Nonlocal fractional equations from random walks

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A creative component presented for the degree of
Master of Science of Applied Mathematics

Department of Mathematics
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United States of America
December 2023

NONLOCAL FRACTIONAL EQUATIONS FROM RANDOM WALKS

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Abstract. If a particle is undergoing a continuous random walk, it is of interest to model the probability of observing said particle at some position and observing it at some time. Under "normal" circumstances, this probability function satisfies the heat equation. However, there are natural phenomena where particles get stuck in one spot before moving again or make jumps of arbitrary length. Such behaviors are examples of anomalous diffusion, and we are interested in modeling this same probability under these scenarios and how it impacts the heat equation. The fractional left derivative and fractional Laplacian are developed and utilized in our formulations. Finally, we compare the kernels that are obtained from experimental observations with kernels that result from the computations of the discrete left fractional derivative and discrete fractional Laplacian. We show that the difference between the restriction of the original fractional derivative to the mesh and the corresponding discrete derivative can be made arbitrarily small. This proves that the discrete fractional derivative converges to its corresponding continuous fractional derivative, and it allows us to compute the kernels without relying on experiments. Extra care is taken in preserving coefficients in computations to show the importance of the gamma function in these models.

Contents

2010 *Mathematics Subject Classification.* 26A33, 33B15, 35Q70, 42A38, 60K40.

Key words and phrases. Fractional derivatives, Gamma function, PDEs, Fourier transform, Random processes.

Master's creative component advised by Dr. Pablo Raúl Stinga.

1. Setting the scene

Consider a particle of unit mass on the lattice $h\mathbb{Z}^n = \{hx : x \in \mathbb{Z}^n\}$, where $h > 0$, that is moving step-by-step at random. We call h the *space step*. Also, we fix $\tau > 0$ to be the *time step*. Without loss of generality, we suppose that the particle starts at the origin. For every step, the particle moves in one of the $2n$ possible directions with uniform probability, and each step is independent of the previous one.

For the duration of the paper, we assume $u : \mathbb{R}^n \times \mathbb{R} \to [0, 1]$ is Schwartz class to maintain a sufficient level of smoothness. For $x \in \mathbb{R}^n$ and $t \in \mathbb{R}$, we say that $u(x, t)$ is the probability of observing the particle in position x and observing the particle at time t. If h, τ are made arbitrarily small, then we obtain a continuous random walk in time and space. With this, the goal is to derive a partial differential equation that models the behavior of u.

1.1. **Classical random walk.**

To illustrate these ideas, we start with the simple case where we require the particle to move one space step of size h for every time step of size τ, which is called a classical random walk. In the context of random walks throughout time, we are only able to consider past and present times. Obviously, we cannot draw on information from the future. When we talk about a change in some quantity with respect to time, it only makes sense to consider the infinitesimal change coming from the left. As such, we do not talk about derivatives of functions but rather left derivatives of functions.

Definition 1.1. Let $u : \mathbb{R} \to \mathbb{R}$ be a function. We say that u is *left-differentiable* if

$$D_{\text{left}} u(t) = \lim_{\tau \to 0^+} \frac{u(t) - u(t - \tau)}{\tau}$$

exists, and $D_{\text{left}} u$ is the *left derivative of u.*

It is worth mentioning that all left derivatives will be taken with respect to time, so the components of the spacial variable will be treated as constants. Another tool that will be used routinely in calculations is the fact that second derivatives can be written as limits of second-order incremental quotients:

Proposition 1.2. *Let $u : \mathbb{R}^n \to \mathbb{R}$ be a twice-differentiable function, and let $1 \leq k \leq n$. Then,*

$$u_{x_k x_k}(x) = \lim_{h \to 0} \frac{u(x + he_k) + u(x - he_k) - 2u(x)}{h^2},$$

where $\{e_k\}_{k=1}^n$ is the standard orthonormal basis for $\mathbb{R}^n$.

Note that the direction of the limit above is inconsequential because of the symmetry of the first two terms in the numerator and the square in the denominator.

Since there are $2n$ possible directions for the particle to move, the probability of choosing one of them is $\frac{1}{2n}$. For each $1 \leq k \leq n$, the particle moves in the direction of he_k or $-he_k$ with equal probability. Finally, we require that taking a time step of size τ results in a space step of size h. With all of these observations, by the law of total probability, we have that

$$u(x, t) = \sum_{k=1}^{n} \frac{u(x + he_k, t - \tau) + u(x - he_k, t - \tau)}{2n}. \tag{1.1}$$

Next, by properties of finite sums, we can rearrange and divide both sides by τ:

$$\begin{aligned}\frac{u(x,t)-u(x,t-\tau)}{\tau} &= \sum_{k=1}^{n} \frac{u(x+he_k,t-\tau)+u(x-he_k,t-\tau)-2u(x,t-\tau)}{2\tau n} \\ &= \frac{h^2}{2\tau n}\sum_{k=1}^{n} \frac{u(x+he_k,t-\tau)+u(x-he_k,t-\tau)-2u(x,t-\tau)}{h^2}.\end{aligned}$$

To obtain a continuous random walk, we take the limits as both $h, \tau \to 0^+$. However, in order for us to arrive at a meaningful result, we must require that

$$\frac{h^2}{2\tau n} \to \frac{K}{2n}$$

for some $K > 0$. On the left, we recognize the limit in τ as the left derivative of u. Inside the sum on the right, we have the limit in h on the spacial component of the second-order incremental quotient. Thus, after taking limits, the previous equation is reduced to

$$\begin{aligned} D_{\text{left}} u &= \frac{K}{2n}\sum_{k=1}^{n} u_{x_k x_k} \\ &= \frac{K}{2n}\Delta u, \end{aligned} \tag{1.2}$$

which we know to be the heat equation. We can summarize our results as follows:

Theorem 1.3. *A particle undergoes a continuous classical random walk. The probability of observing the particle at position $x \in \mathbb{R}^n$ and observing the particle at time $t \in \mathbb{R}$ is given by the following PDE:*

$$D_{\text{left}} u = \frac{K}{2n}\Delta u \tag{1.3}$$

for some $K > 0$.

1.2. **Introduction to waiting times and jumps.**

Requiring the particle to move one unit space step for each time interval is rather specific, so we now consider three more realistic cases. What if the particle gets stuck and has a probability of waiting some amount of time before moving again? What if the particle makes a longer jump than just a step of size h? What if both of these events happen? These are examples of a process called anomalous diffusion. In particular, we can identify what these cases are:

(1) There is a probability of the particle getting stuck and undergoing a waiting time while still taking steps of size h.
(2) There is a probability of the particle making a jump of arbitrary size while still moving for every time step τ.
(3) There is a probability of the particle getting stuck and another probability of making a jump of arbitrary size.

These considerations will require us to modify (1.1) to fit the situation in question. The task now is to mathematically describe these modifications.

Experimental results have shown that not all random motion can be modeled by (1.2). Proteins moving through cell membranes may get stuck, and electrons can get stuck on semiconductors until they obtain enough potential energy to move again. These processes with waiting times of the particles are examples of subdiffusion. If this happens, then it has been observed that the probability of waiting some time before moving follows a Pareto power law, resulting in the following definition:

Definition 1.4. Let $\psi : (0, \infty) \to [0, 1]$ be defined by $\psi(\tau) = c_\alpha \tau^{-(1+\alpha)}$ for $0 < \alpha < 1$, where $c_\alpha > 0$ is chosen such that

$$\sum_{m=1}^{\infty} \psi(m) = 1.$$

We say that $\psi(m)$ is the *probability of waiting m units of time between steps.*

Similarly, it is natural that particles in random motion may experience jumps of various lengths. A particle can move along the surface of a crystal, and due to the symmetry of the crystal, the restriction to the surface can cause the particle to make unpredicatble jumps. This type of process is an example of superdiffusion. In this scenario, experimental observations have shown that the probability of making a jump of some size follows a similar Pareto power law from before, so we have the following definition:

Definition 1.5. Let $\phi : \mathbb{R}^n \to [0, 1]$ be defined by $\phi(y) = d_s \left|y\right|^{-(n+2s)}$ for $y \neq 0$ and $\phi(y) = 0$ for $y = 0$, where $0 < s < 1$, and where $d_s > 0$ is chosen such that

$$\sum_{k \in \mathbb{Z}^n} \phi(k) = 1.$$

We say that $\phi(k)$ is the *probability of making a jump of size $|k|$.*

Note that ϕ is even, which corresponds to the particle having an equal probability of moving left or right.

Remark 1.6. Due to the definitions of ψ, ϕ, the normalization constants c_α,d_s respectively can be written in terms of the Riemann zeta function. While this choice may lead to more precise calculations of coefficients, we will instead only use c_α,d_s to retain the physical context of the problem.

These two probabilities change the model in (1.2) in subtle yet fundamental ways. Now that we have established what they are, we must find out how they impact the heat equation. To discuss these changes, we shift our attention to the analytical tools that we will use.

Notes:

The setup of the classical random walk is described in [5]. To derive the heat equation, we instead use methods similar to those in [6]. The three cases regarding waiting times and jumps are explored further in Section 3. Additional context for the anomalous processes of subdiffusion and superdiffusion is given in [6] and [4], and the proposed formulas for ψ and ϕ that describe these phenomena are given in [6] and [8] respectively.

2. Deriving the fractional derivatives

The introduction of waiting times and jumps requires the implementation of fractional derivatives to model these changes. In this section, we discuss two types of these derivatives and derive convenient integral expressions for them. It is implied that the main tool behind the formulation of these fractional derivatives is the gamma function, whose key properties used here are given in Section A.1.

2.1. **Fractional left derivative.**

To begin to define the fractional left derivative, we start by rewriting the left derivative in terms of the Fourier transform. While it is well established how derivatives and Fourier transforms relate, it would not hurt to take care when considering left derivatives.

Proposition 2.1. *Let $\omega \in \mathbb{R}$. Then,*

$$\widehat{D_{\text{left}}u}(\omega) = (i\omega)\widehat{u}(\omega).$$

Proof: Applying the Fourier transform to $D_{\text{left}}u$ and integrating by parts gives that

$$\begin{aligned}\widehat{D_{\text{left}}u}(\omega) &= \frac{1}{(2\pi)^{\frac{1}{2}}}\int_{\mathbb{R}}(D_{\text{left}}u)e^{-i\omega r}dr\\ &= \frac{-1}{(2\pi)^{\frac{1}{2}}}\int_{\mathbb{R}}\big(D_{\text{left}}[e^{-i\omega r}]\big)udr.\end{aligned}$$

Note that the exponential is differentiable, so its left derivative would be the familiar regular derivative, and

$$\begin{aligned}\widehat{D_{\text{left}}u}(\omega) &= \frac{i\omega}{(2\pi)^{\frac{1}{2}}}\int_{\mathbb{R}}e^{-i\omega r}udr\\ &= (i\omega)\widehat{u}(\omega).\end{aligned}$$

■

Since a single copy of $i\omega$ was removed from the exponent and placed in front of $\widehat{u}$, this suggests that we can also take α copies of $i\omega$, so inverting the Fourier transform gives the following definition:

Definition 2.2 (Fourier)**.** Let $0 < \alpha < 1$. The *fractional left derivative of order α of u* is defined to be

$$(D_{\text{left}})^{\alpha}u = \mathcal{F}^{-1}\big((i\omega)^{\alpha}\widehat{u}(\omega)\big).$$

We restrict α to be less than 1 because that is where the "interesting" behavior occurs. If $\alpha > 1$, then we could simply take $\lfloor\alpha\rfloor$ integer order derivatives to reattain the otherwise typical fractional behavior. Observe that the factor $(i\omega)^{\alpha}$ is well-defined by (A.7). If we take that identity and multiply by $\widehat{u}(\omega)$, then we get that

$$(i\omega)^{\alpha}\widehat{u}(\omega) = \frac{1}{\Gamma(-\alpha)}\int_0^{\infty}\big(e^{-i\omega r}\widehat{u}(\omega) - \widehat{u}(\omega)\big)\frac{dr}{r^{1+\alpha}},$$

and taking the inverse transform of both sides gives

$$(D_{\text{left}})^{\alpha}u(t) = \frac{1}{\Gamma(-\alpha)}\int_0^{\infty}\Big(\mathcal{F}^{-1}\big(e^{-i\omega r}\widehat{u}(\omega)\big)(t) - u(t)\Big)\frac{dr}{r^{1+\alpha}}. \tag{2.1}$$

We pause here to note that these manipulations have allowed us to move from a power of $i\omega$ to a power of D_{left}. We now use the method of semigroups to rewrite the expression above as the exponential of the left derivative operator.

Proposition 2.3. *Let $R_r : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ be defined such that*

$$\widehat{R_r u}(\omega) = e^{-i\omega r}\widehat{u}(\omega)$$

for $\omega \in \mathbb{R}$ and $r \geq 0$. Then, R_r defines a semigroup.

This will show that $v := R_r u$ is a solution to the initial value problem

$$\begin{cases} v_r = -D_{\text{left}} u & r \neq 0 \\ v = u & r = 0 \end{cases}$$

and provide meaning to $v = e^{-rD_{\text{left}}}u$.

