Fractional derivatives and applications

Undergraduate Dissertation

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FRACTIONAL DERIVATIVES AND APPLICATIONS

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Abstract. A natural question in the topic of classical differentiation and integration is how these ideas exist in a fractional sense. What happens when a function is differentiated to a non-integer order? This thesis discusses the motivation behind such a question, as well as the motivation for a closely related function: the gamma function. We then discuss desirable properties of fractional derivatives and integrals based on what their classical counterparts satisfy. This allows us to define a fractional integral in terms of the gamma function and the classical integral, and from there, two definitions of a fractional derivative. Next, we apply these ideas to solving fractional differential equations in regards to a fractional variation of Picard's theorem. Finally, we discuss an application of fractional calculus to the tautochrone problem, which requires finding a unique curve such that the time for objects to slide down the curve from rest at any point is conserved.

Contents

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1. Introduction

In this paper, we discuss fractional integration, differentiation, and applications. Sections 2 through 4 must be read in order, as all preceding information is necessary for further understanding. The final two are independent of each other and either can be read before the other, but still after reading the preliminary sections. A decent background in single and multivariable calculus, basic real analysis, and elementary physics are required for reading this paper.

A large emphasis is placed on the importance of definitions. The next section builds up the motivation and basic theory behind the gamma function, which is an interpolation of the positive integer factorials to the real numbers. This function is the primary choice to model this behavior because it exhibits certain desirable properties, which will be described in detail later. The groundwork of finding a best definition for the gamma function leads to the importance of finding a good definition for the fractional integral, which is introduced in the following section. There are many possible ways that fractional integrals may be defined, but they should have specific properties that are also satisfied by integer order integrals. The first definition of a fractional integral to be found is now the most widely accepted variant, and it not only satisfies these properties, but it utilizes the gamma function as well. In the following section, this definition of fractional integration is used to define two similar, but different, methods of fractional differentiation. Contrary to classical calculus, this sequence of sections is the most intuitive approach with our definitions used. This section concludes with a fractional analogue of the fundamental theorem of calculus, which is used in the final two application sections. A further historical account of fractional integration and differentiation can be found in [5]. Another definition not studied here is the Marchaud derivative, which can be shown to be related to the Fourier transform.

The first application is solving a simple fractional differential equation with an initial value. Existence of a solution is not immediately guaranteed, so we state and prove a fractional analogue of Picard's iterations theorem. From concepts in continuity to various results centering around sequence convergence, the proof is quite lengthy and utilizes a great deal of real analysis. However, not only does it allow us to prove existence and uniqueness of a solution, but it provides a framework for how to structure our solution. Some fractional models that arise in this paper may appear to be long, but the solution to this equation can actually be presented in a very elegant way.

The second application and final section deals with an physics problem. A tautochrone is a special, frictionless curve such that any two objects released from rest at any points on the curve will reach the origin in equal times. The formulation begins with the utilization of the conservation of energy, and from here, an integral equation is derived. Methods of fractional calculus are then used to solve for the equation of this curve, and it too takes a rather simple form. A problem for further study would be allowing the curve to retain its coefficient of friction and investigating how the solution would change, or if it could even be found analytically. Another application not studied here is one in bioengineering, and that is modeling the behavior of viscoelastic materials, which loosely are materials that retain both elastic properties of solids and viscous properties of liquids.

Throughout this paper, it is of maximum importance to mind the notations used. Effort is made to distinguish between various derivative definitions and the integral definition, but a careless read will cause one to lose their way. The difference between notating a fractional integral or derivative comes down to sign of the order, and it is not always explicitly stated. We try to minimize complex notation when possible, but what is being omitted or simplified should not be forgotten.

2. The gamma function

In this section, we motivate the definition of the gamma function and present several important properties that will be used later. Part of the material of this section is taken from [6] and [1].

2.1. **Definition and uniqueness.**

The most notable property of the gamma function is that it interpolates the positive integer factorials to positive real numbers, but we need an idea of what that could mean. We want satisfied $\Gamma(n+1) = n!$ for $n \in \mathbb{N}$. The best strategy would be to find an expression for integer factorials and then check to see if it is well-defined for positive real arguments. To begin, we start by calculating the following integral:

$$\begin{aligned} \int_0^\infty e^{-at}\,dt &= -\frac{1}{a}e^{-at}\Big|_0^\infty \\ &= \frac{1}{a}. \end{aligned}$$

Now, we differentiate both sides of the equation with respect to a once:

$$\begin{aligned} \frac{\partial}{\partial a}\int_0^\infty e^{-at}\,dt &= \frac{\partial}{\partial a}\frac{1}{a} \\ \int_0^\infty -te^{-at}\,dt &= \frac{-1}{a^2} \\ \int_0^\infty te^{-at}\,dt &= \frac{1}{a^2}. \end{aligned}$$

Differentiate both sides of this equality with respect to a again:

$$\begin{aligned} \frac{\partial}{\partial a}\int_0^\infty te^{-at}\,dt &= \frac{\partial}{\partial a}\frac{1}{a^2} \\ \int_0^\infty -t^2e^{-at}\,dt &= \frac{-2}{a^3} \\ \int_0^\infty t^2e^{-at}\,dt &= \frac{2}{a^3}. \end{aligned}$$

Repeating this process n times will ultimately yield

$$\int_0^\infty t^n e^{-at}\,dt = \frac{n!}{a^{n+1}}.$$

Let $a = 1$, and replace n with $n-1$:

$$\int_0^\infty t^{n-1}e^{-t}\,dt = (n-1)!$$

Since we require that $\Gamma(n+1) = n!$ for $n \in \mathbb{N}$, we can say that

$$\Gamma(n) = \int_0^\infty t^{n-1}e^{-t}\,dt.$$

Now, it makes sense to extend the domain of this function to any positive real number, as the integral will converge. We get the following definition:

Definition 2.1. For $x > 0$, we define the *gamma function* to be

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t}\, dt. \tag{2.1}$$

With this definition, we must make sure that the recurrence relation is still satisfied for arbitrary positive real and integer arguments. Thus, we have the following theorem:

Theorem 2.2. *Let* $0 < x < \infty$.

(1) $\Gamma(x+1) = x\Gamma(x)$,
(2) *For any* $n \in \mathbb{N}$, $\Gamma(n+1) = n!$

Proof:

(1) We prove this statement by integration by parts:

$$\begin{aligned}\Gamma(x+1) &= \int_0^\infty t^x e^{-t}\, dt \\ &= t^x e^{-t}\Big|_0^\infty + \int_0^\infty x t^{x-1} e^{-t}\, dt \\ &= \lim_{t\to\infty} t^x e^{-t} - 0^x e^{-0} + x \int_0^\infty t^{x-1} e^{-t}\, dt \\ &= x\Gamma(x).\end{aligned}$$

(2) We use induction on n. For $n = 1$, we immediately have that $\Gamma(1) = 1$ from the definition. Now, suppose that the relation holds for any n, so we must show that it holds for $n+1$:

$$\begin{aligned}\Gamma(n+2) &= (n+1)\Gamma(n+1) \\ &= (n+1)n! \\ &= (n+1)!\end{aligned}$$

■

In the previous theorem, (1) gives us the recursive property, and (2) gives us that the gamma function evaluated at any positive integer returns the factorial of one less than that integer. What this theorem does not shed any light on is what happens between the integers. If we did not know the image of the gamma function given by our definition, then for all we know, each positive integer point could be connected by line segments or by sine waves, but we must think about what we want from this curve. It would be nice if our function is differentiable everywhere in its domain, which would result in a smooth curve. Also, if Γ is periodic or a sum or product of periodic functions, we would have a large class of functions for options. For example, $\Gamma(x) + \sin(2\pi x)$ would return the same value as $\Gamma(x)$ at integer arguments, but we lose our wish that the desired function is increasing on $[1, \infty)$, which is something that the positive integer factorials satisfy. One possible remedy for this is requiring Γ to be convex, but we can actually prove that it has a stronger property, which is that Γ is *logarithmically convex*, or that $\ln \Gamma$ is convex:

Theorem 2.3. $\ln \Gamma$ *is convex on* $(0, \infty)$.

Proof: Let $1 < p, q < \infty$ such that $\dfrac{1}{p} + \dfrac{1}{q} = 1$. We have the following:

$$\int_0^\infty t^{\frac{x}{p}+\frac{y}{q}} e^{-t} \frac{dt}{t} = \int_0^\infty \left(t^x \frac{e^{-t}}{t}\right)^{\frac{1}{p}} \left(t^y \frac{e^{-t}}{t}\right)^{\frac{1}{q}} dt$$

$$\leq \left(\int_0^\infty t^x \frac{e^{-t}}{t}\,dt\right)^{\frac{1}{p}} \left(\int_0^\infty t^y \frac{e^{-t}}{t}\,dt\right)^{\frac{1}{q}}$$

by Hölder's inequality. Note that we have the gamma function evaluated at $\frac{1}{p}+\frac{1}{q}$, x, and y respectively in the inequality, so

$$\Gamma\Big(\frac{x}{p}+\frac{y}{q}\Big) \leq \Gamma(x)^{\frac{1}{p}}\Gamma(y)^{\frac{1}{q}}.$$

Let $f = \ln\Gamma$, so the above equation is equivalent to

$$e^{f(\frac{x}{p}+\frac{y}{q})} \leq e^{\frac{x}{p}f(x)}e^{\frac{y}{q}f(y)},$$

so

$$f\Big(\frac{x}{p}+\frac{y}{q}\Big) \leq \frac{x}{p}f(x)+\frac{y}{q}f(y).$$

Since $p \in (1,\infty)$, define $\lambda = \frac{1}{p}$, so $1-\lambda = \frac{1}{q}$, and so

$$f\big(\lambda x+(1-\lambda)y\big) \leq \lambda f(x)+(1-\lambda)f(y),$$

so f is convex, hence $\ln\Gamma$ is convex. ■

Not only does the gamma function satisfy these three properties, but it turns out that it is the only function that does. It is the logarithmic convexity that makes it unique. As such, this is what makes our characterization of the gamma function the most desirable one.

