambda^{2(\alpha+\sigma)},$$

and

$$\frac{1}{\lambda^{\frac{a+1}{2}}} \int_0^{\sqrt{\lambda}} y^a \left[\frac{1}{\lambda^n} \int_{B_\lambda^+} |U-\mathcal{A}|^2 \, dx + \frac{1}{\lambda^{n-1}} \int_{T_\lambda} |U-\mathcal{A}|^2 \, dx' \right] dy < \lambda^{2(\alpha+\sigma)}.$$

□

Theorem 7.2.2. *For each $0<\alpha<1$ and $0<\sigma<1$ such that $0<\alpha+\sigma<1$, there exists a universal constant $C_0>0$ and a small $0<\delta_0<1$ such that if*

$$[f]^2_{L^{2,\alpha}(0)} := \sup_{0<r\leq 1} \frac{1}{r^{n-1+2\alpha}} \int_{T_r} |f-f(0)|^2 \, dx' < \infty,$$

and

$$\sup_{0<r\leq 1} \frac{1}{r^n} \int_{B_r^+} |A(x)-A(0)|^2 \, dx < \delta_0^2,$$

then there exists a constant $\mathcal{A}_\infty \in \mathbb{R}$ *such that*

$$\frac{1}{r^{n-1}}\int_{T_r} |U(x',0,0)-\mathcal{A}_\infty|^2\,dx' \le C_1 r^{2(\alpha+\sigma)},$$

and

$$\frac{1}{r^{\frac{a+1}{2}}}\int_0^{\sqrt{r}} y^a\left[\frac{1}{r^n}\int_{B_r^+}|U-\mathcal{A}_\infty|^2\,dx+\frac{1}{r^{n-1}}\int_{T_r}|U-\mathcal{A}_\infty|^2\,dx'\right]dy < C_1 r^{2(\alpha+\sigma)},$$

for all $r>0$ *sufficiently small, where* $C_1>0$ *and*

$$C_1+|\mathcal{A}_\infty|\le C_0\big([f]_{L^{2,\alpha}(0)}+|f(0)|+\|U\|_{L^2_a}\big).$$

Proof. Recall that $A(0)=I$. By dividing by the appropriate norms we can assume that

$$\int_0^1 y^a\left[\int_{B_1^+}U^2\,dx+\int_{T_1}U^2\,dx'\right]dy+\int_{T_1}U^2\,dx'\le 1, \tag{7.5}$$

and

$$[f]^2_{L^{2,\alpha}(0)}<\delta_0^2,$$

as it is standard. Also, we can suppose that $f(0)=0$. Indeed, consider $\widetilde{U}=U-\frac{1}{1-a}y^{1-a}f(0)$ and observe that $\partial_y(y^a\partial_y(y^{1-a}))=0$ and $y^a\partial_y\big(\frac{1}{1-a}y^{1-a}f(0)\big)=f(0)$.

The proof is done by induction. We show that for every $k\ge 0$ there exists $\mathcal{A}_k$ such that

$$\frac{1}{(\lambda^k)^{n-1}}\int_{T_{\lambda^k}}|U(x',0,0)-\mathcal{A}_k|^2\,dx'<(\lambda^k)^{2(\alpha+\sigma)}, \tag{7.6}$$

and

$$\frac{1}{(\lambda^k)^{\frac{a+1}{2}}}\int_0^{\lambda^{k/2}} y^a\left[\frac{1}{(\lambda^k)^n}\int_{B^+_{\lambda^k}}|U-\mathcal{A}_k|^2\,dx+\frac{1}{(\lambda^k)^{n-1}}\int_{T_{\lambda^k}}|U-\mathcal{A}_k|^2\,dx'\right]dy<(\lambda^k)^{2(\alpha+\sigma)}, \tag{7.7}$$

where $0<\lambda<1$ is as in Lemma 7.2.1, and $|\mathcal{A}_k-\mathcal{A}_{k+1}|\le C_0\lambda^{k(\alpha+\sigma)}$. This implies the conclusion as usual, see [7].