Proof: We first show that R_r is bounded on $L^2(\mathbb{R})$. By Plancherel's identity, we have that

$$\begin{aligned} \|R_r u\|_{L^2(\mathbb{R})} &= \left\|\widehat{R_r u}\right\|_{L^2(\mathbb{R})} \\ &= \left\|e^{-i\omega r}\widehat{u}\right\|_{L^2(\mathbb{R})} \\ &\leq \|\widehat{u}\|_{L^2(\mathbb{R})} \\ &= \|u\|_{L^2(\mathbb{R})}, \end{aligned}$$

which follows again by Plancherel's identity. Rearranging gives us that $\|R_r\|_{L^2(\mathbb{R})} \leq 1$, so R_r is bounded on $L^2(\mathbb{R})$. Next, we show that $R_0 u = u$, but this is true since the inverse Fourier transform cancels out the Fourier transform. It then remains to show that $R_{r_1} \circ R_{r_2} = R_{r_1+r_2}$. We have that

$$\begin{aligned} (R_{r_1} \circ R_{r_2})u &= R_{r_1}\big(\mathcal{F}^{-1}(e^{-i\omega r_2}\widehat{u})\big) \\ &= \mathcal{F}^{-1}\big(e^{-i\omega r_1}\mathcal{F}^{-1}(e^{-i\omega r_2}\widehat{u})\big) \\ &= \mathcal{F}^{-1}\big(\mathcal{F}^{-1}(e^{-i\omega r_1}e^{-i\omega r_2}\widehat{u})\big) \\ &= e^{-i\omega(r_1+r_2)} \\ &= R_{r_1+r_2}u. \end{aligned}$$

Thus, R_r is a semigroup. ■

Now, we can make the following notation:

$$e^{-rD_{\text{left}}}u = \mathcal{F}^{-1}\big(e^{-i\omega r}\widehat{u}(\omega)\big). \tag{2.2}$$

We should not be concerned with any other meaning for this notation. D_{left} is unbounded as an operator, so the matrix exponential given from this linear operator cannot be represented as an infinite series. Plugging this back into (2.1) gives

$$(D_{\text{left}})^\alpha u(t) = \frac{1}{\Gamma(-\alpha)}\int_0^\infty \big(e^{-rD_{\text{left}}}u(t) - u(t)\big)\frac{dr}{r^{1+\alpha}}.$$

On the other hand, if we simplify (2.2), by the property of Fourier transforms, we see that the result is simply the r-translation of u, that is,

$$(D_{\text{left}})^\alpha u(t) = \frac{1}{|\Gamma(-\alpha)|}\int_0^\infty \big(u(t) - u(t-r)\big)\frac{dr}{r^{1+\alpha}}. \tag{2.3}$$

Of course, it is unnatural to consider a function of time that looks arbitrarily into the future, as illustrated in the bounds of integration. Thus, we make the change of variables $s = t - r$, so $ds = -dr$, and

$$\begin{aligned}(D_{\text{left}})^\alpha u(t) &= \frac{-1}{|\Gamma(-\alpha)|}\int_t^{-\infty}\frac{u(t)-u(s)}{(t-s)^{1+\alpha}}ds \\ &= \frac{1}{|\Gamma(-\alpha)|}\int_{-\infty}^{t}\frac{u(t)-u(s)}{(t-s)^{1+\alpha}}ds \\ &= \frac{1}{|\Gamma(-\alpha)|}\int_{-\infty}^{t}\frac{u(t)-u(r)}{(t-r)^{1+\alpha}}dr.\end{aligned}$$

Thus, we can summarize with the following:

Theorem 2.4. *Let $u : \mathbb{R} \to \mathbb{R}$ be sufficiently smooth and $0 < \alpha < 1$. Then,*

$$(D_{\text{left}})^\alpha u(t) = \frac{1}{|\Gamma(-\alpha)|}\int_{-\infty}^{t}\frac{u(t)-u(r)}{(t-r)^{1+\alpha}}dr. \tag{2.4}$$

This form of the fractional left derivative is also known as the Fourier-Weyl-Marchaud fractional derivative.

2.2. **Fractional Laplacian.**

We apply the Fourier transform again here. For this subsection, we are not concerned with one-sided derivatives, but due to the added level of complexity to the Laplacian compared to the first derivative, we show the computations here as well. Let $\xi \in \mathbb{R}^n$ and $1 \le k \le n$. Since u is Schwartz, then integration by parts gives that:

$$\begin{aligned}\widehat{\frac{\partial}{\partial x_k}u}(\xi) &= \frac{1}{(2\pi)^{\frac{n}{2}}}\int_{\mathbb{R}^n}\frac{\partial u}{\partial x_k}e^{-ix\xi}dx \\ &= \frac{i\xi_k}{(2\pi)^{\frac{n}{2}}}\int_{\mathbb{R}^n}u(x)e^{-ix\xi}dx \\ &= (i\xi_k)\widehat{u}(\xi).\end{aligned}$$

We can apply integration by parts again to obtain an expression for the second, non-mixed partial derivative of u.

$$\begin{aligned}\widehat{\frac{\partial^2}{\partial x_k^2}u}(\xi) &= \frac{1}{(2\pi)^{\frac{n}{2}}}\int_{\mathbb{R}^n}\frac{\partial^2 u}{\partial x_k^2}e^{-ix\xi}dx \\ &= \frac{i\xi_k}{(2\pi)^{\frac{n}{2}}}\int_{\mathbb{R}^n}\frac{\partial u}{\partial x_k}e^{-ix\xi}dx \\ &= (i\xi_k)^2\widehat{u}(\xi) \\ &= -(\xi_k^2)\widehat{u}(\xi),\end{aligned} \tag{2.5}$$

and we can take the inverse Fourier transform of both sides to get

$$\frac{\partial^2 u}{\partial x_k^2}(x) = \mathcal{F}^{-1}\big(-(\xi_k^2)\widehat{u}(\xi)\big)(x).$$

We now sum over all $1 \leq k \leq n$. The inverse Fourier transform is linear, so we can move the sum inside and pull out the negative:

$$\begin{aligned}\sum_{k=1}^{n} \frac{\partial^2 u}{\partial x_k^2}(x) &= \sum_{k=1}^{n} \mathcal{F}^{-1}\big(- (\xi_k^2)\widehat{u}(\xi)\big)(x) \\ &= -\mathcal{F}^{-1}\left(\sum_{k=1}^{n} (\xi_k^2)\widehat{u}(\xi) \right)(x) \\ &= -\mathcal{F}^{-1}\big(|\xi|^2 \, \widehat{u}(\xi)\big)(x).\end{aligned}$$

On the other hand, the sum of the non-mixed second partial derivatives is defined to be the Laplacian of u, so

$$-\Delta u = \mathcal{F}^{-1}\big(|\xi|^2 \, \widehat{u}(\xi)\big).$$

Note that taking two partial derivatives in (2.5) corresponded to taking $|\xi|$ to the second power. This suggests that taking a power of the operator $-\Delta$ would lead to raising $|\xi|^2$ to that same power. Therefore, this motivates the following definition of the fractional Laplacian:

Definition 2.5. Let $0 < s < 1$. The *fractional Laplacian of order s of u* is defined to be

$$(-\Delta)^s u = \mathcal{F}^{-1}\big(|\xi|^{2s} \, \widehat{u}(\xi)\big).$$

This definition gives great insight in where the fractional Laplacian comes from, but in order for it to be useful, we must simplify the right side of the above equation. First, we start by letting $\lambda = |\xi|^2$ in (A.4):

$$|\xi|^{2s} = \frac{1}{\Gamma(-s)} \int_0^\infty \frac{e^{-t|\xi|^2} - 1}{t^{1+s}} dt,$$

so multiplying both sides by $\widehat{u}(\xi)$ gives

$$|\xi|^{2s} \, \widehat{u}(\xi) = \frac{1}{\Gamma(-s)} \int_0^\infty \frac{e^{-t|\xi|^2}\widehat{u}(\xi) - \widehat{u}(\xi)}{t^{1+s}} dt,$$

and taking the inverse of the Fourier transform on both sides ultimately gives that

$$(-\Delta)^s u(x) = \frac{1}{\Gamma(-s)} \int_0^\infty \Big(\mathcal{F}^{-1}\big(e^{-t|\xi|^2}\widehat{u}(\xi)\big)(x) - u(x)\Big) \frac{dt}{t^{1+s}}. \tag{2.6}$$

In the same way with the fractional left derivative, these manipulations allow us to move from a power of $|\xi|^2$ to a power of $-\Delta$. We again describe this connection with semigroups, and the proof is identical to the argument for R_r being a semigroup.

Proposition 2.6. *Let $S_t : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ be defined such that*

$$\widehat{S_t u}(\xi) = e^{-t|\xi|^2}\widehat{u}(\xi)$$

for $\xi \in \mathbb{R}^n$ and $t \geq 0$. Then, S_t defines a semigroup.

This shows that $v := S_t u$ is a solution to

$$\begin{cases} v_t = \Delta u & t \neq 0 \\ v = u & t = 0 \end{cases}$$

and provides meaning to the following notation:

$$e^{t\Delta} u = \mathcal{F}^{-1}(e^{-t|\xi|^2}\widehat{u}). \tag{2.7}$$

We can substitute this expression back into (2.6) to get the following:

$$(-\Delta)^s u(x) = \frac{1}{\Gamma(-s)} \int_0^\infty \left(e^{t\Delta}u(x) - u(x)\right) \frac{dt}{t^{1+s}}, \tag{2.8}$$

which shows how we can write the fractional power of the Laplacian using semigroups. If we simplify (2.7), then we see that $e^{t\Delta}u$ is the *continuous convolution* of the Gaussian G_t and u, that is,

$$\begin{aligned} e^{t\Delta}u(x) &= (G_t * u)(x) \\ &= \int_{\mathbb{R}^n} \frac{e^{-\frac{|x-y|^2}{4t}}}{(4\pi t)^{\frac{n}{2}}} u(y)dy. \end{aligned}$$

We continue in the calculation of $(-\Delta)^s u(x)$. Plugging everything back into (2.8) gives

$$\begin{aligned} (-\Delta)^s u(x) &= \frac{1}{\Gamma(-s)} \int_0^\infty \left[\left(\int_{\mathbb{R}^n} \frac{e^{-\frac{|x-y|^2}{4t}}}{(4\pi t)^{\frac{n}{2}}} u(y)dy\right) - u(x)\right] \frac{dt}{t^{1+s}} \\ &= \frac{1}{(4\pi)^{\frac{n}{2}}\Gamma(-s)} \int_0^\infty \int_{\mathbb{R}^n} \frac{e^{-\frac{|x-y|^2}{4t}}}{t^{\frac{n}{2}}} \big(u(y) - u(x)\big)dy \frac{dt}{t^{1+s}}, \end{aligned} \tag{2.9}$$

where this manipulation is justified since the integral of the Gaussian is 1. Make the change of variables $y = x - v$, and thus

$$\begin{aligned} (-\Delta)^s u(x) &= \frac{1}{(4\pi)^{\frac{n}{2}}\Gamma(-s)} \int_0^\infty \int_{\mathbb{R}^n} \frac{e^{-\frac{|v|^2}{4t}}}{t^{\frac{n}{2}}} \big(u(x-v) - u(x)\big)dv \frac{dt}{t^{1+s}} \\ &= \frac{1}{(4\pi)^{\frac{n}{2}}\Gamma(-s)} \int_0^\infty \int_{\mathbb{R}^n} \frac{e^{-\frac{|y|^2}{4t}}}{t^{\frac{n}{2}}} \big(u(x-y) - u(x)\big)dy \frac{dt}{t^{1+s}}. \end{aligned} \tag{2.10}$$

Now, make a second change of variables $y = -w$, so

$$\begin{aligned} (-\Delta)^s u(x) &= \frac{1}{(4\pi)^{\frac{n}{2}}\Gamma(-s)} \int_0^\infty \int_{\mathbb{R}^n} \frac{e^{-\frac{|-w|^2}{4t}}}{t^{\frac{n}{2}}} \big(u(x+w) - u(x)\big)dw \frac{dt}{t^{1+s}} \\ &= \frac{1}{(4\pi)^{\frac{n}{2}}\Gamma(-s)} \int_0^\infty \int_{\mathbb{R}^n} \frac{e^{-\frac{|y|^2}{4t}}}{t^{\frac{n}{2}}} \big(u(x+y) - u(x)\big)dy \frac{dt}{t^{1+s}}. \end{aligned} \tag{2.11}$$

If we add (2.10) and (2.11), then we see that the sum is two times (2.9), or

$$\begin{aligned} (-\Delta)^s u(x) &= \frac{1}{2}\Bigg(\frac{1}{(4\pi)^{\frac{n}{2}}\Gamma(-s)} \int_0^\infty \int_{\mathbb{R}^n} \frac{e^{-\frac{|y|^2}{4t}}}{t^{\frac{n}{2}}} \big(u(x-y) - u(x)\big)dy \frac{dt}{t^{1+s}} \\ &\qquad + \frac{1}{(4\pi)^{\frac{n}{2}}\Gamma(-s)} \int_0^\infty \int_{\mathbb{R}^n} \frac{e^{-\frac{|y|^2}{4t}}}{t^{\frac{n}{2}}} \big(u(x+y) - u(x)\big)dy \frac{dt}{t^{1+s}}\Bigg) \\ &= \frac{1}{2(4\pi)^{\frac{n}{2}}\Gamma(-s)} \int_0^\infty \int_{\mathbb{R}^n} \frac{e^{-\frac{|y|^2}{4t}}}{t^{\frac{n}{2}}} \big(u(x+y) + u(x-y) - 2u(x)\big)dy \frac{dt}{t^{1+s}}. \end{aligned}$$

To continue to simplify this expression, we would like to apply Fubini's theorem to swap the order of the integrals. We will assume that Fubini's theorem holds and solve for the range of s such that it does. Establishing this range will finish the calculation. Therefore, swapping integrals gives

$$\begin{aligned}(-\Delta)^s u(x) &= \frac{1}{2(4\pi)^{\frac{n}{2}}\Gamma(-s)}\int_{\mathbb{R}^n}\int_0^\infty \frac{e^{-\frac{|y|^2}{4t}}}{t^{\frac{n}{2}}}\big(u(x+y)+u(x-y)-2u(x)\big)\frac{dt}{t^{1+s}}dy\\ &= \frac{1}{2(4\pi)^{\frac{n}{2}}\Gamma(-s)}\int_{\mathbb{R}^n}\big(u(x+y)+u(x-y)-2u(x)\big)\int_0^\infty \frac{e^{-\frac{|y|^2}{4t}}}{t^{\frac{n}{2}}}\frac{dt}{t^{1+s}}dy.\end{aligned}$$