Theorem 2.4 (Böhr-Mollerup). *If f is a positive function on $(0,\infty)$ such that*
(1) $f(x+1) = xf(x)$,
(2) $f(1) = 1$,
(3) $\ln f$ *is convex,*

then $f(x) = \Gamma(x)$.

Proof: We assume f is a function with all of the listed properties. Since Γ satisfies (1), (2), and (3) by the previous two theorems, we must show that $f(x)$ is determined uniquely by (1), (2), and (3) for $x > 0$. By the recurrence relation of (1), it suffices to show this uniqueness only for $x \in (0,1)$.

Let $\phi(x) = \ln f(x)$. It follows that

$$\begin{aligned}\phi(x+1) &= \ln f(x+1)\\ &= \ln xf(x)\\ &= \ln x+\ln f(x)\\ &= \ln x+\phi(x).\end{aligned}$$

We also have that ϕ is convex since ln is convex, and $\phi(1) = \ln f(1) = \ln 1 = 0$. Inductively, we would see that the previous process gives

$$\begin{aligned}\phi(n+1+x) &= \phi(x)+\ln x+\ln(x+1)+\cdots+\ln(x+n)\\ &= \phi(x)+\ln\big(x(x+1)\cdots(x+n)\big)\end{aligned} \tag{2.2}$$

for $n \in \mathbb{N}$. In particular, we have that that

$$\begin{aligned}\phi(n+1) &= \ln f(n+1)\\ &= \ln(n!).\end{aligned} \tag{2.3}$$

Next, we will consider the difference quotients of ϕ on the intervals $[n, n+1]$, $[n+1, n+1+x]$, and $[n+1, n+2]$. Since ϕ is convex, it follows by definition that

$$\begin{aligned}
\frac{\phi(n+1)-\phi(n)}{(n+1)-n} &\leq \frac{\phi(n+1+x)-\phi(n+1)}{(n+1+x)-(n+1)} \leq \frac{\phi(n+2)-\phi(n+1)}{(n+2)-(n+1)} \\
\phi(n+1)-\phi(n) &\leq \frac{\phi(n+1+x)-\phi(n+1)}{x} \leq \phi(n+2)-\phi(n+1) \\
\ln(n!)-\ln\big((n-1)!\big) &\leq \frac{\phi(n+1+x)-\phi(n+1)}{x} \leq \ln\big((n+1)!\big)-\ln(n!) \\
\ln n &\leq \frac{\phi(n+1+x)-\phi(n+1)}{x} \leq \ln(n+1).
\end{aligned}$$

Applying equations (2.2) and (2.3) to the middle piece gives us

$$\ln n \leq \frac{\phi(x)+\ln\big(x(x+1)\cdots(x+n)\big)-\ln(n!)}{x} \leq \ln(n+1),$$

so rearranging gives that

$$\begin{aligned}
0 &\leq \phi(x)+\ln\big(x(x+1)\cdots(x+n)\big)-\ln(n!)-x\ln n \leq x\big(\ln(n+1)-\ln n\big) \\
&\leq \phi(x)+\ln\big(x(x+1)\cdots(x+n)\big)-\ln(n!)-\ln(n^x) \leq x\ln\left(\frac{n+1}{n}\right) \\
&\leq \phi(x)+\ln\left[\frac{x(x+1)\cdots(x+n)}{n!n^x}\right] \leq x\ln\left(1+\frac{1}{n}\right) \\
&\leq \phi(x)-\ln\left[\frac{n!n^x}{x(x+1)\cdots(x+n)}\right] \leq x\ln\left(1+\frac{1}{n}\right).
\end{aligned}$$

Finally, we take a limit as $n \to \infty$ on all sides.

$$\begin{aligned}
0 &\leq \phi(x)-\lim_{n\to\infty}\ln\left[\frac{n!n^x}{x(x+1)\cdots(x+n)}\right] \leq \lim_{n\to\infty} x\ln\left(1+\frac{1}{n}\right) \\
&\leq \phi(x)-\ln\left[\lim_{n\to\infty}\frac{n!n^x}{x(x+1)\cdots(x+n)}\right] \leq x\ln(1) \\
&\leq \phi(x)-\ln\left[\lim_{n\to\infty}\frac{n!n^x}{x(x+1)\cdots(x+n)}\right] \leq 0.
\end{aligned}$$

By the squeeze theorem, the middle piece of the inequality is equal to 0. Thus, we have that

$$\phi(x) = \ln\left[\lim_{n\to\infty}\frac{n!n^x}{x(x+1)\cdots(x+n)}\right],$$

and so

$$f(x) = \lim_{n\to\infty}\frac{n!n^x}{x(x+1)\cdots(x+n)}.$$

Since we have uniquely determined f using (1), (2), and (3), it must be that $f(x) = \Gamma(x)$. ■

Although it is of no use for our purposes, the proof of this theorem gives us a second characterization of the gamma function.

Corollary 2.5. *For any $x > 0$,*

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}\,dt = \lim_{n\to\infty}\frac{n!n^x}{x(x+1)\cdots(x+n)}.$$

Let us illustrate the use of the original definition and its properties with several numerical examples.

Example 2.6. *Calculate* $\Gamma\left(\frac{3}{2}\right)$.

$$\Gamma\left(\frac{3}{2}\right) = \int_0^\infty t^{\frac{1}{2}} e^{-t}\, dt.$$

Let $u = t^{\frac{1}{2}}$, so $dt = 2u\, du$. Thus,

$$\begin{aligned}
\Gamma\left(\frac{3}{2}\right) &= \int_0^\infty u e^{-u^2} 2u\, du \\
&= -ue^{-u^2}\Big|_0^\infty + \int_0^\infty e^{-u^2}\, du \\
&= \frac{1}{2}\int_{-\infty}^\infty e^{-u^2}\, du.
\end{aligned}$$

Call this resulting integral A and square it:

$$\begin{aligned}
A^2 &= \frac{1}{4}\left(\int_{-\infty}^\infty e^{-u^2}\, du\right)^2 \\
&= \frac{1}{4}\left(\int_{-\infty}^\infty e^{-x^2}\, dx\right)\left(\int_{-\infty}^\infty e^{-y^2}\, dy\right) \\
&= \frac{1}{4}\int_{-\infty}^\infty \int_{-\infty}^\infty e^{-(x^2+y^2)}\, dx\, dy.
\end{aligned}$$

We now convert to polar coordinates. Let $r^2 = x^2 + y^2$, and we also have that $dx\, dy = r\, dr\, d\theta$. After converting limits of integration, we get the following:

$$\begin{aligned}
A^2 &= \frac{1}{4}\int_0^{2\pi}\int_0^\infty re^{-r^2}\, dr\, d\theta \\
&= \frac{1}{4}\int_0^{2\pi}\left(-\frac{1}{2}e^{-r^2}\Big|_0^\infty\right) d\theta \\
&= \frac{1}{4}\int_0^{2\pi}\frac{1}{2}\, d\theta \\
&= \frac{\pi}{4}.
\end{aligned}$$

Thus, we have that $A = \Gamma\left(\frac{3}{2}\right) = \frac{\sqrt{\pi}}{2}$.

Example 2.7. *Calculate* $\Gamma\left(\frac{1}{2}\right)$.

It is now more efficient to apply Theorem 2.2 instead:

$$\begin{aligned}
\Gamma\left(\frac{1}{2}\right) &= \frac{\Gamma\left(\frac{3}{2}\right)}{\frac{1}{2}} \\
&= \frac{\frac{\sqrt{\pi}}{2}}{\frac{1}{2}} \\
&= \sqrt{\pi}.
\end{aligned}$$

2.2. **Other properties.**

In the next part of this section, we state and prove several properties which will be extremely important later. The first result states a relation between the beta function and the gamma function. For our purposes, we only consider the integral definition of the beta function instead of the function itself.

Theorem 2.8. *If* $\alpha, \beta > 0$*, then*

$$B(\alpha,\beta) := \int_0^1 v^{\alpha-1}(1-v)^{\beta-1}\,dv = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}.$$

Proof: Start by multiplying the following:

$$\begin{aligned}\Gamma(\alpha)\Gamma(\beta) &= \left(\int_0^\infty t^{\alpha-1}e^{-t}\,dt\right)\left(\int_0^\infty s^{\beta-1}e^{-s}\,ds\right)\\ &= \int_0^\infty\int_0^\infty t^{\alpha-1}s^{\beta-1}e^{-t-s}\,dt\,ds.\end{aligned}$$

Let $t = uv$ and $s = u(1-v)$, so $u = t+s$. Note that since $t, s > 0$, it follows that $u > 0$ and $0 < v < 1$. To find $dt\,ds$, we must compute the following Jacobian determinant:

$$\begin{aligned}\frac{\partial(t,s)}{\partial(u,v)} &= \begin{vmatrix}\frac{\partial t}{\partial u} & \frac{\partial t}{\partial v}\\ \frac{\partial s}{\partial u} & \frac{\partial s}{\partial v}\end{vmatrix}\\ &= \begin{vmatrix} v & u\\ 1-v & -u\end{vmatrix}\\ &= u.\end{aligned}$$

Thus, $dt\,ds = u\,du\,dv$, so

$$\begin{aligned}\Gamma(\alpha)\Gamma(\beta) &= \int_0^\infty\int_0^1 (uv)^{\alpha-1}\big(u(v-1)\big)^{\beta-1}e^{-u}u\,du\,dv\\ &= \int_0^\infty\int_0^1 u^{\alpha-1}v^{\alpha-1}u^{\beta-1}(v-1)^{\beta-1}e^{-u}u\,du\,dv\\ &= \left(\int_0^\infty u^{\alpha+\beta-1}e^{-u}\,du\right)\left(\int_0^1 v^{\alpha-1}(v-1)^{\beta-1}\,dv\right)\\ &= \Gamma(\alpha+\beta)\int_0^1 v^{\alpha-1}(v-1)^{\beta-1}\,dv,\end{aligned}$$

so

$$\int_0^1 v^{\alpha-1}(1-v)^{\beta-1}\,dv = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}.$$

■

We know that for large enough arguments, Γ increases quickly. However, we have a useful set of bounds for a particular ratio of the gamma function.

