Cascade Feedback Linearization

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Midwest Geometry Conference, Sunday September 8th, 2019

What to cover

Control Systems

Definition

A **Control System** is an underdetermined ODE system on a manifold M^{n+m+1} such that in local coordiantes we have

$$\frac{dx^i}{dt} = f^i(t, x, u)$$

where the x^i's $1 \leq i \leq n$ are the state variables and the u^j's $1 \leq j \leq m$ are the control parameters and t is the independent variable.

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Definition (As an EDS)

Let M be a manifold of dimension $n + m + 1$ and and ω an Exterior Differential System (EDS) such that in local coordinates

$$\omega = \langle dx^i - f^i(x, u)\, dt \rangle$$

and solutions are integral manifolds to ω (i.e. $f : N \to M$ s.t. $f^*\omega = 0$)

Extended Static Feedback Equivalence

Definition

Let (M^{n+m+1}, ω) and $(\tilde{M}^{n+m+1}, \tilde{\omega})$ be two EDS for control systems. We say the two systems are **Extended Static Feedback Equivalent** (ESFE) if there exists a diffeomorphism $\varphi : \tilde{M} \to M$ of the form

$$\varphi : (t, \tilde{x}, \tilde{u}) \mapsto (t, x(t, \tilde{x}), u(t, \tilde{x}, \tilde{u}))$$

such that $\varphi^* \omega = \tilde{\omega}$

Note: We say **Static Feedback Equivalent** (SFE) if $x = x(\tilde{x})$ and $u = u(\tilde{x}, \tilde{u})$

Extended Static Feedback Linearization

Definition (Brunovsky Normal Form)

Let $J^\kappa = J^\kappa(\mathbb{R}, \mathbb{R}^m)$ be a partial prolongation of a jet space (i.e. nonuniform order across derivatives of components of maps from $\mathbb{R}$ to $\mathbb{R}^m$). Then the EDS

$$\beta^\kappa = \langle \theta^i_{i_l} = dz^i_{l_i} - z^i_{l_i+1} dt \rangle \text{ where } 1 \leq i \leq m \text{ and } 0 \leq l_i \leq \rho_i$$

is the **Brunovsky Normal Form**. This is what is refered to as "linear" in the sense of control theory.

Definition (ESFL)

We'll say that a control system (M, ω) is **Extended Static Feedback Linearizable** (ESFL) if

$$(M, \omega) \cong_{ESF} (J^\kappa, \beta^\kappa)$$

Dynamic Feedback Linearization

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- Very optimistic hope: given $\omega = \langle dx^i - f^i(t, x, u)\, dt\rangle$ do there exist coordinates and functions on a larger manifold $\widehat{M}$ such that the system

$$\hat{\omega} = \langle dx^i - f^i(t, x, u(x, z, v))\, dt, dz^j - g^j(t, x, z, v)\, dt\rangle$$

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The former is a special case of the latter and we'll call either situation from above a **Dynamic Feedback Linearization**.

Cascade Linearization Outline

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- Find a symmetry group G of (M, ω) such that $(M/G, \omega_G)$ is SFL
- Use Anderson and Fels reconstruction theorem to put ω into a form of "Equation of Lie Type+Partial Contact forms" called γ^G
- Restrict γ^G to a submanifold determined by choosing a jet fiber to be a fixed (but arbitrary!) function of the independent variable

Step 1: SFL Quotients

Theorem (J. De Doná, N. Tehseen, P.J. Vassiliou)

Let G be a Lie group of (control admissible) symmetries of a control system (M, ω) such that the action of G on M is free and regular and denote the Lie algebra of G in $\mathfrak{X}(M)$ as Γ. If $ann(\omega) \oplus \Gamma$ satisfies the conditions of being a relative Goursat bundle then $q_* ann(\omega) = ann(\beta^\kappa)$ where β^κ is the canonical contact system on J^κ and $q : M \to M/G$ the quotient map of the G-action.

In other words this theorem says that if the control system plus its symmetries looks like a contact system then the quotient system is in fact a contact system.

Step 2: Construction of γ^G

Theorem (I. Anderson, M. Fels)

Let (M, ω) be an EDS with symmetry group G acting freely and regularly on M such that its infinitesimal symmetries are strongly transverse to ω. Furthermore if $\sigma : M/G \to M$ is a cross section to the quotient map then if s_G is an integral manifold of ω_G then $s = \mu(g(x), \sigma \circ s_G)$ is an integral manifold of ω provided that $g(x)$ satisfies an equation of Lie type.