The case $k=0$ is just the normalization we assumed in (7.5) with $\mathcal{A}_0=0$. For $k=1$, this is the statement of Lemma 7.2.1 with $\mathcal{A}_1=\mathcal{A}$, and $|\mathcal{A}|\le C_0$ from that result. Let us suppose that (7.6) holds for some $k\ge 1$. To prove the estimate for $k+1$ let us look at the rescaled function

$$\bar{U}(\bar{x},\bar{y}):=\frac{U(\lambda^k\bar{x},\lambda^{k/2}\bar{y})-\mathcal{A}_k}{\lambda^{k(\alpha+\sigma)}},$$

which is well defined for $(\bar{x}, \bar{y}) \in B_1^+ \times [0,1)$. Notice that from (4.9) we already have that

$$\int_0^{\lambda^{k/2}} y^a \left[\int_{B^+_{\lambda^k}} A(x)\nabla_x U \nabla_x \Psi \, dx + \int_{T_{\lambda^k}} U_y \Psi_y \, dx' \right] dy = \int_{T_{\lambda^k}} \Psi(x',0) f(x') \, dx',$$

for any test function Ψ defined in $B^+_{\lambda^k} \times [0, \lambda^{k/2})$. By making the change of variables $x = \lambda^k \bar{x}$, $y = \lambda^{k/2}\bar{y}$, with $(\bar{x}, \bar{y}) \in B_1^+ \times \in [0,1)$, and by calling $\bar{A}(\bar{x}) = A(\lambda^k \bar{x})$ and $\bar{\Psi}(\bar{x}, \bar{y}) = \Psi(\lambda^k \bar{x}, \lambda^{k/2}\bar{y})$ we get

$$\begin{aligned} \int_0^1 \lambda^{k(\frac{a+1}{2})} \bar{y}^a \Bigg[\int_{B_1^+} & \bar{A}(\bar{x}) \lambda^{k(\alpha+\sigma)-k} \nabla_{\bar{x}} \bar{U} \lambda^{-k} \nabla_{\bar{x}} \bar{\Psi} \lambda^{kn} \, d\bar{x} + \\ & \int_{T_1} \lambda^{k(\alpha+\sigma)} \lambda^{-k/2} \bar{U}_{\bar{y}} \lambda^{-k/2} \bar{\Psi}_{\bar{y}} \lambda^{k(n-1)} \, d\bar{x}' \Bigg] d\bar{y} = \int_{T_1} \bar{\Psi}(\bar{x}',0) f(\lambda^k \bar{x}') \lambda^{k(n-1)} \, d\bar{x}'. \end{aligned}$$

If we call $\bar{f}(\bar{x}') := \lambda^{-k\alpha} f(\lambda^k \bar{x}')$ then the identity above reads

$$\int_0^1 \bar{y}^a \left[\int_{B_1^+} \bar{A}(\bar{x}) \nabla_{\bar{x}} \bar{U} \nabla_{\bar{x}} \bar{\Psi} \, d\bar{x} + \int_{T_1} \bar{U}_{\bar{y}} \bar{\Psi}_{\bar{y}} \, d\bar{x}' \right] d\bar{y} = \int_{T_1} \bar{\Psi}(\bar{x}',0) \bar{f}(\bar{x}') \, d\bar{x}'.$$

Note that $\bar{A}(0) = I$ and $\bar{f}(0) = 0$. Now,

$$\begin{aligned} \int_{B_1^+} |\bar{A}(\bar{x}) - I|^2 \, d\bar{x} &= \frac{1}{(\lambda^k)^n} \int_{B^+_{\lambda^k}} |A(x) - I|^2 \, dx < \delta_0^2; \\ \int_{T_1} |\bar{f}(\bar{x}')|^2 \, d\bar{x}' &= \frac{1}{(\lambda^k)^{n-1+2\alpha}} \int_{T_{\lambda^k}} |f(x')|^2 \, dx' < \delta_0^2. \end{aligned}$$