Theorem 2.9 (Gautschi's inequality). *For any* $x > 0$ *and* $0 < \alpha < 1$*,*

$$x^{1-\alpha} \le \frac{\Gamma(x+1)}{\Gamma(x+\alpha)} \le (x+1)^{1-\alpha}. \tag{2.4}$$

Proof: Since Γ is logarithmically convex, then for any distinct $z, w > 0$ and $0 < \beta < 1$,

$$\Gamma\big(\beta z + (1-\beta)w\big) < \Gamma(z)^{\beta}\Gamma(w)^{1-\beta}.$$

To achieve the lower bound, let $z = x$, $w = x + 1$, and $\beta = 1 - \alpha$. Plugging into the above equation gives that

$$\begin{aligned}
\Gamma\big((1-\alpha)x + (1-1+\alpha)(x+1)\big) &< \Gamma(x)^{1-\alpha}\Gamma(x+1)^{\alpha} \\
\Gamma(x+\alpha) &< \Gamma(x)^{1-\alpha}x^{\alpha}\Gamma(x)^{\alpha} \\
&= x^{\alpha}\Gamma(x) \\
&= \frac{x^{\alpha+1}}{x}\Gamma(x) \\
&= x^{\alpha-1}\Gamma(x+1).
\end{aligned}$$

Thus,

$$x^{1-\alpha} < \frac{\Gamma(x+1)}{\Gamma(x+\alpha)}.$$

To achieve the upper bound, let $z = x + \alpha$, $w = x + \alpha + 1$, and $\beta = \alpha$. Directly substituting again gives that

$$\begin{aligned}
\Gamma\big(\alpha(x+\alpha) + (1-\alpha)(x+\alpha+1)\big) &< \Gamma(x+\alpha)^{\alpha}\Gamma(x+\alpha)^{1-\alpha} \\
\Gamma(x+1) &< \Gamma(x+\alpha)^{\alpha}\Gamma(x+\alpha+1)^{1-\alpha} \\
&= \Gamma(x+\alpha)^{\alpha}(x+\alpha)^{1-\alpha}\Gamma(x+\alpha)^{1-\alpha} \\
&= (x+\alpha)^{1-\alpha}\Gamma(x+\alpha) \\
&< (x+1)^{1-\alpha}\Gamma(x+\alpha).
\end{aligned}$$

Thus,

$$\frac{\Gamma(x+1)}{\Gamma(x+\alpha)} < (x+1)^{1-\alpha}.$$

■

While these final remarks of the section are mainly of independent interest, we may expand the class of real arguments for the gamma function to certain negative entries. Motivated by the recursion property for the gamma function with positive arguments, we define the following:

Definition 2.10. For $0 < \alpha < 1$, we say that

$$\Gamma(-\alpha) := \frac{\Gamma(1-\alpha)}{-\alpha}.$$

Remark 2.11. If it happens that α is between any two consecutive negative integers, then the recursive property can be applied on $\Gamma(1-\alpha)$ n times until $n - \alpha > 0$, so the original definition may be applied.

Remark 2.12. By the same reasoning, it follows that the gamma function evaluated at any negative integer is undefined, because we would eventually recover the factor $\Gamma(0)$, which is undefined.

3. Defining the fractional integral

Before we proceed, we explain the reason for beginning the paper with an exposition of the gamma function. As previously mentioned, many of the prior results will be used later in calculations and proofs. However, the section as a whole gives a deeper meaning to what we are about to establish in the following two sections. We heavily emphasized the desired behavior of the gamma function through the Böhr-Mollerup theorem. We also acknowledged that as long as the function interpolated the integer factorials to the real numbers, the gamma function could be whatever we wanted it to be. Developing the notion of fractional integrals and derivatives follows a similar thought process: extending relations in terms of integers to real numbers and requiring the implementation of additional desirable properties as listed below. Also, while we may create separate definitions for specific classes of functions, it would be nice to have a definition for the most general class of functions.

3.1. **Formulating a suitable definition.**

It is worth mentioning that in classical calculus, defining the derivative comes before defining the Riemann integral. However, in this fractional sense, we will find that it will be more useful to define fractional integration before fractional differentiation.

By nature of the models we will be working with, we are motivated to state the following definition:

Definition 3.1. Let $c > 0$, and let $u : \mathbb{R} \to \mathbb{R}$ be a function. We say that u is *left-continuous at c* if for every $\varepsilon > 0$, there exists a $\delta > 0$ such that for every x such that $c - \delta < x < c$, we have $\left|u(x) - u(c)\right| < \varepsilon$. The Riemann integral of an integrable left-continuous function u is called the *left integral* of u, notated $I_{left}u$. In context, we may just say Iu instead.

Notation 3.2. Let $u : \mathbb{R} \to \mathbb{R}$ be a function. For $\alpha \geq 0$, the symbol $I^\alpha u$ represents integration of order α of u.

Notation 3.3. The symbols I^α and $D^{-\alpha}$ will be used interchangeably.

Integrals of integer order satisfy certain properties, and we would want our definition of fractional integration to have these as well. If we do not take care in attempting to preserve these properties, then our endeavors risk losing meaning. Thus, we make the following requirements for any bounded and left-continuous function $u : \mathbb{R} \to \mathbb{R}$:

(1) If $\alpha \in \mathbb{Z}_+$, then the fractional order integral of u must coincide with classical α-order integration, and $I^\alpha u(t)$ must vanish along with its $\alpha - 1$ derivatives at some initial value c.
(2) $I^0 u(t) = u(t)$.
(3) For any $\alpha > 0$, I^α is a linear operator.
(4) For any $\alpha, \beta > 0$, $I^\alpha I^\beta u(t) = I^{\alpha+\beta} u(t)$.

Throughout the 1800s, various mathematicians were trying to formalize definitions of fractional differentiation and integration in terms of classical calculus and these properties. A sophisticated formula for fractional integration came first, thanks to Riemann and Liouville, and it is now the widely accepted definition for its kind. We show a derivation for this definition through the following lemma:

Lemma 3.4. *Let $u : \mathbb{R} \to \mathbb{R}$ be a bounded and left-continuous function on $[0, t]$. For $n \in \mathbb{N}$,*

$$I^n u(t) = \frac{1}{(n-1)!} \int_0^t (t-s)^{n-1} u(s)\, ds.$$

Proof: We use induction on n. First, let $n = 1$, so we have that

$$I^1 u(t) = \int_0^t u(s)\, ds,$$

which we know to be true.

For the inductive step, suppose that the relation holds for n, so we must show that it holds for $n + 1$, that is

$$I^{n+1} u(t) = \frac{1}{n!} \int_0^t (t-s)^n u(s)\, ds.$$

We have that

$$\begin{aligned}
I^{n+1} u(t) &= \int_0^t I^n u(s)\, ds \\
&= \frac{1}{(n-1)!} \int_0^t \int_0^s (s-r)^{n-1} u(r)\, dr\, ds \\
&= \frac{1}{(n-1)!} \int_0^t \int_r^t (s-r)^{n-1} u(r)\, ds\, dr \\
&= \frac{1}{(n-1)!} \int_0^t u(r) \int_r^t (s-r)^{n-1}\, ds\, dr \\
&= \frac{1}{(n-1)!} \int_0^t u(r) \frac{(s-r)^n}{n} \bigg|_r^t\, dr \\
&= \frac{1}{n(n-1)!} \int_0^t u(r)(t-r)^n\, dr \\
&= \frac{1}{n!} \int_0^t u(r)(t-r)^n\, dr,
\end{aligned}$$

as required. ■

Note that this is a left integral as in Definition 3.1. Now, just like in defining the gamma function, we may extend n to be any positive real number, because the integral will still converge. Thus, we get the following definition:

Definition 3.5 (Riemann-Liouville). Let $u : \mathbb{R} \to \mathbb{R}$ be bounded and left-continuous on $[c, t]$. The *Riemann-Liouville fractional integral* is defined to be

$$I^\alpha u(t) = \frac{1}{\Gamma(\alpha)} \int_c^t (t-s)^{\alpha-1} u(s)\, ds.$$

Without loss of generality, we can often take $c = 0$. We will assume this simplification unless otherwise noted. We can also see that the left-continuity of u allows this integral to be well-defined, because it is still continuous in a sufficiently large neighborhood near the upper limit of integration. Next, let us begin the journey of realizing that our definition satisfies our desired required properties.

Proposition 3.6. *The Riemann-Liouville integral satisfies the four requirements above.*

Proof:

(1) This follows from the previous lemma.