Let $r = \frac{|y|^2}{4t}$, and $dr = -\frac{|y|^2}{4t^2}dt$, so $dt = -\frac{|y|^2}{4r^2}dr$. Substituting and simplifying gives the following:

(2.12)

$$\begin{aligned}(-\Delta)^s u(x) &= \frac{-1}{2(4\pi)^{\frac{n}{2}}\Gamma(-s)}\int_{\mathbb{R}^n}\big(u(x+y)+u(x-y)-2u(x)\big)\int_\infty^0 e^{-r}\left(\frac{4r}{|y|^2}\right)^{\frac{n}{2}+s}\frac{\frac{|y|^2}{4r^2}dr}{\frac{|y|^2}{4r}}dy\\ &= \frac{4^{\frac{n}{2}+s}}{2(4\pi)^{\frac{n}{2}}\Gamma(-s)}\int_{\mathbb{R}^n}\frac{u(x+y)+u(x-y)-2u(x)}{|y|^{n+2s}}\int_0^\infty e^{-r}r^{\frac{n}{2}+s}\frac{dr}{r}dy\\ &= \frac{4^s\Gamma(\frac{n}{2}+s)}{2\pi^{\frac{n}{2}}\Gamma(-s)}\int_{\mathbb{R}^n}\frac{u(x+y)+u(x-y)-2u(x)}{|y|^{n+2s}}dy,\end{aligned}$$

which follows directly from the definition of the gamma function for positive real numbers. This is the expression we are looking for, but it remains to show that the integral converges absolutely for some desired range of s. Let

$$G(y) = \frac{u(x+y)+u(x-y)-2u(x)}{|y|^{n+2s}},$$

so we have that

$$\begin{aligned}\left|\int_{\mathbb{R}^n} G(y)dy\right| &= \left|\int_{|y|<1} G(y)dy\right| + \left|\int_{|y|\geq 1} G(y)dy\right|\\ &\leq \int_{|y|<1}|G(y)|\,dy + \int_{|y|\geq 1}|G(y)|\,dy\\ &=: A + B.\end{aligned}$$

In the integrand of A, we have that

$$\begin{aligned}|u(x+y)+u(x-y)-2u(x)| &= \big|\big(u(x+y)-u(x)\big)+\big(u(x-y)-u(x)\big)\big|\\ &= |\nabla u(\xi)\cdot y + \nabla u(\eta)\cdot(-y)|\end{aligned}$$

for some $\xi, \eta \in \mathbb{R}^n$ by the mean value theorem. Now,

$$\begin{aligned}|u(x+y)+u(x-y)-2u(x)| &= \big|\big(\nabla u(\xi)-\nabla u(\eta)\big)\cdot y\big|\\ &= \big|D^2u(\nu)(\xi-\eta)\cdot y\big|\end{aligned}\tag{2.13}$$

for some $\nu \in \mathbb{R}^n$ by the mean value theorem again. Note that

$$\begin{aligned}\xi &= \lambda x + (1-\lambda)(x+y)\\ \eta &= \mu x + (1-\mu)(x-y)\end{aligned}$$

for some $\lambda, \mu \in (0,1)$, so

$$\xi - \eta = \lambda x + (1-\lambda)(x+y) - \mu x - (1-\mu)(x-y)$$

$$
\begin{aligned}
&= (\lambda x + x + y - \lambda x - \lambda y) - (\mu x + x - y - \mu x + \mu y) \\
&= (1-\lambda)y + (1-\mu)y \\
&= (2-\lambda-\mu)y.
\end{aligned}
$$

Substituting back into (2.13) gives

$$
\begin{aligned}
|u(x+y) - u(x-y) - 2u(x)| &= \left|D^2u(\nu)(2-\lambda-\mu)y \cdot y\right| \\
&\leq 2\left\|D^2u\right\|_{L^\infty(\mathbb{R}^n)} |y|^2 .
\end{aligned}
$$

Note that this string of arguments relies on the fact that u has bounded first and second derivatives, which we have since u is Schwartz. We now substitute back into A:

$$
\begin{aligned}
A &\leq \int_{|y|<1} \frac{2\left\|D^2u\right\|_{L^\infty(\mathbb{R}^n)} |y|^2}{|y|^{n+2s}} dy \\
&= 2\left\|D^2u\right\|_{L^\infty(\mathbb{R}^n)} \int_{|y|<1} \frac{1}{|y|^{n+2s-2}} dy.
\end{aligned}
$$

Apply spherical coordinates to the integral above. Let $y = r\theta$, where $r \in (0,1)$ and $\theta \in \mathbb{S}^{n-1}$, and so $|\theta| = 1$. Thus,

$$
\begin{aligned}
A &\leq 2\left\|D^2u\right\|_{L^\infty(\mathbb{R}^n)} \int_0^1 \left(\int_{\mathbb{S}^{n-1}} \frac{1}{|r\theta|^{n+2s-2}} d\theta\right) r^{n-1} dr \\
&= 2\left\|D^2u\right\|_{L^\infty(\mathbb{R}^n)} \int_0^1 \left(\int_{\mathbb{S}^{n-1}} d\theta\right) r^{1-2s} dr \\
&= 2\left\|D^2u\right\|_{L^\infty(\mathbb{R}^n)} \left|\mathbb{S}^{n-1}\right| \int_0^1 r^{1-2s} dr \\
&= \frac{2\left\|D^2u\right\|_{L^\infty(\mathbb{R}^n)} \left|\mathbb{S}^{n-1}\right|}{2-2s} r^{2-2s}\Big|_0^1 .
\end{aligned}
$$

Observe that this difference converges if and only if $2-2s > 0$, i.e. $s < 1$. Next, we consider B:

$$
\begin{aligned}
B &= \int_{|y|\geq 1} \frac{|u(x+y) - u(x-y) - 2u(x)|}{|y|^{n+2s}} dy \\
&\leq \int_{|y|\geq 1} \frac{|u(x+y)| + |u(x-y)| + 2\,|u(x)|}{|y|^{n+2s}} dy \\
&\leq 4\,\|u\|_{L^\infty(\mathbb{R}^n)} \int_{|y|\geq 1} \frac{1}{|y|^{n+2s}} dy.
\end{aligned}
$$

Apply spherical coordinates again. Let $y = r\theta$, where $r \in [1,\infty)$ and $\theta \in \mathbb{S}^{n-1}$, and so $|\theta| = 1$. Thus,

$$
\begin{aligned}
B &\leq 4\,\|u\|_{L^\infty(\mathbb{R}^n)} \int_1^\infty \left(\int_{\mathbb{S}^{n-1}} \frac{1}{|r\theta|^{n+2s}} d\theta\right) r^{n-1} dr \\
&= 4\,\|u\|_{L^\infty(\mathbb{R}^n)} \int_1^\infty \left(\int_{\mathbb{S}^{n-1}} d\theta\right) r^{-1-2s} dr \\
&= 4\,\|u\|_{L^\infty(\mathbb{R}^n)} \left|\mathbb{S}^{n-1}\right| \int_1^\infty r^{-1-2s} dr
\end{aligned}
$$

$$= \frac{4\left\|u\right\|_{L^\infty(\mathbb{R}^n)}\left|\mathbb{S}^{n-1}\right|}{-2s} r^{-2s}\Big|_1^\infty ,$$

and this difference converges if and only if $-2s < 0$, i.e. $s > 0$. Therefore, G is integrable for $0 < s < 1$, which is the range we started with. Combining this information with (2.12) gives us the practical formula for the fractional Laplacian:

Theorem 2.7. *Let $u : \mathbb{R}^n \to \mathbb{R}$ be sufficiently smooth and $0 < s < 1$. Then,*

$$(-\Delta)^s u(x) = \frac{4^s\Gamma(\frac{n}{2}+s)}{2\pi^{\frac{n}{2}}\Gamma(-s)} \int_{\mathbb{R}^n} \frac{u(x+y)+u(x-y)-2u(x)}{|y|^{n+2s}} dy. \tag{2.14}$$

Notes:

We follow the derivation of the Fourier-Weyl-Marchaud fractional derivative in [6]. While not essential to the computations, the semigroup method provides insight and meaning into taking fractional powers of operators, and it is explained in [7]. We use a similar method to derive the integral expression for the fractional Laplacian. We make the choice to write the second-order incremental quotient of u instead of the first-order incremental quotient, because it is desirable to use the boundedness of the second derivative of u, as described in [8].

3. Establishing anomalous diffusion models

With the language of fractional derivatives established, we can answer the question of how the probabilities of waiting times and jumps impact the heat equation and create anomalous diffusion models. The computations in the following three subsections are quite similar in structure, but the physical differences in the three cases fundamentally change our initial model from the law of total probability.

3.1. **Random walks with waiting times.**

We pick up from Section 1.2 with the first case, where there is a probability of the particle getting stuck but still taking steps of size h. If we implement the probability of waiting between steps given by ψ, we must consider all past times that the particle came from. Since the size of the step is still bounded to h, we also must take into account the $2n$ directions that these steps can be taken in. Putting this all together, by the law of total probability, we have that

$$u(x,t) = \sum_{m=1}^{\infty}\sum_{k=1}^{n}\psi(m)\frac{u(x+he_k,t-\tau m)+u(x-he_k,t-\tau m)}{2n}.$$

Recall that both $\psi(m), \frac{1}{n}$ both sum to 1 in their respective ranges. Therefore, we can make the following rearrangement:

$$\sum_{m=1}^{\infty}\psi(m)\big[u(x,t)-u(x,t-\tau m)\big]$$
$$= \sum_{m=1}^{\infty}\psi(m)\sum_{k=1}^{n}\frac{u(x+he_k,t-\tau m)+u(x-he_k,t-\tau m)-2u(x,t-\tau m)}{2n}.$$

Now, substitute in the expression for $\psi(m)$ on the left side and make the following manipulations:

$$\sum_{m=1}^{\infty}\frac{c_\alpha\big[u(x,t)-u(x,t-\tau m)\big]}{\tau^{1+\alpha}m^{1+\alpha}}\tau^{1+\alpha}$$
$$= \frac{h^2}{2n}\sum_{m=1}^{\infty}\psi(m)\sum_{k=1}^{n}\frac{u(x+he_k,t-\tau m)+u(x-he_k,t-\tau m)-2u(x,t-\tau m)}{h^2},$$

and rearranging once more gives

$$\sum_{m=1}^{\infty}\frac{u(x,t)-u(x,t-\tau m)}{(\tau m)^{1+\alpha}}\tau$$
$$= \frac{h^2}{2c_\alpha\tau^\alpha n}\sum_{m=1}^{\infty}\psi(m)\sum_{k=1}^{n}\frac{u(x+he_k,t-\tau m)+u(x-he_k,t-\tau m)-2u(x,t-\tau m)}{h^2}.$$

As before, in order to obtain a continuous random walk, we must take the limits as $h,\tau\to 0^+$. We require that

$$\frac{h^2}{2c_\alpha\tau^\alpha n}\to\frac{K_\alpha}{2c_\alpha n}$$

as $h, \tau \to 0^+$ for some $K_\alpha > 0$. On the left-hand side, the limit as $\tau \to 0^+$ is a limit of a Riemann sum, and so after simplifying the right side, we have

$$\int_0^\infty \frac{u(x,t) - u(x,t-r)}{r^{1+\alpha}} dr = \frac{K_\alpha}{2c_\alpha n} \sum_{m=1}^{\infty} \psi(m) \sum_{k=1}^{n} \lim_{h \to 0^+} \frac{u(x + he_k, t) + u(x - he_k, t) - 2u(x,t)}{h^2}.$$

The remaining limit in the sum is the limit of the second-order incremental quotient, which is the second derivative in the k-th coordinate. On the left-hand side, we know that this integral can be manipulated such that the bounds are $-\infty$ and t. Thus,

$$\begin{aligned}\int_{-\infty}^t \frac{u(t) - u(r)}{(t-r)^{1+\alpha}} dr &= \frac{K_\alpha}{2c_\alpha n} \sum_{m=1}^{\infty} \psi(m) \sum_{k=1}^{n} u_{x_k x_k} \\ &= \frac{K_\alpha}{2c_\alpha n} \Delta u \sum_{m=1}^{\infty} \psi(m) \\ &= \frac{K_\alpha}{2c_\alpha n} \Delta u.\end{aligned}$$

Finally, by directly applying (2.4), we conclude with the following result:

Theorem 3.1. *A particle undergoes a continuous random walk with unit space step and arbitrary waiting times between steps. The probability of observing the particle at position $x \in \mathbb{R}^n$ and observing the particle at time $t \in \mathbb{R}$ is given by the following PDE:*

$$(D_{\text{left}})^\alpha u = \frac{K_\alpha}{2c_\alpha n \,|\Gamma(-\alpha)|} \Delta u \tag{3.1}$$

for some $K_\alpha > 0$, where $0 < \alpha < 1$.