Step 2: Construction of γ^G part 2

Theorem (P. Vassiliou)

Let (M,ω) be an EDS for a control system with symmetry group G that acts freely and transitively on M and whose infinitesimal symmetries are strongly transverse to ω. Suppose further that $(M/G,\omega_G)$ is SF equivalent to (J^κ,β^κ) for some κ via $\varphi : J^\kappa \to M/G$. Then $\tilde{\varphi} = \mu(g,\sigma\circ\varphi) : G\times J^\kappa \to M$ is a diffeomorphism such that $\tilde{\varphi}^*\omega = \Theta_G \oplus \beta^\kappa$ where Θ_G is an EDS for the equation of Lie type. Call $\gamma^G = \Theta_G \oplus \beta^\kappa$. Then the following diagram commutes

$$\begin{tikzcd}
(M,\omega) \arrow[d, "q"] & (G\times J^\kappa,\gamma^G) \arrow[l, "\tilde{\varphi}"'] \arrow[d, "q_0"] \\
(M_G,\omega_G) & (J^\kappa,\beta) \arrow[l, "\varphi"']
\end{tikzcd}$$

Step 3: Restrictions to partial contact submanifolds

We can always rewrite the Brunovsky normal form as $\beta^\kappa = \beta^\nu \oplus \beta^{\nu^\perp}$ where β^ν and $\beta^{\nu^\perp}$ are the canonical contact systems on J^ν and $J^{\nu^\perp}$ respectively. Then one can consider

Definition

A **Partial Contact Submanifold** (or "Curve") of β^κ is given by the image of a map $C^\nu_f = j^\nu f \times Id_{J^{\nu^\perp}} : \mathbb{R} \times J^{\nu^\perp} \to J^\kappa$. If $\Sigma^\nu_f = \mathrm{Im}(C_f)$ then $\beta^\nu\big|_{\Sigma_f} \equiv 0$ for all $j^\nu f : \mathbb{R} \to J^\nu$

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Note: $\gamma^G\big|_{\Sigma^\kappa_f}$ is a frobenius system. So perhaps on a partial contact submanifold we can achieve that $\gamma^G\big|_{\Sigma^\nu_f}$ is SFL for arbitrary f...

Reduction Example

Let

$$\gamma^G = \langle \theta_0^i, \theta_1^i, d\epsilon - (z_2^1 z_2^2 \cos(z_0^1 z_2^2) - (z_1^1 z_2^2)^2 \sin(z_0^1 z_2^2))dt \rangle$$

where $\theta_0^i = dz_0^i - z_1^i\, dt$ and $\theta_1^i = dz_1^i - z_2^i\, dt$ and $i = 1, 2$. Reduction along the z^2 fiber means we assign $z_0^2 = f(t) \Rightarrow z_1^2 = \dot{f}(t) \Rightarrow z_2^2 = \ddot{f}(t)$ and thus

$$\bar{\gamma}^G = \langle \theta_0^1, \theta_1^1, d\epsilon - (z_2^1 \ddot{f}(t) \cos(z_0^1 \ddot{f}(t)) - (z_1^1 \ddot{f}(t))^2 \sin(z_0^1 \ddot{f}(t)))dt \rangle$$

Turns out this reduction is SFL! But the γ^G is not
Remark: $p(z) = D_{t_1}^2 (\sin(z_0^1 z_2^2))$

Cascade Static Feedback Linearizable

Definition

If a control system (M, ω) with control symmetry group G has a SFL quotient and if γ^G is ESFL when restricted to a partial contact submanifold then we say that ω is **Cascade Static Feedback Linearizable**

Partial Prolongation of γ^G

Theorem (K.-, J. Clelland, P. Vassiliou)

If a control system ω is cascade feedback linearizable then γ^G is DFL by partial prolongation. In particular in the two control case: Let $\bar{\gamma}^G = \gamma^G\big|_{\Sigma_f^\nu}$ be ESFL. If $\zeta^k(t, \epsilon, j^{\nu^\perp} z_1, f(t), \cdots, f^{(q)}(t))$ is the highest order variable for the Brunovsky form, then $\gamma^G = \Theta_G \oplus \beta^\nu \oplus \beta^{\nu^\perp}$ must be prolonged at least $q - \text{length}(\nu)$ times in β^ν to achieve a SFL for $\hat{\gamma}^G$.

When does γ^G admit an ESFL Reduction?