(2) Note that substituting $\alpha = 0$ yields an indeterminate form since $\Gamma(0) = \infty$ and since the integrand diverges at $s = t$. Thus, we consider the limit

$$\begin{aligned} I^0 u(t) &= \lim_{\alpha \to 0^+} \frac{1}{\Gamma(\alpha)} \int_0^t u(s)(t-s)^{\alpha-1}\, ds \\ &= \lim_{\alpha \to 0^+} \frac{1}{\Gamma(\alpha)} \int_0^t \big(u(s) - u(t)\big)(t-s)^{\alpha-1}\, ds + \lim_{\alpha \to 0^+} \frac{1}{\Gamma(\alpha)} \int_0^t u(t)(t-s)^{\alpha-1}\, ds. \end{aligned}$$

We take to the limit as α goes to zero from the right to ensure that we are still talking about an integral. Define the first term in this sum to be A and the second term to be B. Let $\varepsilon > 0$. Since u is left-continuous on $[0, t]$, there is a $\delta > 0$ such that for any $t - \delta < s < t$, $\big|u(s) - u(t)\big| < \varepsilon$. Note that A can also be written as

$$A = \lim_{\alpha \to 0^+} \frac{1}{\Gamma(\alpha)} \int_0^{t-\delta} \big(u(s) - u(t)\big)(t-s)^{\alpha-1}\, ds + \lim_{\alpha \to 0^+} \frac{1}{\Gamma(\alpha)} \int_{t-\delta}^t \big(u(s) - u(t)\big)(t-s)^{\alpha-1}\, ds.$$

Call the first term in this sum to be C and the second term to be D. Thus, by the triangle inequality, we get that

$$\begin{aligned} |A| &= |C + D| \\ &\le |C| + |D|. \end{aligned}$$

First, we shall investigate the convergence of $|C|$. Since u is bounded, it must be that the difference of u and a constant is bounded, so there is a $M > 0$ such that for every $0 \le s \le t$, $\big|u(s) - u(t)\big| < M$. Thus,

$$\begin{aligned} |C| &= \left| \lim_{\alpha \to 0^+} \frac{1}{\Gamma(\alpha)} \int_0^{t-\delta} \big(u(s) - u(t)\big)(t-s)^{\alpha-1}\, ds \right| \\ &\le \lim_{\alpha \to 0^+} \frac{1}{\Gamma(\alpha)} \int_0^{t-\delta} \left| \big(u(s) - u(t)\big)(t-s)^{\alpha-1} \right| ds \\ &= \lim_{\alpha \to 0^+} \frac{1}{\Gamma(\alpha)} \int_0^{t-\delta} \big|u(s) - u(t)\big|(t-s)^{\alpha-1}\, ds \\ &\le \lim_{\alpha \to 0^+} \frac{M}{\Gamma(\alpha)} \int_0^{t-\delta} (t-s)^{\alpha-1}\, ds \\ &= \lim_{\alpha \to 0^+} \frac{M}{\Gamma(\alpha)} \frac{-(t-s)^\alpha}{\alpha} \Bigg|_0^{t-\delta} \\ &= \lim_{\alpha \to 0^+} \frac{M(t^\alpha - \delta^\alpha)}{\Gamma(\alpha+1)} \\ &= \frac{M(t^0 - \delta^0)}{\Gamma(1)} \\ &= 0. \end{aligned}$$

Note that t^α, δ^α are both well-defined and continuous since both $t, \delta > 0$. Next, we determine the convergence of $|D|$.

$$\begin{aligned}
|D| &= \left| \lim_{\alpha \to 0^+} \frac{1}{\Gamma(\alpha)} \int_{t-\delta}^{t} \big(u(s) - u(t)\big)(t-s)^{\alpha-1}\, ds \right| \\
&\leq \lim_{\alpha \to 0^+} \frac{1}{\Gamma(\alpha)} \int_{t-\delta}^{t} \left| \big(u(s) - u(t)\big)(t-s)^{\alpha-1} \right| ds \\
&= \lim_{\alpha \to 0^+} \frac{1}{\Gamma(\alpha)} \int_{t-\delta}^{t} \big|u(s) - u(t)\big|(t-s)^{\alpha-1}\, ds \\
&\leq \lim_{\alpha \to 0^+} \frac{\varepsilon}{\Gamma(\alpha)} \int_{t-\delta}^{t} (t-s)^{\alpha-1}\, ds \\
&= \lim_{\alpha \to 0^+} \frac{\varepsilon}{\Gamma(\alpha)} \frac{-(t-s)^\alpha}{\alpha} \Big|_{t-\delta}^{t} \\
&= \lim_{\alpha \to 0^+} \frac{\varepsilon \delta^\alpha}{\Gamma(\alpha+1)} \\
&= \frac{\varepsilon \delta^0}{\Gamma(1)} \\
&= \varepsilon.
\end{aligned}$$

Thus, we have that $|A| < \varepsilon$. Since ε was arbitrary, it follows that $A = 0$. Now, we determine the convergence of B.

$$\begin{aligned}
B &= \lim_{\alpha \to 0^+} \frac{1}{\Gamma(\alpha)} \int_0^t u(t)(t-s)^{\alpha-1}\, ds \\
&= \lim_{\alpha \to 0^+} \frac{u(t)}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}\, ds \\
&= \lim_{\alpha \to 0^+} \frac{u(t)}{\Gamma(\alpha)} \frac{-(t-s)^\alpha}{\alpha} \Big|_0^t \\
&= \lim_{\alpha \to 0^+} \frac{u(t)t^\alpha}{\Gamma(\alpha+1)} \\
&= \frac{u(t)t^0}{\Gamma(1)} \\
&= u(t).
\end{aligned}$$

Since $I^0 u(t) = A + B$, it must be that $I^0 u(t) = u(t)$.

(3) Let $\alpha \in \mathbb{R}$, and let $a, b \in \mathbb{R}$. Thus,

$$\begin{aligned}
I^\alpha\big(au(t) + bv(t)\big) &= \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}\big(au(s) + bv(s)\big)\, ds \\
&= \frac{a}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} u(s)\, ds + \frac{b}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} v(s)\, ds \\
&= aI^\alpha u(t) + bI^\alpha v(t).
\end{aligned}$$

(4) Let $\alpha, \beta > 0$. Thus, we have that

$$\begin{aligned}
I^{\alpha} I^{\beta} u(t) &= \frac{1}{\Gamma(\alpha)} \int_0^t \frac{I^{\beta} u(s)}{(t-s)^{1-\alpha}}\, ds \\
&= \frac{1}{\Gamma(\alpha)} \int_0^t \frac{1}{(t-s)^{1-\alpha}} \frac{1}{\Gamma(\beta)} \int_0^s (s-r)^{\beta-1} u(r)\, dr\, ds \\
&= \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_0^t \int_0^s \frac{u(r)}{(t-s)^{1-\alpha}(s-r)^{1-\beta}}\, dr\, ds \\
&= \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_0^t u(r) \int_r^t \frac{1}{(t-s)^{1-\alpha}(s-r)^{1-\beta}}\, ds\, dr.
\end{aligned}$$

Make the substitution $v(s) = \dfrac{t-s}{t-r}$, so $dv = \dfrac{-ds}{t-r}$, hence $ds = (r-t)\, dv$. Since $v(s) = \dfrac{t-s}{t-r}$, we have that $v(t-r) = s-r$, and also $1 - v = \dfrac{s-r}{t-r}$, so $s - r = (1-v)(t-r)$. Thus, by Theorem 2.8,

$$\begin{aligned}
I^{\alpha} I^{\beta} u(t) &= \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_0^t u(r) \int_1^0 \frac{r-t}{\big(v(t-r)\big)^{1-\alpha}\big((1-v)(t-r)\big)^{1-\beta}}\, dv\, dr \\
&= \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_0^t \frac{u(r)(t-r)}{(t-r)^{1-\alpha}(t-r)^{1-\beta}} \int_0^1 \frac{1}{v^{1-\alpha}(1-v)^{1-\beta}}\, dv\, dr \\
&= \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_0^t \frac{u(r)}{(t-r)^{1-(\alpha+\beta)}} \int_0^1 \frac{1}{v^{1-\alpha}(1-v)^{1-\beta}}\, dv\, dr \\
&= \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_0^t \frac{u(r)}{(t-r)^{1-(\alpha+\beta)}} \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}\, dr \\
&= \frac{1}{\Gamma(\alpha+\beta)} \int_0^t \frac{u(r)}{(t-r)^{1-(\alpha+\beta)}}\, dr \\
&= I^{\alpha+\beta} u(t).
\end{aligned}$$

■

3.2. **Examples.**

We may now take the Riemann-Liouville integral to be an acceptable definition. While it satisfies the nice requirements that classical integrals are expected to follow, we lose familiar rules of integration, such as the power rule. In the following examples, we calculate the Riemann-Liouville fractional integral of order $\frac{1}{2}$ of simple monomial functions. Note that our assumption of $c = 0$ will impact the computation, but we will keep it here for further simplification.

Example 3.7. *Calculate $I^{\frac{1}{2}}(1)$.*

We will directly apply the Riemann-Liouville integral definition:

$$\begin{aligned}
I^{\frac{1}{2}}(1) &= \frac{1}{\Gamma(\frac{1}{2})} \int_0^t (t-s)^{\frac{1}{2}-1}\, ds \\
&= \frac{1}{\sqrt{\pi}} \int_0^t (t-s)^{-\frac{1}{2}}\, ds
\end{aligned}$$

$$= -2\frac{1}{\sqrt{\pi}}(t-s)^{\frac{1}{2}}\Big|_0^t$$
$$= \frac{2}{\sqrt{\pi}}t^{\frac{1}{2}}.$$

Example 3.8. *Calculate* $I^{\frac{1}{2}}(t)$*.*

We directly apply the Riemann-Liouville integral definition and use Theorem 2.8:

$$\begin{aligned} I^{\frac{1}{2}}(t) &= \frac{1}{\Gamma(\frac{1}{2})}\int_0^t s(t-s)^{\frac{1}{2}-1}\,ds \\ &= \frac{1}{\Gamma(\frac{1}{2})}t^{-\frac{1}{2}}\int_0^t s\Big(1-\frac{s}{t}\Big)^{-\frac{1}{2}}\,ds. \end{aligned}$$

Let $r = \dfrac{s}{t}$, so $ds = t\,dr$, hence:

$$\begin{aligned} I^{\frac{1}{2}}(t) &= \frac{1}{\Gamma(\frac{1}{2})}t^{-\frac{1}{2}}\int_0^1 tr(1-r)^{-\frac{1}{2}}t\,dr \\ &= \frac{1}{\Gamma(\frac{1}{2})}t^{\frac{3}{2}}\int_0^1 r^{2-1}(1-r)^{\frac{1}{2}-1}\,dr \\ &= \frac{1}{\Gamma(\frac{1}{2})}t^{\frac{3}{2}}\frac{\Gamma(2)\Gamma(\frac{1}{2})}{\Gamma(\frac{5}{2})} \\ &= \frac{1}{\frac{3}{2}\Gamma(\frac{3}{2})}t^{\frac{3}{2}} \\ &= \frac{4}{3\sqrt{\pi}}t^{\frac{3}{2}}. \end{aligned}$$

4. Defining the fractional derivative

Notation 4.1. Let $u : \mathbb{R} \to \mathbb{R}$ be a function. The symbol $D^\alpha u$ represents a derivative of order α of u when $\alpha \geq 0$.