Also, by using the induction hypothesis (that is, (7.6) and (7.7)),

$$\begin{aligned} &\int_0^1 \bar{y}^a \left[\int_{B_1^+} \bar{U}^2 \, d\bar{x} + \int_{T_1} \bar{U}^2 \, d\bar{x}' \right] d\bar{y} + \int_{T_1} \bar{U}^2 \, d\bar{x}' \\ &\quad = \frac{1}{(\lambda^k)^{(\frac{a+1}{2})}} \int_0^{\lambda^{k/2}} y^a \left[\int_{B^+_{\lambda^k}} \frac{|U - \mathcal{A}_k|^2}{(\lambda^k)^{2(\alpha+\sigma)}} \frac{dx}{(\lambda^k)^n} + \int_{T_1} \frac{|U - \mathcal{A}_k|^2}{(\lambda^k)^{2(\alpha+\sigma)}} \frac{dx'}{(\lambda^k)^{(n-1)}} \right] dy \\ &\qquad + \int_{T_{\lambda^k}} \frac{|U - \mathcal{A}_k|^2}{(\lambda^k)^{2(\alpha+\sigma)}} \frac{dx'}{(\lambda^k)^{(n-1)}} \leq 1. \end{aligned}$$

Hence we can apply Lemma 7.2.1 to $\bar{U}$ to find a number $\mathcal{A}$ such that

$$\frac{1}{\lambda^{n-1}} \int_{T_\lambda} |\bar{U}(\bar{x}',0,0) - \mathcal{A}|^2 \, d\bar{x}' < \lambda^{2(\alpha+\sigma)},$$

and

$$\frac{1}{\lambda^{\frac{a+1}{2}}} \int_0^{\sqrt{\lambda}} \bar{y}^a \left[\frac{1}{\lambda^n} \int_{B_\lambda^+} |\bar{U} - \mathcal{A}|^2 \, d\bar{x} + \frac{1}{\lambda^{n-1}} \int_{T_\lambda} |U - \mathcal{A}|^2 \, d\bar{x}' \right] d\bar{y} < \lambda^{2(\alpha+\sigma)}.$$

By changing variables back, we obtain

$$\frac{1}{(\lambda^{k+1})^{n-1}} \int_{T_{\lambda^{k+1}}} |U(x',0,0) - \mathcal{A}_{k+1}|^2 \, dx' < (\lambda^{k+1})^{2(\alpha+\sigma)},$$

and

$$\frac{1}{(\lambda^{k+1})^{\frac{a+1}{2}}} \int_0^{\lambda^{(k+1)/2}} y^a \Bigg[\frac{1}{(\lambda^{k+1})^n} \int_{B^+_{\lambda^{k+1}}} |U - \mathcal{A}_{k+1}|^2 \, dx + \frac{1}{(\lambda^{k+1})^{n-1}} \int_{T_{\lambda^{k+1}}} |U - \mathcal{A}_{k+1}|^2 \, dx' \Bigg] \, dy < (\lambda^{k+1})^{2(\alpha+\sigma)},$$

where $\mathcal{A}_{k+1} := \mathcal{A}_k + \lambda^{k(\alpha+\sigma)} \mathcal{A}$. It is clear that $|\mathcal{A}_k - \mathcal{A}_{k+1}| < C_0 \lambda^{k(\alpha+\sigma)}$. This completes the induction step and the proof of the theorem. □

By following exactly the same steps in the previous proof and changing the exponent α by $-\sigma + \alpha$ we obtain the following result.