3.2. Random walks with jumps.

Next, we investigate the second case listed in Section 1.2, where the particle has a probability of making a jump of arbitrary size while still moving for every time step τ. Due to this requirement in the time coordinate, we lose the consideration of the whole past, but we now allow for the possibility of the particle coming from any point in space. By the law of total probability, we have the following:

$$u(x,t) = \sum_{k \in \mathbb{Z}^n} \phi(k) u(x - hk, t - \tau).$$

Since the sum of $\phi(k)$ over all k is 1, we can make the following rearrangement:

$$\begin{aligned}&\sum_{k \in \mathbb{Z}^n} \phi(k) \big[u(x,t) + u(x + hk, t - \tau) - 2u(x, t - \tau)\big] \\ &\qquad = \sum_{k \in \mathbb{Z}^n} \phi(k) \big[u(x - hk, t - \tau) + u(x + hk, t - \tau) - 2u(x, t - \tau)\big].\end{aligned}$$

Substituting in the expression for $\phi(k)$ on the right side and making necessary adjustments gives that

$$\begin{aligned}&\sum_{k \in \mathbb{Z}^n} \phi(k) \frac{u(x,t) + u(x + hk, t - \tau) - 2u(x, t - \tau)}{\tau} \\ &\qquad = \frac{d_s h^{2s}}{\tau} \sum_{k \in \mathbb{Z}^n} \frac{u(x - hk, t - \tau) + u(x + hk, t - \tau) - 2u(x, t - \tau)}{|hk|^{n+2s}} h^n.\end{aligned}$$

We now must take the limits as $h, \tau \to 0^+$ on both sides. Once again, we make the assumption that the quantity

$$\frac{d_s h^{2s}}{\tau} \to d_s K_s$$

for some $K_s > 0$. In the sum on the left-hand side, taking the limit as $\tau \to 0^+$ gives the left derivative of u. In the sum on the right side, taking the limits as $h, \tau \to 0^+$ is the limit of a Riemann sum, leaving

$$\sum_{k \in \mathbb{Z}^n} \phi(k) D_{\text{left}} u = d_s K_s \int_{\mathbb{R}^n} \frac{u(x+y,t) + u(x-y,t) - 2u(x,t)}{|y|^{n+2s}} dy.$$

The remaining sum on the left side is 1. Finally, by direct comparison with (2.14), we arrive at the following theorem:

Theorem 3.2. *A particle undergoes a continuous random walk with jumps of arbitrary size between steps and no waiting time between steps. The probability of observing the particle at position $x \in \mathbb{R}^n$ and observing the particle at time $t \in \mathbb{R}$ is given by the following PDE:*

$$D_{\text{left}} u = \frac{2 d_s K_s \pi^{\frac{n}{2}} \Gamma(-s)}{4^s \Gamma(\frac{n}{2} + s)} (-\Delta)^s u \tag{3.2}$$

for some $K_s > 0$, where $0 < s < 1$.

3.3. Random walks with waiting times and jumps.

We conclude the modeling of these continuous random walks with the third case from Section 1.2, where there is a probability of the particle getting stuck for some time and another probability of making a jump of arbitrary size. It is interesting to note that given the previous two models, the resulting PDE is exactly what one might expect, as it implements both fractional derivatives and resulting constants. Now, the particle can come from any point in space and any time from the past, and we must implement both probabilities ψ, ϕ. By the law of total probability, we have

$$u(x,t) = \sum_{m=1}^{\infty} \sum_{k \in \mathbb{Z}^n} \psi(m) \phi(k) u(x - hk, t - \tau m).$$

In a similar fashion from the previous subsections, we make the following rearrangement:

$$\begin{aligned} &\sum_{m=1}^{\infty} \sum_{k \in \mathbb{Z}^n} \psi(m) \phi(k) \frac{u(x,t) + u(x+hk, t-\tau m) - 2u(x, t-\tau m)}{\tau^{1+\alpha}} \tau \\ &\quad = \frac{h^{2s}}{t^{\alpha}} \sum_{m=1}^{\infty} \sum_{k \in \mathbb{Z}^n} \psi(m) \phi(k) \frac{u(x-hk, t-\tau m) + u(x+hk, t-\tau m) - 2u(x, t-\tau m)}{|h|^{n+2s}} h^n. \end{aligned}$$

If we substitute the expression for $\psi(m)$ on the left and the expression for $\phi(k)$ on the right, we get

$$\begin{aligned} &\sum_{m=1}^{\infty} \sum_{k \in \mathbb{Z}^n} \phi(k) \frac{u(x,t) + u(x+hk, t-\tau m) - 2u(x, t-\tau m)}{(\tau m)^{1+\alpha}} \tau \\ &\quad = \frac{d_s h^{2s}}{c_\alpha t^{\alpha}} \sum_{m=1}^{\infty} \sum_{k \in \mathbb{Z}^n} \psi(m) \frac{u(x-hk, t-\tau m) + u(x+hk, t-\tau m) - 2u(x, t-\tau m)}{|hk|^{n+2s}} h^n. \end{aligned} \tag{3.3}$$

As $h, \tau \to 0^+$, we once more assume that

$$\frac{d_s h^{2s}}{c_\alpha t^\alpha} \to \frac{d_s K_{\alpha,s}}{c_\alpha}$$

for some $K_{\alpha,s} > 0$. On the left side, we recognize the limit of a Riemann sum as $h, \tau \to 0^+$:

$$\sum_{m=1}^{\infty} \frac{u(x,t) + u(x+hk, t-\tau m) - 2u(x, t-\tau m)}{(\tau m)^{1+\alpha}} \tau \to \int_0^\infty \frac{u(x,t) - u(x, t-r)}{r^{1+\alpha}} dr. \tag{3.4}$$

On the right side, we also have the limit of another Riemann sum as $h, \tau \to 0^+$:

$$\begin{aligned} &\sum_{k \in \mathbb{Z}^n} \frac{u(x-hk, t-\tau m) + u(x+hk, t-\tau m) - 2u(x, t-\tau m)}{|hk|^{n+2s}} h^n \\ &\quad \to \int_{\mathbb{R}^n} \frac{u(x+y,t) + u(x-y,t) - 2u(x,t)}{|y|^{n+2s}} dy. \end{aligned} \tag{3.5}$$

Substituting in all these limits to (3.3) gives that

$$\begin{aligned} &\sum_{k \in \mathbb{Z}^n} \phi(k) \int_0^\infty \frac{u(x,t) - u(x,t-r)}{r^{1+\alpha}} dr \\ &\quad = \sum_{m=1}^{\infty} \psi(m) \frac{d_s K_{\alpha,s}}{c_\alpha} \int_{\mathbb{R}^n} \frac{u(x+y,t) + u(x-y,t) - 2u(x,t)}{|y|^{n+2s}} dy, \end{aligned}$$

and since both of the remaining sums are equal to 1,

$$\int_0^\infty \frac{u(x,t) - u(x,t-r)}{r^{1+\alpha}} dr = \frac{d_s K_{\alpha,s}}{c_\alpha} \int_{\mathbb{R}^n} \frac{u(x+y,t) + u(x-y,t) - 2u(x,t)}{|y|^{n+2s}} dy.$$

Up to respective constants, the integral on the left is the fractional left derivative of order α of u, and the integral on the right is the fractional Laplacian of order s of u, so by (2.4) and (2.14),

$$|\Gamma(-\alpha)| \, (D_{\text{left}})^\alpha u = \frac{d_s K_{\alpha,s}}{c_\alpha} \frac{2\pi^{\frac{n}{2}} \Gamma(-s)}{4^s \Gamma(\frac{n}{2} + s)} (-\Delta)^s u.$$

Rearranging gives us the desired theorem:

Theorem 3.3. *A particle undergoes a continuous random walk with jumps of arbitrary size between steps and arbitrary waiting times between steps. The probability of observing the particle at position $x \in \mathbb{R}^n$ and observing the particle at time $t \in \mathbb{R}$ is given by the following PDE:*

$$(D_{\text{left}})^\alpha u = \frac{2\pi^{\frac{n}{2}} d_s K_{\alpha,s} \Gamma(-s)}{4^s c_\alpha \Gamma(\frac{n}{2} + s) \, |\Gamma(-\alpha)|} (-\Delta)^s u \tag{3.6}$$

for some $K_{\alpha,s} > 0$, where $0 < \alpha, s < 1$.

Notes:

The computations in the first case are drawn from [6], but using ideas in [5], we are able to extend this formula for any spacial dimension $n \geq 1$. For the second case, we reference the ideas of [8] but instead use the formulation involving the second-order incremental quotient. Additionally, we take greater care in keeping track of constants so that there is no confusion with negative signs. Finally, the third case is a combination of the previous two, and the resulting PDE was developed independently.

4. Finding the kernels via discrete derivatives

In the previous section, the process of taking the limits as $h, \tau \to 0^+$ could not have worked better for us. In particular, up to respective constants, it was very nice that the limits of the sums in (3.4) and (3.5) gave us the specific Riemann integrals that define the fractional left derivative and fractional Laplacian. The reason why this convergence was allowed was because of our choices of ψ and ϕ being reciprocal power functions. These definitions were motivated by experimental observations, but how could we have derived our anomalous diffusion models without knowing these beforehand? Can we solve the problem purely mathematically without having to guess what the probability kernels are?

In this final section, we explore the answer to this question by using discrete calculus. We will find that if we derive the fractional derivatives in a discrete sense and wait to take the limit in the derivatives until the very end, we can arrive at the formulas without having to guess the probability kernels that were ψ and ϕ.

Let $\mathbb{Z}_h = \{hj : j \in \mathbb{Z}\}$, and we still let u be Schwartz class. We say that the *restriction of u to the mesh at $j \in \mathbb{Z}$* is $r_h u_j = u(hj)$.

4.1. **Discrete fractional left derivative.**

After we restrict u to the mesh, we can no longer consider the continuous left derivative, so we introduce the discrete left derivative:

Definition 4.1. The *discrete left derivative of u* is defined to be

$$\delta_{\text{left}} r_h u_j = \frac{r_h u_j - r_h u_{j-1}}{h}.$$

As motivated by previous sections, we know we can raise this operator acting on $r_h u_j$ to the $0 < \alpha < 1$ power, so we have the *discrete fractional left derivative of order α of u*:

$$(\delta_{\text{left}})^\alpha r_h u_j = \frac{1}{\Gamma(-\alpha)} \int_0^\infty (e^{-r\delta_{\text{left}}} r_h u_j - r_h u_j) \frac{dr}{r^{1+\alpha}}. \tag{4.1}$$

In Section 2, we found that the operator exponential acting on u results in the continuous convolution of u and some other function. In this case, we assume that the result will be the discrete convolution between $r_h u_j$ some function. Furthermore, we can show that this function is a discrete semigroup:

Proposition 4.2. *Define $v_j : \mathbb{Z}_h \to \mathbb{R}$ such that*

$$r_h v_j(r) = e^{-r\delta_{\text{left}}} r_h u_j := \sum_{m=0}^{\infty} G_m\left(\frac{r}{h}\right) r_h u_{j-m},$$

where $G_m(r) = e^{-r} \frac{r^m}{m!}$. Then, $G_m(\frac{r}{h})$ sums to 1, and $\{v_{j_r}\}$ is a discrete semigroup.

Proof: First, we have the following:

$$\begin{aligned}
\sum_{m=0}^{\infty} G_m\left(\frac{r}{h}\right) &= \sum_{m=0}^{\infty} e^{-\frac{r}{h}} \frac{(\frac{r}{h})^m}{m!} \\
&= e^{-\frac{r}{h}} \sum_{m=0}^{\infty} \frac{(\frac{r}{h})^m}{m!} \\
&= e^{-\frac{r}{h}} e^{\frac{r}{h}} \\
&= 1.
\end{aligned}$$

Now, to show that $\{v_{j_r}\}$ is a discrete semigroup, it suffices to show that v_j is the solution to

$$\begin{cases} \partial_r r_h v_j = -\delta_{\text{left}} r_h v_j & r \neq 0 \\ r_h v_j = r_h u_j & r = 0 \end{cases}.$$

We perform the following differentiation. Note that the derivative and sum can be interchanged because the series is absolutely convergent:

$$\begin{aligned}
\partial_r r_h v_j &= -\sum_{m=0}^{\infty} \frac{1}{h} e^{-\frac{r}{h}} \frac{(\frac{r}{h})^m}{m!} r_h u_{j-m} + \sum_{m=1}^{\infty} \frac{m}{h} e^{-\frac{r}{h}} \frac{(\frac{r}{h})^{m-1}}{m!} r_h u_{j-m} \\
&= -\frac{1}{h}\left(\sum_{m=0}^{\infty} e^{-\frac{r}{h}} \frac{(\frac{r}{h})^m}{m!} r_h u_{j-m} - \sum_{m=1}^{\infty} e^{-\frac{r}{h}} \frac{(\frac{r}{h})^{m-1}}{(m-1)!} r_h u_{j-m} \right) \\
&= -\frac{1}{h}\left(\sum_{m=0}^{\infty} e^{-\frac{r}{h}} \frac{(\frac{r}{h})^m}{m!} r_h u_{j-m} - \sum_{m=0}^{\infty} e^{-\frac{r}{h}} \frac{(\frac{r}{h})^{m}}{m!} r_h u_{j-(m+1)} \right) \\
&= -\frac{1}{h}\left(\sum_{m=0}^{\infty} e^{-\frac{r}{h}} \frac{(\frac{r}{h})^m}{m!} r_h u_{j-m} - \sum_{m=0}^{\infty} e^{-\frac{r}{h}} \frac{(\frac{r}{h})^{m}}{m!} r_h u_{(j-1)-m} \right) \\
&= -\frac{r_h v_j - r_h v_{j-1}}{h} \\
&= -\delta_{\text{left}} r_h v_j.
\end{aligned}$$

Also, at $r = 0$, we have that

$$\begin{aligned}
r_h v_j(0) &= \sum_{m=0}^{\infty} e^{-\frac{0}{h}} \frac{0^m}{m!} r_h u_{j-m} \\
&= r_h u_j + \sum_{m=1}^{\infty} \frac{0^m}{m!} r_h u_{j-m} \\
&= r_h u_j.
\end{aligned}$$