We now move to create a definition for fractional differentiation. The most natural way of attempting to define the fractional derivative of u is one that implements elementary functions. One such method is through considering the power function: $u(x) = x^n$. The m-th derivative is calculated to be

$$D^m u(x) = \frac{n!}{(n-m)!} x^{n-m}.$$

From here, we replace m, n by α, β respectively, where $\alpha, \beta > 0$. This is allowed due to the gamma function's capability of interpolating the integer factorials to the real numbers. Thus,

$$D^\alpha u(x) = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)} x^{\beta-\alpha}. \tag{4.1}$$

Let us give some examples to show how this definition differs from ordinary differentiation.

Example 4.2. *Let* $\alpha = \frac{1}{2}$ *and* $\beta = 1$.

We have that $u(x) = x$, so:

$$\begin{aligned} D^{\frac{1}{2}} u(x) &= \frac{\Gamma(2)}{\Gamma(\frac{3}{2})} x^{\frac{1}{2}} \\ &= \frac{1}{\frac{\sqrt{\pi}}{2}} x^{\frac{1}{2}} \\ &= \frac{2}{\sqrt{\pi}} x^{\frac{1}{2}}. \end{aligned}$$

Example 4.3. *Let* $\alpha = \frac{1}{2}$ *and* $\beta = 0$.

We have that $u(x) = 1$, so:

$$\begin{aligned} D^{\frac{1}{2}} u(x) &= \frac{\Gamma(1)}{\Gamma(\frac{1}{2})} x^{-\frac{1}{2}} \\ &= \frac{1}{\sqrt{\pi}} x^{-\frac{1}{2}}. \end{aligned}$$

It is important to note that a derivative of fractional order of a constant might not be 0, as shown in the latter example. The obvious disadvantage to this definition is that it can be used for linear combinations of power functions at most, and this is due to the linearity of the Riemann integral. It excludes a wide range of functions. We could even define fractional derivatives for exponential functions or functions with Fourier series representations, but they would have the same problem. The challenge lies in choosing a larger class of functions and then formulating our fractional derivative definition.

4.1. **Two fractional derivative definitions.**

One of the biggest advantages to defining the fractional integral first is that we can define fractional derivatives in terms of its integral counterpart and ordinary differentiation. Like before, it would be nice if our definition retained some properties of classical differentiation.

First, we must introduce another definition to make sure that our fractional derivatives make sense.

Definition 4.4. We say that a function $u : \mathbb{R} \to \mathbb{R}$ is *left-differentiable at x* if the following limit exists:

$$\lim_{h \to 0^+} \frac{u(x) - u(x-h)}{h},$$

and we notate the *left derivative* of u as $D_{left} u$. In context, we may just say Du instead.

For any bounded and left-differentiable function $u : \mathbb{R} \to \mathbb{R}$, we can now introduce a list of similar requirements:

(1) If $\alpha \in \mathbb{Z}_+$, then the fractional order derivative of u must coincide with the classical integer order derivative of u, i.e. $D^\alpha u(t) = u^{(\alpha)}(t)$.
(2) $D^0 u(t) = u(t)$.
(3) For any $\alpha > 0$, D^α is a linear operator.

Note that the fourth property from the list of fractional integral requirements is not included. For our fractional derivative definitions, this is not true in general – see Section 2.3.3 of [2] for further details. For a specific example of this failure, see Example 4.10.

Because the Riemann-Liouville integral satisfies these properties, it would be helpful to utilize it in our definitions. The first presented here is loosely defined as the first derivative of the Riemann-Liouville fractional integral. The second essentially means that we are taking the Riemann-Liouville fractional integral of the first derivative.

Definition 4.5 (Riemann-Liouville). Let $0 < \alpha < 1$. For any bounded and left-continuous function $u : \mathbb{R} \to \mathbb{R}$, we define the *Riemann-Liouville fractional derivative* of order α to be

$$D_R^\alpha u = D D^{\alpha - 1} u.$$

Definition 4.6 (Caputo). Let $0 < \alpha < 1$. For any bounded and left-differentiably continuous function $u : \mathbb{R} \to \mathbb{R}$, we define the *Caputo fractional derivative* of order α to be

$$D_C^\alpha u = D^{\alpha - 1} D u.$$

In both definitions, D represents a first-order derivative, and $D^{\alpha-1}$ represents Riemann-Liouville integration of order $1 - \alpha$, because $\alpha - 1 < 0$. Also, both of these definitions represent left derivatives due to the x in the upper limit of integration in the Riemann-Liouville integrals.

Remark 4.7. If we did not require that $0 < \alpha < 1$, then $D^{\alpha-1}$ would still represent a derivative. We would have to iterate the definition $\lceil \alpha \rceil$ times to retrieve the Riemann-Liouville integral of order $\lceil \alpha \rceil - \alpha$ and have the definition make sense. Due to this complication, we restrict ourselves to $0 < \alpha < 1$.

At this point, it is of interest to introduce the idea of talking about fractional integrals and derivatives as operators acting on u. This makes it easier to keep track of the order of the derivative or integral. Keeping track of the orders of our operators is like climbing a ladder: if we differentiate a function by some order, then we climb down that many rungs of the ladder. If we integrate a function by some order, then we climb up that may rungs. By this intuition, we see that these two derivatives really are of order α.

Like in the previous section, we must make sure that these two definitions satisfy our desired properties. It turns out that one will work while the other one fails. The proof for the definition that works is a quicker process due to the fact that the Riemann-Liouville integral satisfies them all by Proposition 3.6. It will utilize the fundamental theorem of calculus, so we must first show that it still holds for left-continuous functions.

Lemma 4.8. *Let $u : \mathbb{R} \to \mathbb{R}$ be bounded and left-continuous. Then, $D_{left} I_{left} u = u$.*

Proof: Let $U = I_{left} u$ and $\varepsilon > 0$. Since $u : \mathbb{R} \to \mathbb{R}$ is left-continuous, then there is a $\delta > 0$ such that for any $x - \delta < t < x$, $\big|u(x) - u(t)\big| < \varepsilon$. By the definition of left-differentiability,

$$\begin{aligned} D_{left} I_{left} u(x) &= D_{left} U(x) \\ &= \lim_{h \to 0^+} \frac{U(x) - U(x-h)}{h} \\ &= \lim_{h \to 0^+} \frac{1}{h} \left[\int_0^x u(t)\,dt - \int_0^{x-h} u(t)\,dt \right] \\ &= \lim_{h \to 0^+} \frac{1}{h} \int_{x-h}^x u(t)\,dt \\ &= u(x) + \lim_{h \to 0^+} \frac{1}{h} \int_{x-h}^x \big(u(t) - u(x)\big)\,dt. \end{aligned}$$

It suffices to show that this limit is 0. We see that

$$\begin{aligned} \left| \lim_{h \to 0^+} \frac{1}{h} \int_{x-h}^x \big(u(t) - u(x)\big)\,dt \right| &\leq \lim_{h \to 0^+} \frac{1}{h} \int_{x-h}^x \big|u(t) - u(x)\big|\,dt \\ &< \lim_{h \to 0^+} \frac{1}{h} \int_{x-h}^x \varepsilon\,dt \\ &= \lim_{h \to 0^+} \frac{1}{h} \big(\varepsilon x - \varepsilon(x-h)\big) \\ &= \varepsilon \end{aligned}$$

for all $h < \delta$. Since ε was arbitrary, it must be that

$$\lim_{h \to 0^+} \frac{1}{h} \int_{x-h}^x \big(u(t) - u(x)\big)\,dt = 0,$$

and therefore $D_{left} I_{left} u(x) = u(x)$. ■

Proposition 4.9. *The Riemann-Liouville fractional derivative satisfies the three requirements above.*

Proof:

(1) There is nothing to prove, since we apply a first derivative to the Riemann-Liouville integral, which we know to satisfy this property.

(2) We have that

$$\begin{aligned} D_R^0 u &= D D^{0-1} u \\ &= D I u. \end{aligned}$$

Since these are left derivatives and integrals respectively, by the previous lemma, $D_R^0 u = u$.

(3) D_R^α is linear since the Riemann-Liouville integral and derivative operators are both linear.

