# Positive curvature and fundamental group

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Iowa State University

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**Theorem (Rong, 1999):**
$S^1 \subseteq \mathrm{Isom}(M^n, g) \Longrightarrow \pi_1(M)$ has a cyclic subgroup of index $\leq w(n)$.

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**Corollary (K.):** If $T^3 \subseteq \mathrm{Isom}(M, g)$ and $H^*(\widetilde{M}; \mathbb{Q}) \cong H^*(B^{13}_{(q_1,\ldots,q_5)}; \mathbb{Q})$, then $\pi_1(M)$ has a cyclic subgroup of index at most 3.

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**Lemma (K.):** $\mathbb{Z}_p^2$ cannot act freely on a $\mathbb{Z}_p$-cohomolgy $\mathbb{S}^2 \times \mathbb{S}^3$.

Proof: Refine Heller's argument by computing the differentials.