■

Therefore, we can substitute this expression back into (4.1):

$$\begin{aligned}
(\delta_{\text{left}})^\alpha r_h u_j &= \frac{1}{\Gamma(-\alpha)} \int_0^\infty \left(\sum_{m=0}^{\infty} e^{-\frac{r}{h}} \frac{(\frac{r}{h})^m}{m!} r_h u_{j-m} - r_h u_j \right) \frac{dr}{r^{1+\alpha}} \\
&= \frac{1}{\Gamma(-\alpha)} \int_0^\infty \sum_{m=0}^{\infty} e^{-\frac{r}{h}} \frac{(\frac{r}{h})^m}{m!} (r_h u_{j-m} - r_h u_j) \frac{dr}{r^{1+\alpha}}.
\end{aligned}$$

Fubini's theorem applies to swap the integral and sum since the integrand is absolutely convergent, so

$$\begin{aligned}
(\delta_{\text{left}})^\alpha r_h u_j &= \frac{1}{\Gamma(-\alpha)} \sum_{m=0}^{\infty} \frac{r_h u_{j-m} - r_h u_j}{m!} \int_0^\infty e^{-\frac{r}{h}} \left(\frac{r}{h}\right)^m \frac{dr}{r^{1+\alpha}} \\
&= \frac{1}{h^\alpha \Gamma(-\alpha)} \sum_{m=0}^{\infty} \frac{r_h u_{j-m} - r_h u_j}{m!} \int_0^\infty e^{-\frac{r}{h}} \left(\frac{r}{h}\right)^m \left(\frac{h}{r}\right)^\alpha \frac{dr}{r}.
\end{aligned}$$

Make the change of variables $\tau = \frac{r}{h}$, so $dr = hd\tau$, and thus

$$
\begin{aligned}
(\delta_{\text{left}})^{\alpha} r_h u_j &= \frac{1}{h^{\alpha}\Gamma(-\alpha)} \sum_{m=0}^{\infty} \frac{r_h u_{j-m} - r_h u_j}{\Gamma(m+1)} \int_0^{\infty} e^{-\tau} \tau^m \frac{1}{\tau^{\alpha}} \frac{h d\tau}{h\tau} \\
&= \frac{1}{h^{\alpha}\Gamma(-\alpha)} \sum_{m=0}^{\infty} \frac{r_h u_{j-m} - r_h u_j}{\Gamma(m+1)} \int_0^{\infty} e^{-\tau} \tau^{m-\alpha} \frac{d\tau}{\tau} \\
&= \frac{1}{h^{\alpha}\Gamma(-\alpha)} \sum_{m=0}^{\infty} \frac{\Gamma(m-\alpha)}{\Gamma(m+1)} (r_h u_{j-m} - r_h u_j) \\
&= \frac{1}{h^{\alpha}\,|\Gamma(-\alpha)|} \sum_{m=0}^{\infty} \frac{\Gamma(m-\alpha)}{\Gamma(m+1)} (r_h u_j - r_h u_{j-m}).
\end{aligned}
$$

This serves as a suitable expression, so we summarize below:

Theorem 4.3. *Let $u : \mathbb{R} \to \mathbb{R}$ be a sufficiently smooth function restricted to the mesh and $0 < \alpha < 1$. Then, the discrete fractional left derivative of order α of u is given by*

$$
(\delta_{left})^{\alpha} r_h u_j = \frac{1}{h^{\alpha}\,|\Gamma(-\alpha)|} \sum_{m=0}^{\infty} \frac{\Gamma(m-\alpha)}{\Gamma(m+1)} (r_h u_j - r_h u_{j-m}). \tag{4.2}
$$

4.2. **Convergence of the discrete to continuous fractional left derivative.**

Now that we have an expression for the discrete fractional left derivative of u, one would hope that taking the limit as $h \to 0^+$ would give us the Riemann integral in (2.3). However, this is not immediately apparent, because the kernel in (4.2) does not even lead the way to a Riemann sum, unlike the kernel in the sum of (3.4). Since the limit cannot be calculated directly despite knowing what it should be, we will instead compare (4.2) to the restriction of (2.3) to the mesh. If we can show that the size of their difference can be controlled by $h^{1-\alpha}$, then taking the limit as $h \to 0^+$ will prove that they are indeed equal.

Theorem 4.4. *Let $0 < \alpha < 1$, and let $u : \mathbb{R} \to \mathbb{R}$ be sufficiently smooth. Then,*

$$
\left| r_h\big((D_{\text{left}})^{\alpha} u\big)_j - (\delta_{\text{left}})^{\alpha} r_h u_j \right| \leq \frac{K_{\alpha} h^{1-\alpha} \, \|D_{\text{left}} u\|_{L^{\infty}(\mathbb{R})}}{|\Gamma(-\alpha)|}.
$$

for some $K_{\alpha} > 0$. Moreover, it follows that

$$
\lim_{h \to 0^+} \left| r_h\big((D_{\text{left}})^{\alpha} u\big)_j - (\delta_{\text{left}})^{\alpha} r_h u_j \right| = 0.
$$

Proof: Since $j \in \mathbb{Z}$ simply results in translation, then without loss of generality, we may let $j = 0$. First, note that (2.3) restricted to $\mathbb{Z}_h$ is given by

$$
\begin{aligned}
r_h\big((D_{\text{left}})^{\alpha} u\big)_0 &= \frac{1}{|\Gamma(-\alpha)|} \sum_{m=0}^{\infty} \int_{mh}^{(m+1)h} \frac{u(0) - u(-r)}{r^{1+\alpha}} dr \\
&= \frac{1}{|\Gamma(-\alpha)|} \sum_{m=0}^{\infty} \big[u(0) - u(-mh)\big] \int_{mh}^{(m+1)h} \frac{1}{r^{1+\alpha}} dr \\
&\quad + \frac{1}{|\Gamma(-\alpha)|} \sum_{m=0}^{\infty} \int_{mh}^{(m+1)h} \frac{u(-mh) - u(-r)}{r^{1+\alpha}} dr \\
&=: A + B.
\end{aligned}
$$

We also have that

$$(\delta_{\text{left}})^{\alpha} r_h u_0 = \frac{1}{h^{\alpha}\,|\Gamma(-\alpha)|} \sum_{m=0}^{\infty} \frac{\Gamma(m-\alpha)}{\Gamma(m+1)} [u(0) - u(-mh)] =: C.$$

We start by bounding $|A - C|$:

$$|A - C| \le \frac{1}{|\Gamma(-\alpha)|} \sum_{m=0}^{\infty} |u(0) - u(-mh)| \left| \int_{mh}^{(m+1)h} \frac{1}{r^{1+\alpha}} dr - \frac{\Gamma(m-\alpha)}{h^{\alpha}\Gamma(m+1)} \right|.$$

In the integral, start by letting $r = h\tau$, so $dr = hd\tau$, and

$$\begin{aligned} |A - C| &\le \frac{1}{|\Gamma(-\alpha)|} \sum_{m=0}^{\infty} |u(0) - u(-mh)| \left| \int_{m}^{m+1} \frac{h}{(h\tau)^{1+\alpha}} d\tau - \frac{\Gamma(m-\alpha)}{h^{\alpha}\Gamma(m+1)} \right| \\ &= \frac{1}{h^{\alpha}\,|\Gamma(-\alpha)|} \sum_{m=0}^{\infty} |u(0) - u(-mh)| \left| \int_{m}^{m+1} \frac{1}{\tau^{1+\alpha}} d\tau - \frac{\Gamma(m-\alpha)}{\Gamma(m+1)} \right|. \end{aligned}$$

Since u is left-differentiable on $(0, mh)$, it follows that $|u(0) - u(-mh)| \le (mh)\,\|D_{\text{left}} u\|_{L^{\infty}(\mathbb{R})}$, so

$$|A - C| \le \frac{h^{1-\alpha}\,\|D_{\text{left}} u\|_{L^{\infty}(\mathbb{R})}}{|\Gamma(-\alpha)|} \sum_{m=0}^{\infty} m \left| \int_{m}^{m+1} \frac{1}{\tau^{1+\alpha}} d\tau - \frac{\Gamma(m-\alpha)}{\Gamma(m+1)} \right|.$$

Note that the expression cancels out when $m = 0$, so we instead sum over all $m \ge 1$. In the case when $m = 1$, we see that

$$\begin{aligned} \left| \int_{1}^{2} \frac{1}{\tau^{1+\alpha}} d\tau - \frac{\Gamma(1-\alpha)}{\Gamma(2)} \right| &= \left| \frac{1}{\alpha} - \frac{2^{-\alpha}}{\alpha} + \frac{\alpha^2\Gamma(1-\alpha)}{-\alpha^2} \right| \\ &< \left| \frac{1}{2\alpha} + \frac{\alpha^2\Gamma(-\alpha)}{\alpha} \right| \\ &< \frac{1}{2\alpha} + \frac{|\Gamma(-\alpha)|}{\alpha}. \end{aligned}$$

We now consider $m \ge 2$. Since the length of the integral is 1, we may make the following manipulation:

$$\sum_{m=2}^{\infty} m \left| \int_{m}^{m+1} \frac{1}{\tau^{1+\alpha}} d\tau - \frac{\Gamma(m-\alpha)}{\Gamma(m+1)} \right| = \sum_{m=2}^{\infty} m \left| \int_{m}^{m+1} \left(\frac{1}{\tau^{1+\alpha}} - \frac{1}{m^{1+\alpha}} \right) d\tau + \frac{1}{m^{1+\alpha}} - \frac{\Gamma(m-\alpha)}{\Gamma(m+1)} \right|.$$

By applying the mean value theorem again, we have that

$$\begin{aligned} \frac{1}{\tau^{1+\alpha}} - \frac{1}{m^{1+\alpha}} &\le \sup_{m<\tau<m+1} \frac{d}{d\tau}\left[\frac{1}{\tau^{1+\alpha}}\right] \\ &= (-1-\alpha) \sup_{m<\tau<m+1} \left[\frac{1}{\tau^{2+\alpha}}\right] \\ &= \frac{-1-\alpha}{m^{2+\alpha}}, \end{aligned}$$

and therefore,

$$\sum_{m=2}^{\infty} m \left| \int_{m}^{m+1} \frac{1}{\tau^{1+\alpha}} d\tau - \frac{\Gamma(m-\alpha)}{\Gamma(m+1)} \right| \le \sum_{m=2}^{\infty} m \left| \int_{m}^{m+1} \frac{-1-\alpha}{m^{2+\alpha}} d\tau + \frac{1}{m^{1+\alpha}} - \frac{\Gamma(m-\alpha)}{\Gamma(m+1)} \right|$$

$$= \sum_{m=2}^{\infty} m \left| \frac{-1-\alpha}{m^{2+\alpha}} + \frac{1}{m^{1+\alpha}} - \frac{\Gamma(m-\alpha)}{\Gamma(m+1)} \right|$$
$$= \sum_{m=2}^{\infty} m \left| \frac{1+\alpha}{m^{2+\alpha}} - \frac{1}{m^{1+\alpha}} + \frac{\Gamma(m-\alpha)}{m\Gamma(m)} \right|$$
$$\leq \sum_{m=2}^{\infty} \left| \frac{1+\alpha}{m^{1+\alpha}} - \frac{1}{m^{\alpha}} + \frac{\Gamma(m-\alpha)}{\Gamma(m)} \right|$$
$$\leq \sum_{m=2}^{\infty} \frac{1+\alpha}{m^{1+\alpha}} + \sum_{m=2}^{\infty} \left| \frac{\Gamma(m-\alpha)}{\Gamma(m)} - \frac{1}{m^{\alpha}} \right|.$$

The first sum is a p-series that converges to some $C_\alpha > 0$ since $\alpha > 0$. In the second sum, we make the shift of $k = m - 1$, so

$$\sum_{m=2}^{\infty} m \left| \int_m^{m+1} \frac{1}{\tau^{1+\alpha}} d\tau - \frac{\Gamma(m-\alpha)}{\Gamma(m+1)} \right| \leq C_\alpha + \sum_{k=1}^{\infty} \left| \frac{\Gamma(k+1-\alpha)}{\Gamma(k+1)} - \frac{1}{(k+1)^{\alpha}} \right|.$$

By our corollary to Gautschi's inequality (A.8), we have that

$$\sum_{m=2}^{\infty} m \left| \int_m^{m+1} \frac{1}{\tau^{1+\alpha}} d\tau - \frac{\Gamma(m-\alpha)}{\Gamma(m+1)} \right| \leq C_\alpha + \sum_{k=1}^{\infty} \left(\frac{1}{k^{\alpha}} - \frac{1}{(k+1)^{\alpha}} \right).$$

This remaining sum is a telescoping series that converges to 1, so

$$\sum_{m=2}^{\infty} m \left| \int_m^{m+1} \frac{1}{\tau^{1+\alpha}} d\tau - \frac{\Gamma(m-\alpha)}{\Gamma(m+1)} \right| \leq C_\alpha + 1,$$

and ultimately,

$$|A - C| \leq \frac{h^{1-\alpha} \|D_{\text{left}} u\|_{L^\infty(\mathbb{R})}}{|\Gamma(-\alpha)|} \Big(\frac{1}{2\alpha} + \frac{|\Gamma(-\alpha)|}{\alpha} + C_\alpha + 1 \Big).$$