■

Example 4.10. *Show that the Riemann-Liouville fractional derivative does not satisfy the following property: For any $\alpha, \beta > 0$, $D_R^{\alpha} D_R^{\beta} u(t) = D_R^{\alpha+\beta} u(t)$.*

Let $u(t) = 1$, $\alpha = \dfrac{1}{2}$, and $\beta = \dfrac{1}{4}$. We must show that $D_R^{\frac{1}{2}} D_R^{\frac{1}{4}}(1) \neq D_R^{\frac{3}{4}}(1)$. We find $D_R^{\frac{3}{4}}(1)$:

$$\begin{aligned}
D_R^{\frac{3}{4}}(1) &= DD^{\frac{3}{4}-1}(1) \\
&= \frac{d}{dx}\frac{1}{\Gamma(\frac{1}{4})}\int_0^x (x-t)^{-\frac{3}{4}}\, dt \\
&= \frac{1}{\Gamma(\frac{1}{4})}\frac{d}{dx}\left.\frac{-(x-t)^{\frac{1}{4}}}{\frac{1}{4}}\right|_0^x \\
&= \frac{1}{\Gamma(\frac{1}{4})}\frac{d}{dx}4x^{\frac{1}{4}} \\
&= \frac{1}{\Gamma(\frac{1}{4})}x^{-\frac{3}{4}}.
\end{aligned}$$

We now compute $D_R^{\frac{1}{2}} D_R^{\frac{1}{4}}(1)$:

$$\begin{aligned}
D_R^{\frac{1}{2}} D_R^{\frac{1}{4}}(1) &= DD^{\frac{1}{2}-1}(DD^{\frac{1}{4}-1})(1) \\
&= \frac{1}{\Gamma(\frac{1}{2})}\frac{d}{dx}\int_0^x (x-t)^{-\frac{1}{2}}\left(\frac{1}{\Gamma(\frac{1}{4})}\frac{d}{dt}\int_0^t (t-s)^{-\frac{1}{4}}\, ds\right) dt \\
&= \frac{1}{\Gamma(\frac{1}{2})\Gamma(\frac{1}{4})}\frac{d}{dx}\int_0^x (x-t)^{-\frac{1}{2}}\left(\frac{d}{dt}\left.\frac{-(t-s)^{\frac{3}{4}}}{\frac{3}{4}}\right|_0^t\right) dt \\
&= \frac{1}{\Gamma(\frac{1}{2})\Gamma(\frac{1}{4})}\frac{d}{dx}\int_0^x (x-t)^{-\frac{1}{2}}\left(\frac{d}{dt}\frac{4}{3}t^{\frac{3}{4}}\right) dt \\
&= \frac{1}{\Gamma(\frac{1}{2})\Gamma(\frac{1}{4})}\frac{d}{dx}\int_0^x (x-t)^{-\frac{1}{2}} t^{-\frac{1}{4}}\, dt \\
&= \frac{1}{\Gamma(\frac{1}{2})\Gamma(\frac{1}{4})}\frac{d}{dx}x^{-\frac{1}{2}}\int_0^x \left(1-\frac{t}{x}\right)^{-\frac{1}{2}} t^{-\frac{1}{4}}\, dt.
\end{aligned}$$

Let $r = \dfrac{t}{x}$, so $dt = x\,dr$. We also see that $t^{-\frac{1}{4}} = r^{-\frac{1}{4}}x^{-\frac{1}{4}}$. Hence,

$$\begin{aligned}
D_R^{\frac{1}{2}} D_R^{\frac{1}{4}}(1) &= \frac{1}{\Gamma(\frac{1}{2})\Gamma(\frac{1}{4})}\frac{d}{dx}x^{-\frac{1}{2}}\int_0^1 (1-r)^{-\frac{1}{2}} r^{-\frac{1}{4}} x^{-\frac{1}{4}} x\, dr \\
&= \frac{1}{\Gamma(\frac{1}{2})\Gamma(\frac{1}{4})}\frac{d}{dx}x^{\frac{1}{4}}\int_0^1 (1-r)^{\frac{1}{2}-1} r^{\frac{3}{4}-1}\, dr \\
&= \frac{1}{\Gamma(\frac{1}{2})\Gamma(\frac{1}{4})}\frac{d}{dx}x^{\frac{1}{4}}\frac{\Gamma(\frac{1}{2})\Gamma(\frac{3}{4})}{\Gamma(\frac{5}{4})} \\
&= \frac{\Gamma(\frac{3}{4})}{4\Gamma(\frac{1}{4})\Gamma(\frac{5}{4})}x^{-\frac{3}{4}}.
\end{aligned}$$

Clearly, these two expressions are not equal, so the property does not hold in general.

We point out that in the previous example, we chose the simplest case by letting u be a constant function, but there was still a lot of work in finding the fractional derivatives. However, we note that Theorem 2.8 was of incredible use, and this would be the case for finding the Riemann-Liouville fractional derivative of any linear combination of power functions. For other functions, different integration methods must be used.

As it turns out, the Caputo derivative does not satisfy (2), because the fundamental theorem of calculus would recover the $u(0)$ term. Unless we made the requirement that $u(0) = 0$, this would be a problem. This begins to highlight the key differences between the two definitions. In the Caputo derivative, the fractional integral recovers initial values of u, whereas in the Riemann-Liouville definition, the first derivative eliminates any initial condition. Order matters, and the operators do not always commute. Also, the Caputo derivative requires that u is left-differentiably continuous, that is, u' is left-continuous, because it is necessary for the integral to be well-defined. The Riemann-Liouville derivative requires only left-continuity. Observe the following examples:

Example 4.11. *Find the Riemann-Liouville and Caputo derivatives of order* $\frac{1}{2}$ *of* $u(x) = 1$.

$$\begin{aligned} D_R^{\frac{1}{2}}(1) &= DD^{\frac{1}{2}-1}(1) \\ &= \frac{d}{dx}\frac{1}{\Gamma(\frac{1}{2})}\int_0^x (x-t)^{-\frac{1}{2}}\,dt \\ &= \frac{d}{dx}\frac{-2}{\sqrt{\pi}}(x-t)^{\frac{1}{2}}\Big|_0^x \\ &= \frac{d}{dx}\frac{2}{\sqrt{\pi}}x^{\frac{1}{2}} \\ &= \frac{1}{\sqrt{\pi}}x^{-\frac{1}{2}}. \end{aligned} \tag{4.2}$$

However, we also have that

$$\begin{aligned} D_C^{\frac{1}{2}}(1) &= D^{\frac{1}{2}-1}D(1) \\ &= \frac{1}{\Gamma(\frac{1}{2})}\int_0^x \frac{0}{(x-t)^{\frac{1}{2}}}\,dt \\ &= 0. \end{aligned}$$

Example 4.12. *Find the Riemann-Liouville and Caputo derivatives of order* $\frac{1}{2}$ *of* $u(x) = x$.

$$\begin{aligned} D_R^{\frac{1}{2}}(x) &= DD^{\frac{1}{2}-1}(x) \\ &= \frac{d}{dx}\frac{1}{\Gamma(\frac{1}{2})}\int_0^x t(x-t)^{-\frac{1}{2}}\,dt \\ &= \frac{d}{dx}\frac{4}{3\sqrt{\pi}}x^{\frac{3}{2}} \\ &= \frac{2}{\sqrt{\pi}}x^{\frac{1}{2}}. \end{aligned} \tag{4.3}$$

However, we also have that

$$\begin{aligned} D_C^{\frac{1}{2}}(x) &= D^{\frac{1}{2}-1}D(x) \\ &= \frac{1}{\Gamma(\frac{1}{2})}\int_0^x \frac{1}{(x-t)^{\frac{1}{2}}}\,dt \\ &= \frac{-2}{\sqrt{\pi}}(x-t)^{\frac{1}{2}}\Big|_0^x \\ &= \frac{2}{\sqrt{\pi}}x^{\frac{1}{2}}. \end{aligned}$$

We can see here that the Riemann-Liouville derivative applied to these power functions yields the same result as the definition given in (4.1). This suggests that the Riemann-Liouville derivative is a general case of the power function derivative, and this can be shown by applying the Riemann-Liouville derivative to any power function and using Theorem 2.8. We also see that in the previous example, we have a special case where the Riemann-Liouville and Caputo derivatives yield the same result. Just because the Caputo derivative does not satisfy requirement (2) does not not mean that it should be dismissed completely, as there are still other uses for it. However, for the remainder of this paper, we will focus on the Riemann-Liouville derivative because of its use in applications later.

4.2. **A fractional fundamental theroem of calculus.**

Finally, we show how the Riemann-Liouville fractional integral and derivative relate, and we do so via a fractional analogue of the fundamental theorem of calculus. Let us restate the classical theorem here for context:

Theorem 4.13 (**Fundamental theorem of calculus**). *Let $u : \mathbb{R} \to \mathbb{R}$ be left-continuous on $[0, x]$ and left-differentiable on $(0, x)$. Then,*

$$DIu(x) = \frac{d}{dx}\int_0^x u(t)\,dt = u(x), \tag{4.4}$$

$$IDu(x) = \int_0^x u'(t)\,dt = u(x) - u(0). \tag{4.5}$$

We see that the integral and derivative operators generally do not commute. This observation also holds in the fractional sense, but we will see that there is a clear discrepancy in the recovered initial condition.