Theorem 7.2.3. *For each $0 < \alpha < 1$ and $0 < \sigma < 1$ there exists a universal constant $C_0 > 0$ and a small $0 < \delta_0 < 1$ such that if*

$$[f]^2_{L^{2,-\sigma+\alpha}(0)} := \sup_{0<r\le 1} \frac{1}{r^{n-1+2(-\sigma+\alpha)}} \int_{T_r} |f(x')|^2 \, dx' < \infty,$$

and

$$\sup_{0<r\le 1} \frac{1}{r^n} \int_{B_r^+} |A(x) - A(0)|^2 \, dx < \delta_0^2,$$

then there exists a constant $\mathcal{A}_\infty \in \mathbb{R}$ such that

$$\frac{1}{r^{n-1}} \int_{T_r} |U(x',0,0) - \mathcal{A}_\infty|^2 \, dx' \le C_1 r^{2\alpha},$$

and

$$\frac{1}{r^{\frac{a+1}{2}}} \int_0^{\sqrt{r}} y^a \left[\frac{1}{r^n} \int_{B_r^+} |U - \mathcal{A}_\infty|^2 \, dx + \frac{1}{r^{n-1}} \int_{T_r} |U - \mathcal{A}_\infty|^2 \, dx' \right] dy < C_1 r^{2\alpha},$$

for all $r > 0$ sufficiently small, where $C_1 > 0$ and

$$C_1 + |\mathcal{A}_\infty| \le C_0 \big([f]_{L^{2,-\sigma+\alpha}(0)} + \|U\|_{L^2_a} \big).$$

We are now in a position to prove regularity for solutions to the fractional Neumann problem.

Theorem 7.2.4. *Let $0 < \sigma < 1$ and $f \in H_0^{1/2-\sigma}(\partial\Omega)$. Suppose that $u \in H^1(\Omega)$ is a weak solution to (2.1). Let $0 < \alpha < 1$ and $1 < p < \infty$.*

(1) If f is in $L^p(\partial\Omega)$ for some $p > (n-1)/\sigma$, and $\partial\Omega$ is C^1, then u is in $C^{0,\beta}(\overline{\Omega})$ where $\beta \in (0,1)$ is defined as $\beta := \sigma - (n-1)/p$, with

$$[u]_{C^\beta(\overline{\Omega})} \leq C\big(\|u\|_{H^1(\Omega)} + \|f\|_{L^p(\partial\Omega)}\big).$$

(2) Let $f \in C^\alpha(\partial\Omega)$ for $\alpha + \sigma < 1$, and $\partial\Omega \in C^1$. Then $u \in C^{0,\alpha+\sigma}(\overline{\Omega})$ and

$$[u]_{C^{\alpha+2\sigma}(\overline{\Omega})} \leq C\big(\|u\|_{H^1(\Omega)} + \|f\|_{C^\alpha(\partial\Omega)}\big)$$

The constants $C > 0$ above depend only on $n, \sigma, p, \alpha, \Omega$.

Proof. Recall that we can use the extension problem and flatten near the boundary. Since the coefficients $A(x)$ are continuous functions, after a stretching of the variables we can always assume that

$$\sup_{0<r\leq 1} \frac{1}{r^n} \int_{B_r^+} |A(x) - A(0)|^2 \, dx < \delta_0^2.$$

By Hölder's inequality, if f is in $L^p(T_1)$ for $p > (n-1)/\sigma$ then

$$[f]_{L^{2,-\sigma+\alpha}(0)} \leq \|f\|_{L^p(T_1)},$$

for $\alpha := \sigma - (n-1)/p$. As a consequence of Theorem 7.2.3, if f is in $L^p(\partial\Omega)$ for $p > (n-1)/\sigma$ then $U(x', 0, 0)$ is in $C^{\sigma-(n-1)/p}$, which is exactly part (1). Part (2) follows from Theorem 7.2.2, which says that if $f \in C^\alpha$ then $U(x', 0, 0) \in C^{\alpha+\sigma}$. $\square$

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