Theorem (K. -)

Let $\gamma^G = \beta^\nu \oplus \beta^{\nu^\perp} \oplus \langle d\epsilon - p(z)\,dt\rangle$ be a partial contact subconnection on $J^n(\mathbb{R}, \mathbb{R}^m) \times G$ where $\dim G = 1$ with local coordinate ϵ and $p(z)$ is a C^1 function on $J^n(\mathbb{R}, \mathbb{R}^m)$. Furthermore, denote

$$D_{t_i} = \partial_t + \sum_{l=0}^{n-1} z^i_{l+1}\partial_{z^i_l}$$

to be the total derivative operator along the jet fiber for coordiante z^i_0. If

$$p(z) = D_{t_i}A \text{ where } \frac{\partial A}{\partial t} = \frac{\partial A}{\partial z^i_l} = 0 \text{ for all } l \geq 1$$

then γ^G is ESFL when reduced along all fiber coordinates except z^i

Details About ESFL for a P.C.C. Reduction

Let $\bar{\mathcal{H}}_G = \text{span}\{\bar{X} = \partial_t + \sum_{l=0}^{n-1} z_{l+1}\partial_{z_l} + \bar{p}(z)\partial_\epsilon, \bar{Y}_0 = \partial_{z_n}\}$
Then the kth derived flag of this ditribution is

$$\bar{\mathcal{H}}_G^{(k)} = \{\bar{X}, \bar{Y}_0, \cdots, \bar{Y}_k\}$$

where

$$\bar{Y}_k = Q_k\partial_\epsilon + (-1)^k\partial_{z_{n-k}}$$
$$Q_{k+1} = \bar{D}_t(Q_k) + (-1)^{k+1}\frac{\partial\bar{p}}{\partial z_{n-k}}$$

initialized with $Q_0 = 0$. To be ESFL we need that $Q_{n+1} \neq 0$ and

$$\frac{\partial Q_{k+1}}{\partial z_{n-k}} + \frac{\partial Q_k}{\partial z_{n-k-1}} = 0$$

Truncated Euler Operators

Definition

Let E_k denote the Euler-Operator on $J^k(\mathbb{R}, \mathbb{R})$ which is defined by

$$E_k(f) = \frac{\partial f}{\partial z_0} - D_t\left(\frac{\partial f}{\partial z_1}\right) + D_t^2\left(\frac{\partial f}{\partial z_2}\right) - \cdots + (-1)^k D_t^k\left(\frac{\partial f}{\partial z_k}\right)$$

for any $f \in C^{k+1}(J^k, \mathbb{R})$ and where D_t is the total derivative operator on J^n and $n \geq k$.

More EO facts

Proposition

The kernel of an Euler-Operator of order k contains the set

$$K_{E_k} := \{f(z) = D_t g(z) | g \in C^1(J^k(\mathbb{R}, \mathbb{R}))\ \& \textit{ is linear in } z_k\}$$

where D_t is the total derivative operator on $J^k(\mathbb{R}, \mathbb{R})$

Lemma

The Euler-Operators are related to the derived flag of $\bar{\mathcal{H}}_G$ by

$$(-1)^{n-k-1} \bar{D}_t^{n-k}(Q_{k+1}) = E_n(\bar{p}) - E_{n-k-1}(\bar{p})$$

Computing $\bar{p}$ versus p and the Q_k

In the theorem we have that $p(z) = D_{t_i} A$. In this case it turns out that

$$\bar{p} = \left(\bar{D}_t - \frac{d}{dt} \right) \bar{A}$$

This immediately gives that $\bar{D}_t^{n-k}(Q_{k+1}) = 0$ for all $0 \leq k \leq n-1$ and $Q_{n+1} = -\frac{\partial \bar{A}_t}{\partial z_0}$ and naturally the PDE

$$\frac{\partial Q_{k+1}}{\partial z_{n-k}} + \frac{\partial Q_k}{\partial z_{n-k-1}} = 0$$

is satisfied.

Conjectures

- The theorem holds if $p(z) = D^r_{t_i} A$ for any $1 \leq r \leq n$ and in fact these are the only such possible functions of p to give an ESFL reduction

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- If $\dim G > 1$ and is abelian then each associated function p_a must have the form of the above to guarantee ESFL reduction

Conjectures

- The theorem holds if $p(z) = D^r_{t_i} A$ for any $1 \leq r \leq n$ and in fact these are the only such possible functions of p to give an ESFL reduction
- If $\dim G > 1$ and is abelian then each associated function p_a must have the form of the above to guarantee ESFL reduction
- If G is nonabelian then all results will be the same with modified euler operators and total derivatives designed to be invariant under the group action in some sense (yes very vauge will be more precise later).

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Thanks for coming!