For $|B|$, we begin by applying the mean value theorem once again:

$$
\begin{aligned}
|B| &\leq \frac{1}{|\Gamma(-\alpha)|} \left| \sum_{m=0}^{\infty} \int_{mh}^{(m+1)h} \frac{u(-mh) - u(-r)}{r^{1+\alpha}} dr \right| \\
&\leq \frac{1}{|\Gamma(-\alpha)|} \sum_{m=0}^{\infty} \left| \int_{mh}^{(m+1)h} \frac{(r - mh) \|D_{\text{left}} u\|_{L^\infty(\mathbb{R})}}{r^{1+\alpha}} dr \right| \\
&\leq \frac{\|D_{\text{left}} u\|_{L^\infty(\mathbb{R})}}{|\Gamma(-\alpha)|} \left| \int_0^h \frac{r}{r^{1+\alpha}} dr \right| + \frac{\|D_{\text{left}} u\|_{L^\infty(\mathbb{R})}}{|\Gamma(-\alpha)|} \sum_{m=1}^{\infty} \int_{mh}^{(m+1)h} \left| \frac{(m+1)h - mh}{(mh)^{1+\alpha}} \right| dr \\
&\leq \frac{\|D_{\text{left}} u\|_{L^\infty(\mathbb{R})}}{|\Gamma(-\alpha)|} \left| \int_0^h \frac{1}{r^{\alpha}} dr \right| + \frac{\|D_{\text{left}} u\|_{L^\infty(\mathbb{R})}}{h^{\alpha} |\Gamma(-\alpha)|} \sum_{m=1}^{\infty} \int_{mh}^{(m+1)h} \frac{1}{m^{1+\alpha}} dr \qquad (4.3) \\
&= \frac{h^{1-\alpha} \|D_{\text{left}} u\|_{L^\infty(\mathbb{R})}}{(1-\alpha) |\Gamma(-\alpha)|} + \frac{\|D_{\text{left}} u\|_{L^\infty(\mathbb{R})}}{h^{\alpha} |\Gamma(-\alpha)|} \sum_{m=1}^{\infty} \frac{h}{m^{1+\alpha}} \\
&= \frac{h^{1-\alpha} \|D_{\text{left}} u\|_{L^\infty(\mathbb{R})}}{|\Gamma(-\alpha)|} \Big(\frac{1}{1-\alpha} + \sum_{m=1}^{\infty} \frac{1}{m^{1+\alpha}} \Big),
\end{aligned}
$$

and this sum also converges to some $D_\alpha > 0$ for $\alpha > 0$. Finally, we have the following:

$$\begin{aligned}\left|r_h\big((D_{\text{left}})^\alpha u\big)_0 - (\delta_{\text{left}})^\alpha r_h u_0\right| &\le \frac{h^{1-\alpha}\,\|D_{\text{left}}u\|_{L^\infty(\mathbb{R})}}{|\Gamma(-\alpha)|}\left(\frac{1}{2\alpha} + \frac{|\Gamma(-\alpha)|}{\alpha} + C_\alpha + 1 + \frac{1}{1-\alpha} + D_\alpha\right)\\ &\le \frac{K_\alpha h^{1-\alpha}\,\|D_{\text{left}}u\|_{L^\infty(\mathbb{R})}}{|\Gamma(-\alpha)|},\end{aligned}$$

and taking the limit as $h \to 0^+$ implies that

$$\lim_{h\to 0^+}\left|r_h\big((D_{\text{left}})^\alpha u\big)_0 - (\delta_{\text{left}})^\alpha r_h u_0\right| = 0.$$

■

4.3. Discrete fractional Laplacian.

We now prove similar results for the discrete fractional Laplacian. Due to the mathematical complexity that will follow, we shall only work in the spatial dimension $n = 1$. Additionally, we will use the formulation of the fractional Laplacian in terms of the first-order incremental quotient. The construction of the discrete fractional Laplacian also relies on properties of the modified Bessel function of the first kind, and these can be found in Section A.2.

Definition 4.5. The *discrete second derivative of u* is

$$-\Delta_h r_h u_j = \frac{r_h u_{j+1} + r_h u_{j-1} - 2r_h u_j}{-h^2}.$$

Note that since the spacial dimension is $n = 1$, this discrete second derivative of u is the discrete Laplacian of u. Once more, we can raise this object to the $0 < s < 1$ power, giving us a *discrete fractional Laplacian of order s of u*:

$$(-\Delta_h)^s r_h u_j = \frac{1}{\Gamma(-s)}\int_0^\infty (e^{t\Delta_h} r_h u_j - r_h u_j)\frac{dt}{t^{1+s}}. \tag{4.4}$$

To determine the meaning of $e^{t\Delta_h} r_h u_j$, we choose a particular definition for it and show that it is a discrete semigroup.

Proposition 4.6. *Define $v_j : \mathbb{Z}_h \to \mathbb{R}$ such that*

$$r_h v_j(t) = e^{t\Delta_h} r_h u_j := \sum_{k\in\mathbb{Z}} G_k\Big(\frac{t}{h^2}\Big) r_h u_{j-k},$$

where $G_k(t) = e^{-2t} I_k(2t)$, and I_k is the modified Bessel function of the first kind. Then, $r_h v_j(t)$ is well-defined, and $\{v_{j_t}\}$ is a discrete semigroup.

Proof: Since u is bounded, we immediately have that

$$\begin{aligned}|r_h v_j(t)| &\le \|u\|_{L^\infty(\mathbb{R})} \sum_{k\in\mathbb{Z}} e^{-\frac{2t}{h^2}} I_k\Big(\frac{2t}{h^2}\Big)\\ &= \|u\|_{L^\infty(\mathbb{R})},\end{aligned}$$

since the leftover sum is equal to 1 by (A.11). Next, we make the following differentiation, which is allowed due to the absolute convergence of the series.

$$\partial_t r_h v_j = \sum_{k\in\mathbb{Z}} \frac{\partial}{\partial t}\left[e^{-\frac{2t}{h^2}} I_k\Big(\frac{2t}{h^2}\Big)\right] r_h u_{j-k}$$

$$= \frac{1}{h^2}\sum_{k\in\mathbb{Z}} e^{-\frac{2t}{h^2}}\left[I_{k+1}\Big(\frac{2t}{h^2}\Big) + I_{k-1}\Big(\frac{2t}{h^2}\Big) - 2I_k\Big(\frac{2t}{h^2}\Big)\right] r_h u_{j-k}$$
$$= \frac{1}{h^2}\sum_{k\in\mathbb{Z}} e^{-\frac{2t}{h^2}} I_k\Big(\frac{2t}{h^2}\Big) r_h u_{(j+1)-k} + \frac{1}{h^2}\sum_{k\in\mathbb{Z}} e^{-\frac{2t}{h^2}} I_k\Big(\frac{2t}{h^2}\Big) r_h u_{(j-1)-k}$$
$$- \frac{2}{h^2}\sum_{k\in\mathbb{Z}} e^{-\frac{2t}{h^2}} I_k\Big(\frac{2t}{h^2}\Big) r_h u_{j-k}$$
$$= \frac{r_h u_{j+1} + r_h u_{j-1} - 2 r_h u_j}{h^2}$$
$$= \Delta_h r_h v_j,$$

where the second line is due to (A.10). Also, $r_h v_j(0) = r_h u_j$ by direct substitution. ■

Therefore, we have shown that v_j is a solution to

$$\begin{cases} \partial_t r_h v_j = \Delta_h r_h v_j & r \neq 0 \\ r_h v_j = r_h u_j & r = 0 \end{cases},$$

and so we can substitute back into (4.4):

$$(-\Delta_h)^s r_h u_j = \frac{1}{\Gamma(-s)}\int_0^\infty \left(\sum_{k\in\mathbb{Z}} G_k\Big(\frac{t}{h^2}\Big) r_h u_{j-k} - r_h u_j\right)\frac{dt}{t^{1+s}}$$
$$= \frac{1}{\Gamma(-s)}\int_0^\infty \sum_{k\in\mathbb{Z}} G_k\Big(\frac{t}{h^2}\Big)(r_h u_{j-k} - r_h u_j)\frac{dt}{t^{1+s}},$$

since the sum in the integrand is equal to 1. Fubini applies once again to give us

$$= \frac{1}{\Gamma(-s)}\sum_{k\in\mathbb{Z}}(r_h u_{j-k} - r_h u_j)\int_0^\infty G_k\Big(\frac{t}{h^2}\Big)\frac{dt}{t^{1+s}}$$
$$= \frac{1}{\Gamma(-s)}\sum_{k\in\mathbb{Z}}(r_h u_{j-k} - r_h u_j)\int_0^\infty e^{-\frac{2t}{h^2}} I_k\Big(\frac{2t}{h^2}\Big)\frac{dt}{t^{1+s}}$$
$$= \frac{4^s\Gamma(\frac{1}{2}+s)}{h^{2s}\sqrt{\pi}\Gamma(-s)}\sum_{k\in\mathbb{Z}}\frac{\Gamma(|k|-s)}{\Gamma(|k|+1+s)}(r_h u_{j-k} - r_h u_j),$$

where the last line follows from the integral identity (A.12). We can pause here and summarize:

Theorem 4.7. *Let $u : \mathbb{R} \to \mathbb{R}$ be a sufficiently smooth function restricted to the mesh and $0 < s < 1$. Then, the discrete fractional Laplacian of order s of u is given by*

$$(-\Delta_h)^s r_h u_j = \frac{4^s\Gamma(\frac{1}{2}+s)}{h^{2s}\sqrt{\pi}\Gamma(-s)}\sum_{k\in\mathbb{Z}}\frac{\Gamma(|k|-s)}{\Gamma(|k|+1+s)}(r_h u_{j-k} - r_h u_j). \tag{4.5}$$

4.4. **Convergence of the discrete to continuous fractional Laplacian.**

Again, if we take the limit as $h \to 0^+$, this does not give us a Riemann integral on the right-hand side. However, if we restrict the original fractional Laplacian to $\mathbb{Z}_h$ and compare it with the discrete fractional Laplacian above, we can see that the difference between the two is controlled by some positive power of h. Due to the form of the discrete fractional Laplacian that we just derived, we have two cases for this final result:

Theorem 4.8. *Let $u : \mathbb{R} \to \mathbb{R}$ be sufficiently smooth. If $0 < s < \frac{1}{2}$, then,*

$$\left| r_h\big((-\Delta)^s u\big)_j - (-\Delta_h)^s (r_h u_j) \right| \leq \frac{4^s \Gamma(\frac{1}{2}+s) K_s h^{1-2s} \left\|Du\right\|_{L^\infty(\mathbb{R})}}{\sqrt{\pi}\left|\Gamma(-s)\right|}$$

for some $K_s > 0$. If $\frac{1}{2} < s < 1$, then

$$\left| r_h\big((-\Delta)^s u\big)_j - (-\Delta_h)^s (r_h u_j) \right| \leq \frac{4^s \Gamma(\frac{1}{2}+s) K_s h^{2-2s} \left\|Du\right\|_{L^\infty(\mathbb{R})}}{\sqrt{\pi}\left|\Gamma(-s)\right|}$$

for some $K_s > 0$. Moreover,

$$\lim_{h\to 0^+} \left| r_h\big((-\Delta)^s u\big)_j - (-\Delta_h)^s (r_h u_j) \right| = 0.$$

Proof: We only prove the case for $0 < s < \frac{1}{2}$, and we will find that the some of the following calculations are similar or identical to those of the corresponding proof for the fractional left derivative. By the same reasoning as before, without loss of generality, let $j = 0$. If we use the formulation of the fractional Laplacian using just the first-order incremental quotient, then we lose the factor of $\frac{1}{2}$ in (2.14) and instead have

$$(-\Delta)^s u(x) = \frac{4^s\Gamma(\frac{1}{2}+s)}{\sqrt{\pi}\Gamma(-s)} \int_{\mathbb{R}} \frac{u(x-y)-u(x)}{|y|^{1+2s}} dy,$$

and restricting this to the mesh gives us

$$\begin{aligned} r_h\big((-\Delta)^s u\big)_0 &= \frac{4^s\Gamma(\frac{1}{2}+s)}{\sqrt{\pi}\Gamma(-s)} \sum_{k\in\mathbb{Z}} \int_{kh}^{(k+1)h} \frac{u(-y)-u(0)}{|y|^{1+2s}} dy \\ &= \frac{4^s\Gamma(\frac{1}{2}+s)}{\sqrt{\pi}\Gamma(-s)} \Bigg[\int_{-h}^{h} \frac{u(-y)-u(0)}{|y|^{1+2s}} dy + \sum_{k=1}^{\infty} \int_{kh}^{(k+1)h} \frac{u(-y)-u(0)}{|y|^{1+2s}} dy \\ &\quad + \sum_{k=1}^{\infty} \int_{-(k+1)h}^{-kh} \frac{u(-y)-u(0)}{|y|^{1+2s}} dy \Bigg] \\ &=: S_0 + S_1 + S_2. \end{aligned}$$

Note that we can decompose $(-\Delta_h)^s r_h u_0$ in a similar way:

$$\begin{aligned} (-\Delta_h)^s r_h u_0 &= \frac{4^s\Gamma(\frac{1}{2}+s)}{h^{2s}\sqrt{\pi}\Gamma(-s)} \Bigg[\sum_{k=1}^{\infty} \frac{\Gamma(k-s)}{\Gamma(k+1+s)} \big[u(-hk)-u(0)\big] \\ &\quad + \sum_{k=1}^{\infty} \frac{\Gamma(k-s)}{\Gamma(k+1+s)} \big[u(hk)-u(0)\big] \Bigg] \\ &=: C_1 + C_2. \end{aligned}$$

Next, we break up S_1 as we did in the previous corresponding proof:

$$\begin{aligned} S_1 &= \frac{4^s\Gamma(\frac{1}{2}+s)}{\sqrt{\pi}\Gamma(-s)} \Bigg[\sum_{k=1}^{\infty} \big[u(-hk)-u(0)\big] \int_{kh}^{(k+1)h} \frac{1}{|y|^{1+2s}} dy + \sum_{k=1}^{\infty} \int_{kh}^{(k+1)h} \frac{u(-y)-u(-hk)}{|y|^{1+2s}} dy \Bigg] \\ &=: A_1 + B_1. \end{aligned}$$