Theorem 4.14 (**Fractional fundamental theorem of calculus**). *Let $u : \mathbb{R} \to \mathbb{R}$ be left-continuous and bounded on $[0, x]$ and left-differentiable on $(0, x)$. Then,*

$$D_R^\alpha I^\alpha u(x) = u(x), \tag{4.6}$$

$$I^\alpha D_R^\alpha u(x) = u(x) - \frac{I^{1-\alpha}u(0)}{\Gamma(\alpha)}x^{\alpha-1}. \tag{4.7}$$

Proof: The proof of (4.6) is a simple calculation with differential and integral operators:

$$\begin{aligned} D_R^\alpha I^\alpha u &= D^1 D^{\alpha-1} D^{-\alpha} u \\ &= D^{1+\alpha-1-\alpha} u \\ &= D^0 u \\ &= u. \end{aligned}$$

The proof of (4.7) is more complex. By definition, we have that

$$\begin{aligned}
I^{\alpha} D_R^{\alpha} u(x) &= \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} D_R^{\alpha} u(t)\, dt \\
&= \frac{1}{\Gamma(\alpha+1)} \int_0^x \alpha (x-t)^{\alpha-1} D_R^{\alpha} u(t)\, dt \\
&= \frac{d}{dx} \frac{1}{\Gamma(\alpha+1)} \int_0^x (x-t)^{\alpha} D_R^{\alpha} u(t)\, dt,
\end{aligned}$$

which follows from the Leibniz integral rule. However, we also have that by the classical fundamental theorem of calculus,

$$\begin{aligned}
\frac{1}{\Gamma(\alpha+1)} \int_0^x (x-t)^{\alpha} D_R^{\alpha} u(t)\, dt &= \frac{1}{\Gamma(\alpha+1)} \int_0^x (x-t)^{\alpha} D D^{-1} D_R^{\alpha} u(t)\, dt \\
&= \frac{1}{\Gamma(\alpha+1)} \int_0^x (x-t)^{\alpha} D D^{-(1-\alpha)} u(t)\, dt \\
&= \frac{1}{\Gamma(\alpha+1)} \Big((x-t)^{\alpha} I^{1-\alpha} u(t) \Big) \Big|_0^x \\
&\quad + \frac{1}{\Gamma(\alpha+1)} \int_0^x \alpha (x-t)^{\alpha-1} I^{1-\alpha} u(t)\, dt \\
&= -\frac{1}{\Gamma(\alpha+1)} x^{\alpha} I^{1-\alpha} u(0) + \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} I^{1-\alpha} u(t)\, dt \\
&= -\frac{I^{1-\alpha} u(0)}{\Gamma(\alpha+1)} x^{\alpha} + I^{\alpha} I^{1-\alpha} u(x) \\
&= I u(x) - \frac{I^{1-\alpha} u(0)}{\Gamma(\alpha+1)} x^{\alpha}.
\end{aligned}$$

Finally, we have that

$$\begin{aligned}
I^{\alpha} D_R^{\alpha} u(x) &= D\Big[I u(x) - \frac{I^{1-\alpha} u(0)}{\Gamma(\alpha+1)} x^{\alpha} \Big] \\
&= u(x) - \frac{\alpha I^{1-\alpha} u(0)}{\Gamma(\alpha+1)} x^{\alpha-1} \\
&= u(x) - \frac{I^{1-\alpha} u(0)}{\Gamma(\alpha)} x^{\alpha-1}.
\end{aligned}$$

■

Remark 4.15. If $\alpha = 1$ in (4.7), then we see that

$$\begin{aligned}
I D u(x) &= u(x) - \frac{I^{1-1} u(0)}{\Gamma(1)} x^{1-1} \\
&= u(x) - I^0 u(0) x^0 \\
&= u(x) - u(0),
\end{aligned}$$

as expected from the second classical fundamental theorem of calculus.

5. Solving a fractional differential equation

Most of the material from this section is covered in Chapter 8 of [3]. It is well-known that the one function whose classical derivative is equal to itself is the natural exponential function. In particular, the unique solution to the initial value problem

$$\begin{cases} u'(x) = \lambda u(x) & x \in \mathbb{R} \\ u(0) = C \end{cases} \tag{5.1}$$

is $u(x) = Ce^{\lambda x}$. Now, the natural question is as follows: For $0 < \alpha < 1$, what is the unique function whose fractional derivative of order $0 < \alpha < 1$ returns the original function, that is, what is the unique solution to the initial value problem

$$\begin{cases} D^{\alpha}u(x) = \lambda u(x) & x > 0, \quad 0 < \alpha < 1 \\ D^{\alpha-1}u(0) = C \end{cases}? \tag{5.2}$$

We consider $D^{\alpha-1}u(0)$ because it is synonymous to the initial condition in (5.1). Note that the initial condition in (5.1) utilizes a derivative of degree one less than that of the ordinary differential equation, which is 0 in particular. Similarly, we require that the initial condition in (5.2) also has a derivative of degree one less than the degree of the corresponding fractional differential equation. With respect to the ordinary case, this choice of an initial condition is the most natural.

5.1. Existence and uniqueness of a solution.

Before we consider the initial problem as a whole, let us first define a function that will be of use later in this section:

Definition 5.1. The *Mittag-Leffler function* is defined as

$$E_{\alpha,\beta}(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(\alpha k + \beta)}, \tag{5.3}$$

where $\alpha, \beta > 0$.

We can think of this function as a generalized exponential function, because if we have that both $\alpha, \beta = 1$, then the resulting power series is the one that matches the Maclaurin series of e^x. We first must check the radius of convergence to determine where $E_{\alpha,\beta}$ is well-defined. As it turns it out, the Mittag-Leffler function has an infinite radius of convergence like the exponential function does.

Proposition 5.2. *The series for $E_{\alpha,\beta}(x)$ converges for any $x \in \mathbb{R}$.*

Proof: We apply the ratio test, and we note that any absolute value needed is obsolete. Let $d_k = \dfrac{x^k}{\Gamma(\alpha k + \beta)}$, so

$$\begin{aligned} \lim_{k\to\infty} \frac{d_{k+1}}{d_k} &= \lim_{k\to\infty} \frac{x^{k+1}}{\Gamma\big(\alpha(k+1)+\beta\big)} \frac{\Gamma(\alpha k + \beta)}{x^k} \\ &= x \lim_{k\to\infty} \frac{\Gamma(\alpha k + \beta)}{\Gamma(\alpha k + \alpha + \beta)}. \end{aligned}$$

The β in the argument of both gamma functions may be removed, because the limit will render them inconsequential anyway. Let $\alpha k = z + (1-\alpha)$ for some $z > 0$. Thus, we have that

$$\begin{aligned}\lim_{k\to\infty} \frac{d_{k+1}}{d_k} &= x \lim_{z\to\infty} \frac{\Gamma\big(z+(1-\alpha)\big)}{\Gamma\big(z+(1-\alpha)+\alpha\big)} \\ &= x \lim_{z\to\infty} \frac{\Gamma\big(z+(1-\alpha)\big)}{\Gamma(z+1)}.\end{aligned}$$

By Gautschi's inequality (2.4), we have that

$$\begin{aligned} x \lim_{z\to\infty} z^{-\alpha} \le x \lim_{z\to\infty} &\frac{\Gamma(z+1-\alpha)}{\Gamma(z+1)} \le x \lim_{z\to\infty} (z+1)^{-\alpha} \\ 0 \le x \lim_{z\to\infty} &\frac{\Gamma(z+1-\alpha)}{\Gamma(z+1)} \le 0.\end{aligned}$$

Therefore, $x \lim_{z\to\infty} \dfrac{\Gamma(z+1-\alpha)}{\Gamma(z+1)} = 0$, so $\lim_{k\to\infty} \dfrac{d_{k+1}}{d_k} = 0$, and the series given by $E_{\alpha,\beta}(x)$ has an infinite radius of convergence. ∎

Next, let us recall a classical result in the theory of ordinary differential equations: Picard's theorem. Then, we will construct and prove a fractional order version of Picard's theorem in order to suit our needs. We will restrict ourselves to the space of continuous functions instead of left-continuous functions for simplicity, but all of our left-calculus theorems still apply.

Theorem 5.3. *Let $f : A \times B \subset \mathbb{R}^2 \to \mathbb{R}$ be continuous, where $(x, y) \in A \times B$. Suppose also that for every $x \in A, y_1, y_2 \in B$,*

$$\big|f(x, y_1) - f(x, y_2)\big| \le L\,|y_1 - y_2|\,.$$

Finally, suppose that

$$\sup_{(x,y)\in A\times B} \big|f(x, y)\big| = b_0 < \infty.$$

Then, there is a unique continuous solution to the initial value problem

$$\begin{cases} u'(x) = f\big(x, u(x)\big), & x > 0 \\ u(0) = b_1 \end{cases}$$

with domain $D = \Big\{(x, y) \in A \times B : \, 0 < x \le h, y = u(x), \big|u(x) - b_1\big| \le a\Big\}$ for some $a, h > 0$.

Note that the restrictions on the domain define the neighborhood that we would like our solution to live in. We must define a similar domain in the fractional variant, and we will see why it is of importance in the proof.

Theorem 5.4. *Let $0 < \alpha < 1$. Let $f : A \times B \subset \mathbb{R}^2 \to \mathbb{R}$ be continuous, where $(x, y) \in A \times B$. Suppose also that for every $x \in A, y_1, y_2 \in B$,*

$$\big|f(x, y_1) - f(x, y_2)\big| \le L\,|y_1 - y_2|\,.$$

Finally, suppose that

$$\sup_{(x,y)\in A\times B} \big|f(x, y)\big| = b_0 < \infty.$$

Then, there is a unique continuous solution to the initial value problem

$$\begin{cases} D^{\alpha}(x) = f\big(x, u(x)\big), & x > 0 \\ D^{\alpha-1}u(0) = b_1 \end{cases} \tag{5.4}$$

with domain

$$D = \Big\{(x,y) \in A \times B : \; 0 < x \leq h, y = u(x), \Big|x^{1-\alpha}u(x) - \frac{b_1}{\Gamma(\alpha)}\Big| \leq \frac{b_0 h}{\Gamma(\alpha+1)} < a < \infty\Big\}$$

for some $a, h > 0$.

Proof: Start by integrating (5.4) with order α:

$$\begin{aligned} I^{\alpha} f\big(x, u(x)\big) &= I^{\alpha} D^{\alpha} u(x) \\ &= u(x) - \frac{I^{1-\alpha}u(0)}{\Gamma(\alpha)} x^{\alpha-1}, \end{aligned}$$

which follows from (4.7). Thus,

$$\begin{aligned} u(x) &= \frac{I^{1-\alpha}u(x)}{\Gamma(\alpha)} x^{\alpha-1} + I^{\alpha} f\big(x, u(x)\big) \\ &= b_1 \frac{x^{\alpha-1}}{\Gamma(\alpha)} + \frac{1}{\Gamma(\alpha)} \int_0^x \frac{f\big(t, u(t)\big)}{(x-t)^{1-\alpha}}\, dt, \end{aligned} \tag{5.5}$$

which follows from the initial condition in (5.4) and the definition of the Riemann-Liouville integral. We have just derived this from our original problem (5.4), but we must verify that for any continuous function f that satisfies (5.5), it also satisfies (5.4).