Make the change of variables of negating y for S_2 to get

$$S_2 = -\frac{4^s\Gamma(\frac{1}{2}+s)}{\sqrt{\pi}\Gamma(-s)} \sum_{k=1}^{\infty} \int_{(k+1)h}^{kh} \frac{u(y)-u(0)}{|y|^{1+2s}} dy$$

$$= \frac{4^s\Gamma(\frac{1}{2}+s)}{\sqrt{\pi}\Gamma(-s)} \sum_{k=1}^{\infty} \int_{kh}^{(k+1)kh} \frac{u(y)-u(0)}{y^{1+2s}} dy,$$

and we can do the same decomposition for S_2:

$$S_2 = \frac{4^s\Gamma(\frac{1}{2}+s)}{\sqrt{\pi}\Gamma(-s)} \Bigg[\sum_{k=1}^{\infty} \big[u(hk)-u(0)\big] \int_{kh}^{(k+1)h} \frac{1}{y^{1+2s}} dy + \sum_{k=1}^{\infty} \int_{kh}^{(k+1)h} \frac{u(y)-u(hk)}{y^{1+2s}} dy \Bigg]$$
$$=: A_2 + B_2.$$

Therefore, we have that

$$\left| r_h\big((-\Delta)^s u\big)_0 - (-\Delta_h)^s (r_h u_0) \right| = |S_0 + A_1 + B_1 + A_2 + B_2 - C_1 - C_2|$$
$$\leq |S_0| + |A_1 - C_1| + |A_2 - C_2| + |B_1| + |B_2| .$$

If we can show that this sum is controlled above by the factor h^{1-2s}, then we are done. For S_0, by the mean value theorem, we quickly see that

$$|S_0| \leq \frac{4^s\Gamma(\frac{1}{2}+s)}{\sqrt{\pi}\,|\Gamma(-s)|} \int_{-h}^{h} \frac{|u(-y)-u(0)|}{|y|^{1+2s}} dy$$
$$\leq \frac{4^s\Gamma(\frac{1}{2}+s)\,\|Du\|_{L^\infty(\mathbb{R})}}{\sqrt{\pi}\,|\Gamma(-s)|} \int_{-h}^{h} |y|^{-2s}\, dy$$
$$= \frac{2(4^s)\Gamma(\frac{1}{2}+s)\,\|Du\|_{L^\infty(\mathbb{R})}}{\sqrt{\pi}\,|\Gamma(-s)|} \int_{0}^{h} y^{-2s} dy$$
$$= \frac{2(4^s)\Gamma(\frac{1}{2}+s)\,\|Du\|_{L^\infty(\mathbb{R})}\, h^{1-2s}}{\sqrt{\pi}\,|\Gamma(-s)|\,(1-2s)}.$$

For both $|B_1|, |B_2|$, the computation to bound these is the exact same process as in (4.3), so we immediately have that

$$|B_1| + |B_2| \leq \frac{2(4^s)\,\|Du\|_{L^\infty(\mathbb{R})}\,\Gamma(\frac{1}{2}+s)h^{1-2s}}{\sqrt{\pi}\,|\Gamma(-s)|} \left(\frac{1}{1-2s} + \sum_{k=1}^{\infty} \frac{1}{k^{1+2s}} \right),$$

and the series converges to some $D_s > 0$ since $s > 0$. Furthermore, bounding $|A_1 - C_1|$ is very similar as well, but there is a slight difference towards the end the process due to a different ratio of gamma functions. To begin, we have that

$$|A_1 - C_1| \leq \frac{4^s\Gamma(\frac{1}{2}+s)}{\sqrt{\pi}\,|\Gamma(-s)|} \sum_{k=1}^{\infty} |u(-hk)-u(0)| \left| \int_{kh}^{(k+1)h} \frac{1}{y^{1+2s}} dy - \frac{\Gamma(k-s)}{h^{2s}\Gamma(k+1+s)} \right|$$
$$\leq \frac{4^s\,\|Du\|_{L^\infty(\mathbb{R})}\,\Gamma(\frac{1}{2}+s)h}{\sqrt{\pi}\,|\Gamma(-s)|} \sum_{k=1}^{\infty} k \left| \int_{kh}^{(k+1)h} \frac{1}{y^{1+2s}} dy - \frac{\Gamma(k-s)}{h^{2s}\Gamma(k+1+s)} \right|.$$

Let $y = hz$ in the integral, so

$$|A_1 - C_1| \leq \frac{4^s\,\|Du\|_{L^\infty(\mathbb{R})}\,\Gamma(\frac{1}{2}+s)h^{1-2s}}{\sqrt{\pi}\,|\Gamma(-s)|} \sum_{k=1}^{\infty} k \left| \int_{k}^{k+1} \frac{1}{z^{1+2s}} dz - \frac{\Gamma(k-s)}{\Gamma(k+1+s)} \right|$$
$$= \frac{4^s\,\|Du\|_{L^\infty(\mathbb{R})}\,\Gamma(\frac{1}{2}+s)h^{1-2s}}{\sqrt{\pi}\,|\Gamma(-s)|}$$
$$\cdot \sum_{k=1}^{\infty} k \left| \int_{k}^{k+1} \Big(\frac{1}{z^{1+2s}} - \frac{1}{k^{1+2s}} \Big) dz + \frac{1}{k^{1+2s}} - \frac{\Gamma(k-s)}{\Gamma(k+1+s)} \right|.$$

As before, by applying the mean value theorem to the integrand and evaluating the integral, we get

$$|A_1 - C_1| \leq \frac{4^s \, \|Du\|_{L^\infty(\mathbb{R})} \, \Gamma(\frac{1}{2}+s)h^{1-2s}}{\sqrt{\pi}\,|\Gamma(-s)|} \sum_{k=1}^{\infty} k \left| \frac{-1-2s}{k^{2+2s}} + \frac{1}{k^{1+2s}} - \frac{\Gamma(k-s)}{\Gamma(k+1+s)} \right|$$

$$\leq \frac{4^s \, \|Du\|_{L^\infty(\mathbb{R})} \, \Gamma(\frac{1}{2}+s)h^{1-2s}}{\sqrt{\pi}\,|\Gamma(-s)|} \sum_{k=1}^{\infty} \frac{1+2s}{k^{1+2s}} + \sum_{k=1}^{\infty} k \left| \frac{\Gamma(k-s)}{\Gamma(k+1+s)} - \frac{1}{k^{1+2s}} \right|.$$

Applying (A.9) on the rightmost summand gives us

$$|A_1 - C_1| \leq \frac{4^s \, \|Du\|_{L^\infty(\mathbb{R})} \, \Gamma(\frac{1}{2}+s)h^{1-2s}}{\sqrt{\pi}\,|\Gamma(-s)|} \sum_{k=1}^{\infty} \frac{1+2s}{k^{1+2s}} + \sum_{k=1}^{\infty} \frac{kE_s}{k^{2+2s}}$$

$$= \frac{4^s \, \|Du\|_{L^\infty(\mathbb{R})} \, \Gamma(\frac{1}{2}+s)h^{1-2s}}{\sqrt{\pi}\,|\Gamma(-s)|} \sum_{k=1}^{\infty} \frac{1+2s+E_s}{k^{1+2s}},$$

and this sum converges to some $F_s > 0$ since $s > 0$, giving us an upper bound for our penultimate object. However, we are done, because finding the upper bound for $|A_2 - C_2|$ is an identical calculation. Finally, we compile our information:

$$\left| r_h\big((-\Delta)^s u\big)_0 - (-\Delta_h)^s (r_h u_0) \right| \leq \frac{2(4^s)h^{1-2s} \, \|Du\|_{L^\infty(\mathbb{R})} \, \Gamma(\frac{1}{2}+s)}{\sqrt{\pi}\,|\Gamma(-s)|} \left| \frac{2}{1-2s} + D_s + F_s \right|$$

$$\leq \frac{4^s K_s h^{1-2s} \, \|Du\|_{L^\infty(\mathbb{R})} \, \Gamma(\frac{1}{2}+s)}{\sqrt{\pi}\,|\Gamma(-s)|}.$$

Therefore, taking the limit as $h \to 0^+$ yields

$$\lim_{h \to 0^+} \left| r_h\big((-\Delta)^s u\big)_j - (-\Delta_h)^s (r_h u_j) \right| = 0.$$

■

Notes:

There are a few open problems in this last subsection. It would be desirable to compare the discrete fractional Laplacian using a second-order incremental quotient with the restriction of (2.14) to the mesh, because this could possibly eliminate the need for cases in the last theorem. Additionally, it may allow for the case of $s = \frac{1}{2}$, which we do not currently have. Extending the spacial dimension to any $n \geq 1$ would be a challenging problem, because it would likely involve multidimensional modified Bessel functions of the first kind.

The development of the discrete fractional left derivative and its convergence to the continuous fractional left derivative is given in [1], but the argument utilizing the corollary of Gautschi's inequality was constructed independently. Likewise, [3] gives the derivation for the discrete fractional Laplacian. However, for the discrete to continuous convergence, we constructed the restriction of the continuous fractional Laplacian to the mesh in such a way that would mimic the same proof for the fractional left derivative. The proof of convergence in the second case of the last theorem is also given in [3].

5. Future work

In this paper, we discussed the probability of locating a particle at some location and some time that is undergoing a continuous random walk in Euclidean space. We derived various PDEs that model the probability while considering different types of anomalous diffusion. With these models, some initial condition may be implemented, allowing one to investigate the existence, uniqueness, and regularity of solutions. [6] makes an effort to find a solution of the problem where the particle might get stuck but still move with unit space step while assuming a Dirac delta initial condition.

It may also be of interest to add boundaries on the random walk, because it is reasonable to expect that such a particle is contained in some way. The particle might under go a drift force, which would result in a non-symmetric random walk, and the probability of moving left or right would not be equal.

One final scenario we may consider is for a particle to undergo a random walk in non-Euclidean space, such as in a sphere or on its surface. This example would likely require the use of spherical coordinates, so one would be finding the probability of observing the particle at some radius, angle, and time.

Appendix A. Some special functions

A.1. **Gamma function.**

The importance of the content in the following section cannot be understated. The gamma function is the key to defining our fractional derivative operators. Also, the fractional left derivative revolves around the fact that we can define unique fractional powers of complex numbers. The aim of this section is to establish important definitions and relevant properties.

Definition A.1. Define $\Gamma : (0, \infty) \to \mathbb{R}$ by:

$$\Gamma(s) := \int_0^\infty e^{-t} t^s \frac{dt}{t}. \tag{A.1}$$

We call Γ the *gamma function* at $s > 0$.

Note that this function is always positive. It is well known that the gamma function interpolates integer factorials to the real numbers, so it maintains the property that the factorial of some number is equal to that number multiplied by the factorial of one less than it, i.e.

$$\Gamma(s+1) = s\Gamma(s).$$

As given above, the gamma function is defined only for positive real numbers, so suppose we would like to extend this definition to some subset of negative numbers. In particular, the goal is to find $\Gamma(-s)$. If we were to substitute in $-s$ in place of s in (A.1), we would have

$$\Gamma(-s) = \int_0^\infty e^{-t} t^{-s} \frac{dt}{t},$$

but this is not integrable near 0, so it is not well-defined. Despite this, we still have the recursive formula above, so

$$\Gamma(-s) = \frac{\Gamma(1-s)}{-s}. \tag{A.2}$$

Here, it must be that $0 < s < 1$ so $\Gamma(-s)$ is well-defined. Also, since $\Gamma(1-s) > 0$, it follows that $\Gamma(-s) < 0$. Now, we present the formula for $\Gamma(-s)$. While the following reads as a theorem, since we are technically considering a different domain, this should also be thought of as a definition:

Theorem A.2. *Let $0 < s < 1$. Then,*

$$\Gamma(-s) = \int_0^\infty \frac{e^{-t} - 1}{t^{1+s}} dt. \tag{A.3}$$

Proof: From the relation above, applying (A.1) gives that

$$\begin{aligned}
\Gamma(-s) &= \frac{1}{-s} \int_0^\infty e^{-t} t^{1-s-1} dt \\
&= \frac{1}{s} \int_0^\infty \frac{d}{dt}\big[e^{-t}\big] t^{-s} dt \\
&= \frac{1}{s} \int_0^\infty \frac{d}{dt}\big[e^{-t} - 1\big] t^{-s} dt.
\end{aligned}$$

Integration by parts gives that

$$\Gamma(-s) = \frac{e^{-t} - 1}{st^s}\bigg|_0^\infty - \frac{1}{s} \int_0^\infty \frac{-s(e^{-t} - 1)}{t^{1+s}} dt$$

$$= \lim_{t\to\infty} \frac{e^{-t}-1}{st^s} - \lim_{t\to 0} \frac{e^{-t}-1}{st^s} + \int_0^\infty \frac{e^{-t}-1}{t^{1+s}} dt.$$

The first limit clearly goes to 0. For the second one, we apply L'Hôpital's rule:

$$\begin{aligned} \lim_{t\to 0} \frac{e^{-t}-1}{st^s} &= \lim_{t\to 0} \frac{-e^{-t}}{s^2 t^{s-1}} \\ &= \lim_{t\to 0} \frac{-t^{1-s}}{s^2 e^t} \\ &= 0, \end{aligned}$$

since $s < 1$. Therefore,

$$\Gamma(-s) = \int_0^\infty \frac{e^{-t}-1}{t^{1+s}} dt.$$

■

Remark A.3. For any positive integer n and $n < s < n+1$, a formula can be found for $\Gamma(-s)$ by repeatedly using the recursive formula (A.2) n times. This has its uses, but it is not focused on in this paper.

From here, it does not take much effort to derive a formula for fractional powers of positive real numbers from (A.3).

Proposition A.4. *Let $\lambda > 0$ and $0 < s < 1$. Then,*

$$\lambda^s = \frac{1}{\Gamma(-s)} \int_0^\infty \frac{e^{-\lambda t}-1}{t^{1+s}} dt. \tag{A.4}$$

Proof: From the integral on the right-hand side, we make the substitution $r = \lambda t$, so $dt = \frac{dr}{\lambda}$, and we have

$$\begin{aligned} \frac{1}{\Gamma(-s)} \int_0^\infty \frac{e^{-\lambda t}-1}{t^{1+s}} dt &= \frac{1}{\Gamma(-s)} \int_0^\infty \frac{e^{-r}-1}{(\frac{r}{\lambda})^{1+s}} \frac{dr}{\lambda} \\ &= \frac{\lambda^s}{\Gamma(-s)} \int_0^\infty \frac{e^{-r}-1}{r^{1+s}} dr \\ &= \frac{\lambda^s}{\Gamma(-s)} \Gamma(-s) \\ &= \lambda^s. \end{aligned}$$

■

Given this integral formula above, one might ask if λ can take a value from a different domain. To answer this question, we must go back to (A.3). In this definition, it turns out that we can actually choose any ray in the first quadrant of the complex plane whose initial point is the origin as the path of integration. It is interesting to note that we still retain the value of $\Gamma(-s)$ for $0 < s < 1$. This gives us the following result:

Theorem A.5. *Let $z \in \mathbb{C}$, $0 < s < 1$, and let* ray $\varphi_0 \subset \mathbb{C}$ *for* $0 \le \varphi_0 \le \frac{\pi}{2}$*. Then,*

$$\Gamma(-s) = \int_{\text{ray}\,\varphi_0} \frac{e^{-z}-1}{z^{1+s}} dz. \tag{A.5}$$

Proof: To start, let $F(z) = \dfrac{e^{-z}-1}{z^{1+s}}$. For $t \in \mathbb{R}$, observe that

$$\int_0^\infty F(t)dt = \lim_{\varepsilon\to 0}\int_\varepsilon^{\frac{1}{\varepsilon}} F(t)dt.$$

F is holomorphic on the positive real axis but has a singularity at the origin. Let $\varepsilon > 0$ and γ be a contour given by

$$\gamma := \Big\{t \,|\, \varepsilon \le t \le \frac{1}{\varepsilon}\Big\} \cup \Big\{\frac{1}{\varepsilon}e^{it} \,|\, 0 \le t \le \varphi_0\Big\} \cup \Big\{te^{i\varphi_0} \,|\, \varepsilon \le t \le \frac{1}{\varepsilon}\Big\} \cup \big\{\varepsilon e^{it} \,|\, 0 \le t \le \varphi_0\big\},$$

and call each curve $\gamma_1, \gamma_2, \gamma_3, \gamma_4$ respectively. Let $U \subset \mathbb{C}$ be open such that $U \supset \gamma$ and $0 \notin U$. F is holomorphic in all of U, so by Cauchy's integral theorem,

$$\oint_U F(z)dz = 0.$$

On the other hand,

$$0 = \int_{\gamma_1} F(z)dz + \int_{\gamma_2} F(z)dz + \int_{\gamma_3} F(z)dz + \int_{\gamma_4} F(z)dz.$$

As we take the limit as $\varepsilon \to 0$, note that the integral along γ_1 is $\Gamma(-s)$ and the integral along γ_3 is the opposite of the integral along ray φ_0. That is,

$$0 = \Gamma(-s) - \int_{\text{ray }\varphi_0} F(z)dz + \lim_{\varepsilon\to 0}\bigg(\int_{\gamma_2} F(z)dz + \int_{\gamma_4} F(z)dz\bigg). \tag{A.6}$$

We investigate the convergence of these two remaining integrals. For γ_2,

$$\begin{aligned}
\left|\int_{\gamma_2} F(z)dz\right| &= \left|\int_0^{\varphi_0} \frac{e^{-\frac{1}{\varepsilon}e^{it}}-1}{\left(\frac{1}{\varepsilon}e^{it}\right)^{1+s}}\frac{ie^{it}dt}{\varepsilon}\right| \\
&= \left| i\int_0^{\varphi_0} \frac{\varepsilon^s\left(e^{-\frac{1}{\varepsilon}e^{it}}-1\right)}{e^{ist}}dt\right| \\
&\le \int_0^{\varphi_0} \varepsilon^s\left|e^{-\frac{1}{\varepsilon}e^{it}}-1\right|dt \\
&\le \int_0^{\varphi_0} \varepsilon^s\left(\left|e^{-\frac{1}{\varepsilon}e^{it}}\right|+1\right)dt \\
&= \int_0^{\varphi_0} \varepsilon^s\left(\left|e^{-\frac{1}{\varepsilon}\cos(t)}\right|+1\right)dt.
\end{aligned}$$

This last line follows since $|e^z| = e^{\Re(z)}$. Since $t \le \varphi_0 \le \frac{\pi}{2}$, then $\cos(t) \ge 0$, and so $e^{-\frac{1}{\varepsilon}\cos(t)} \le e^{-\frac{1}{\varepsilon}(0)} = 1$. Thus,

$$\begin{aligned}
\left|\int_{\gamma_2} F(z)dz\right| &\le \int_0^{\varphi_0} 2\varepsilon^s dt \\
&= 2\varphi_0\varepsilon^s \\
&\to 0
\end{aligned}$$

as $\varepsilon \to 0$. For γ_4, we have

$$\left|\int_{\gamma_4} F(z)dz\right| = \left|\int_{\varphi_0}^0 \frac{e^{-\varepsilon e^{it}}-1}{\left(\varepsilon e^{it}\right)^{1+s}}i\varepsilon e^{it}dt\right|$$

$$= \left| -i \int_0^{\varphi_0} \frac{e^{-\varepsilon e^{it}} - 1}{\varepsilon^s e^{ist}} dt \right|$$
$$\leq \int_0^{\varphi_0} \varepsilon^{-s} \left| e^{-\varepsilon e^{it}} - 1 \right| dt.$$

Consider the exponential inside the absolute value as a function $G(\varepsilon)$. Here, note that $\left| e^{-\varepsilon e^{it}} - 1 \right| = |G(\varepsilon) - G(0)|$. G is continuous on $[0, \varepsilon]$ and differentiable on $(0, \varepsilon)$, so by the mean value theorem, there exists some $\zeta \in (0, \varepsilon)$ such that

$$\begin{aligned} |G(\varepsilon) - G(0)| &= (\varepsilon - 0) \left| G'(\zeta) \right| \\ &= \varepsilon \left| -e^{it} e^{-\zeta e^{it}} \right| \\ &= \varepsilon \left| e^{-\zeta e^{it}} \right| \\ &= \varepsilon \left| e^{-\zeta \cos(t)} \right| \\ &\leq \varepsilon, \end{aligned}$$

which follows from the reasoning in the previous integral. Thus,

$$\begin{aligned} \left| \int_{\gamma_4} F(z) dz \right| &\leq \int_0^{\varphi_0} \varepsilon^{1-s} dt \\ &= \varphi \varepsilon^{1-s} \\ &\to 0 \end{aligned}$$

as $\varepsilon \to 0$. Plugging back into (A.6) gives

$$0 = \Gamma(-s) - \int_{\text{ray } \varphi_0} F(z) dz,$$

and the result follows. ■

While this result is interesting in its own right, we can use it to obtain a quick result that is essential in defining the fractional left derivative. In particular, if we integrate the ray along the angle $\varphi_0 = \frac{\pi}{2}$, (A.4) can be extended to include fractional powers of imaginary numbers with positive imaginary part.

Corollary A.6. *Let* $\omega > 0$*, and let* $0 < s < 1$*. Then,*

$$(i\omega)^s = \frac{1}{\Gamma(-s)} \int_0^{\infty} \frac{e^{-i\omega t} - 1}{t^{1+s}} dt. \tag{A.7}$$

Proof: In (A.5), use the parameterization $z(t) = i\omega t$ for $t > 0$. Differentiating gives $dz = i\omega dt$, and proper substitution gives that

$$\begin{aligned} \Gamma(-s) &= \int_0^{\infty} \frac{e^{-i\omega t} - 1}{(i\omega t)^{1+s}} i\omega dt \\ &= \frac{1}{(i\omega)^s} \int_0^{\infty} \frac{e^{-i\omega t} - 1}{t^{1+s}} dt, \end{aligned}$$

so rearranging gives the result. ■

It is worth pausing here to discuss the implications of this result. First, if we compare (A.4) and (A.7), we notice that the only difference is that we substitute an imaginary number with positive imaginary part $i\omega$ in place of a positive real number λ. This begs the question of if we could just make this substitution without working through the complex analysis, and the answer is no. Recall that in order to define $\Gamma(-s)$, we could not simply substitute $-s$ in place of s in (A.1), because it would no longer be well-defined. We had to use properties of the gamma function to derive a different expression for $\Gamma(-s)$. This is why we have to check what happens when we extend the domain in (A.4).

This brings us to the second observation. Powers of complex numbers are multi-valued, so we might ask which value of $(i\omega)^s$ are we taking. By considering the ray along φ_0, we are taking the branch that corresponds to the angle φ_0. Thus, $(i\omega)^s$ is given by the unique value of the integral given in (A.7). Therefore, the previous corollary is as much of a result as it is a definition.

Finally, we can extend the idea of raising numbers to fractional powers to raising operators to fractional powers. If L is a differential operator, then the previous work suggests that we can say that

$$L^s = \frac{1}{\Gamma(-s)} \int_0^\infty \frac{e^{-Lt} - 1}{t^{1+s}} dt,$$

provided we can deduce a meaning for e^{-Lt}. We can do this via semigroups, which is explored in Section 2.

This process can be replicated to include fractional powers of negative operators, which is useful for deriving expressions for the continuous and discrete fractional Laplacian.

The gamma function is the unique function that interpolates integer factorials to real numbers that is also logarithmically convex. A consequence of this property, known as Gautschi's inequality, gives lower and upper bounds for a quotient of gamma functions.

Theorem A.7 (Gautschi's inequality)**.** *For any $x > 0$ and $0 < s < 1$,*

$$x^{1-s} < \frac{\Gamma(x+1)}{\Gamma(x+s)} < (x+1)^{1-s}.$$

It is beneficial for us to instead consider an alternate form of Gautschi's inequality, so we have the following corollary:

Corollary A.8. *For any $x > 0$ and $0 < \alpha < 1$,*

$$\frac{1}{(x+1)^\alpha} < \frac{\Gamma(x+1-\alpha)}{\Gamma(x+1)} < \frac{1}{x^\alpha}. \tag{A.8}$$

Proof: Let $s = 1 - \alpha$ in Gautschi's inequality. Since $0 < 1 - \alpha < 1$, we have that

$$x^\alpha < \frac{\Gamma(x+1)}{\Gamma(x+1-\alpha)} < (x+1)^\alpha,$$

and taking reciprocals on all three sides gives the result. ■

We also have another estimate when considering quotients of gamma functions that may be applicable when Gautschi's inequality is not:

Proposition A.9. *For any $x > 0$ and $0 < s < 1$, then there exists some $E_s > 0$ such that*

$$\left| \frac{\Gamma(x-s)}{\Gamma(x+1+s)} - \frac{1}{x^{1+2s}} \right| \leq \frac{E_s}{x^{2+2s}}. \tag{A.9}$$

A.2. **Modified Bessel function of the first kind.**

The construction of the discrete fractional Laplacian requires the use of the modified Bessel function of the first kind. In this subsection, we list without proof a few of its properties.

Definition A.10. Let $\nu \in \mathbb{Z}$. The *modified Bessel function of the first kind* $I_\nu : \mathbb{R} \to \mathbb{R}$ is defined by

$$I_\nu(t) = \sum_{m=0}^{\infty} \frac{(\frac{t}{2})^{2m+\nu}}{\Gamma(m+1)\Gamma(m+\nu+1)}.$$

From the definition, it is clear that $I_\nu(t) \geq 0$. For the boundary values, we see that $I_\nu(0) = 0$ for $\nu \geq 1$, and we take $I_0(0) = 1$. It is proven that this function is even in ν, that is, $I_\nu = I_{-\nu}$. This function also satisfies the relation

$$\frac{\partial}{\partial t}\big[e^{-2t} I_\nu(2t)\big] = e^{-2t}\big[I_{\nu+1}(2t) - 2I_\nu(2t) + I_{\nu-1}(2t)\big]. \tag{A.10}$$

We can view $\{I_\nu\}$ as a sequence of functions and thus can be described via generating functions. In particular, a Laurent series generating function for I_ν leads the way to the following property:

Proposition A.11. *For all* $t \in \mathbb{R}$,

$$\sum_{\nu \in \mathbb{Z}} I_\nu(t) = e^t. \tag{A.11}$$

Finally, an important integral identity that contains the modified Bessel function of the first kind is given below:

Proposition A.12. *If* $c > 0$ *and* $-\frac{1}{2} < s < \nu$, *then*

$$\int_0^\infty e^{-ct} I_\nu(ct) \frac{dt}{t^{1+s}} = \frac{(4c)^s \Gamma(\frac{1}{2}+s)\Gamma(\nu - s)}{\sqrt{\pi}\,\Gamma(\nu+1+s)}. \tag{A.12}$$

Notes:

The results concerning the gamma function evaluation at negative entries, including the integration along a ray on the complex plane, are found in [2]. Gautschi's inequality is a generally known result, but its slightly modified form was developed independently. The final inequality regarding the quotient of gamma functions and all the properties of the modified Bessel function of the first kind are found in [3].

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