First, we show that $D^{\alpha}u(x) = f\big(x, u(x)\big)$ still holds. From (5.5), we get that

$$\begin{aligned} D^{\alpha}u(x) &= D^{\alpha}\left[b_1 \frac{x^{\alpha-1}}{\Gamma(\alpha)} + \frac{1}{\Gamma(\alpha)} \int_0^x \frac{f\big(t, u(t)\big)}{(x-t)^{1-\alpha}}\, dt\right] \\ &= \frac{b_1}{\Gamma(\alpha)} D^{\alpha} x^{\alpha-1} + f\big(x, u(x)\big) \\ &= \frac{b_1}{\Gamma(\alpha)} \frac{\Gamma(\alpha+1-1)}{\Gamma(\alpha-1-\alpha+1)} x^{\alpha-1-\alpha} + f\big(x, u(x)\big) \\ &= \frac{b_1}{\Gamma(0)x} + f\big(x, u(x)\big) \\ &= f\big(x, u(x)\big), \end{aligned}$$

since $\Gamma(0) \to +\infty$ as $x \to 0^{+}$. Now, we show that $D^{\alpha-1}u(0) = b_1$. Also from (5.5), we get that

$$\begin{aligned} D^{\alpha-1}u(x) &= D^{\alpha-1}\left[b_1 \frac{x^{\alpha-1}}{\Gamma(\alpha)} + \frac{1}{\Gamma(\alpha)} \int_0^x \frac{f\big(t, u(t)\big)}{(x-t)^{1-\alpha}}\, dt\right] \\ &= \frac{b_1}{\Gamma(\alpha)} D^{\alpha-1} x^{\alpha-1} + D^{\alpha-1} I^{\alpha} u(x) \\ &= \frac{b_1}{\Gamma(\alpha)} \frac{\Gamma(\alpha+1-1)}{\Gamma(\alpha-1-\alpha+1+1)} x^{\alpha-1-\alpha+1} + D^{-1} D^{\alpha} I^{\alpha} u(x) \\ &= b_1 + D^{-1}u(x) \\ &= b_1 + \int_0^x u(t)\, dt. \end{aligned}$$

Evaluating at $x = 0$ gives that $D^{\alpha-1}u(0) = b_1$, as required. Thus, (5.4) is equivalent to (5.5), so we will use the latter formulation from here on out.

Next, we wish to find a fixed point of (5.5), and we will use successive approximations. Let

$$\begin{aligned} u_0(x) &= b_1 \frac{x^{\alpha-1}}{\Gamma(\alpha)}, \\ u_n(x) &= b_1 \frac{x^{\alpha-1}}{\Gamma(\alpha)} + \frac{1}{\Gamma(\alpha)} \int_0^x \frac{f\big(t, u_{n-1}(t)\big)}{(x-t)^{1-\alpha}}\, dt. \end{aligned} \tag{5.6}$$

Consider the following estimate:

$$\begin{aligned} \left| x^{1-\alpha} u_n(x) - x^{1-\alpha} u_0(x) \right| &= \left| \frac{b_1}{\Gamma(\alpha)} + \frac{x^{1-\alpha}}{\Gamma(\alpha)} \int_0^x \frac{f\big(t, u(t)\big)}{(x-t)^{1-\alpha}}\, dt - \frac{b_1}{\Gamma(\alpha)} \right| \\ &= \frac{x^{1-\alpha}}{\Gamma(\alpha)} \left| \int_0^x \frac{f\big(t, u(t)\big)}{(x-t)^{1-\alpha}}\, dt \right| \\ &\le \frac{x^{1-\alpha}}{\Gamma(\alpha)} \int_0^x \left| \frac{f\big(t, u(t)\big)}{(x-t)^{1-\alpha}} \right| dt \\ &\le \frac{b_0 x^{1-\alpha}}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1}\, dt \\ &= \frac{b_0 x^{1-\alpha}}{\Gamma(\alpha)} \frac{-(x-t)^{\alpha}}{\alpha} \Bigg|_0^x \\ &= \frac{b_0 x^{1-\alpha}}{\Gamma(\alpha)} \frac{x^{\alpha}}{\alpha} \\ &= \frac{b_0 x}{\Gamma(\alpha+1)} \\ &\le \frac{b_0 h}{\Gamma(\alpha+1)}. \end{aligned} \tag{5.7}$$

for some $h \ge x$. Therefore, $\big(x, u_n(x)\big) \in D$. We now move on to successive estimates for each n. For $n = 1$, we have a bound due to (5.7):

$$\begin{aligned} \left| u_1(x) - u_0(x) \right| &\le \frac{b_0 x}{\Gamma(\alpha+1) x^{1-\alpha}} \\ &= \frac{b_0 x^{\alpha}}{\Gamma(\alpha+1)} \\ &\le \frac{b_0 h^{\alpha}}{\Gamma(\alpha+1)}. \end{aligned}$$

For $n = 2$ and on, we use preceding estimates and the Lipschitz continuity of f:

$$\begin{aligned} \left| u_2(x) - u_1(x) \right| &= \left| b_1 \frac{x^{\alpha-1}}{\Gamma(\alpha)} + \frac{1}{\Gamma(\alpha)} \int_0^x \frac{f\big(t, u_1(t)\big)}{(x-t)^{1-\alpha}}\, dt - b_1 \frac{x^{\alpha-1}}{\Gamma(\alpha)} - \frac{1}{\Gamma(\alpha)} \int_0^x \frac{f\big(t, u_0(t)\big)}{(x-t)^{1-\alpha}}\, dt \right| \\ &= \left| \frac{1}{\Gamma(\alpha)} \int_0^x \frac{f\big(t, u_1(t)\big)}{(x-t)^{1-\alpha}}\, dt - \frac{1}{\Gamma(\alpha)} \int_0^x \frac{f\big(t, u_0(t)\big)}{(x-t)^{1-\alpha}}\, dt \right| \\ &= \frac{1}{\Gamma(\alpha)} \left| \int_0^x \frac{f\big(t, u_1(t)\big) - f\big(t, u_0(t)\big)}{(x-t)^{1-\alpha}}\, dt \right| \end{aligned}$$

$$\leq \frac{1}{\Gamma(\alpha)}\int_0^x \frac{\big|f\big(t,u_1(t)\big)-f\big(t,u_0(t)\big)\big|}{(x-t)^{1-\alpha}}\,dt$$
$$\leq \frac{L}{\Gamma(\alpha)}\int_0^x \frac{\big|u_1(t)-u_0(t)\big|}{(x-t)^{1-\alpha}}\,dt$$
$$\leq \frac{Lb_0}{\Gamma(\alpha)\Gamma(\alpha+1)}\int_0^x t^{\alpha}(x-t)^{\alpha-1}\,dt$$
$$= \frac{Lb_0x^{\alpha-1}}{\Gamma(\alpha)\Gamma(\alpha+1)}\int_0^x t^{\alpha}\Big(1-\frac{t}{x}\Big)^{\alpha-1}\,dt.$$

Let $r=\dfrac{t}{x}$, so $dt = x\,dr$, and $t^{\alpha}=r^{\alpha}x^{\alpha}$. Thus,

$$\begin{aligned}
\big|u_2(x)-u_1(x)\big| &= \frac{Lb_0x^{\alpha-1}}{\Gamma(\alpha)\Gamma(\alpha+1)}\int_0^1 r^{\alpha}x^{\alpha}(1-r)^{\alpha-1}x\,dr\\
&= \frac{Lb_0x^{2\alpha}}{\Gamma(\alpha)\Gamma(\alpha+1)}\int_0^1 r^{(\alpha+1)-1}(1-r)^{\alpha-1}\,dr\\
&= \frac{Lb_0x^{2\alpha}}{\Gamma(\alpha)\Gamma(\alpha+1)}\frac{\Gamma(\alpha+1)\Gamma(\alpha)}{\Gamma(2\alpha+1)}\\
&= \frac{Lb_0x^{2\alpha}}{\Gamma(2\alpha+1)}\\
&\leq \frac{Lb_0h^{2\alpha}}{\Gamma(2\alpha+1)}.
\end{aligned}$$

By induction, we eventually see that for $n\in\mathbb{N}$,

$$\big|u_n(x)-u_{n-1}(x)\big| \leq \frac{L^{n-1}b_0h^{n\alpha}}{\Gamma(n\alpha+1)}. \tag{5.8}$$

We claim that the sequence $\big\{u_n(x)\big\}$ converges uniformly. Let $0<j\leq n$, take an indexed sum from 1 to n on both sides of (5.8), and let $n\to\infty$:

$$\sum_{j=1}^{\infty}\big|u_j(x)-u_{j-1}(x)\big| \leq \sum_{j=1}^{\infty}\frac{L^{j-1}b_0h^{j\alpha}}{\Gamma(j\alpha+1)}. \tag{5.9}$$

We test the convergence of the right-hand side of (5.9) by the ratio test. Since every term in the sum is positive, we neglect the absolute value and compute the following limit:

$$\begin{aligned}
\lim_{j\to\infty}\frac{\dfrac{L^jb_0h^{(j+1)\alpha}}{\Gamma\big((j+1)\alpha+1\big)}}{\dfrac{L^{j-1}b_0h^{j\alpha}}{\Gamma(j\alpha+1)}} &= \lim_{j\to\infty}\frac{L^jb_0h^{(j+1)\alpha}}{L^{j-1}b_0h^{j\alpha}}\frac{\Gamma(j\alpha+1)}{\Gamma\big((j+1)\alpha+1\big)}\\
&= Lh^{\alpha}\lim_{j\to\infty}\frac{\Gamma(j\alpha+1)}{\Gamma\big((j+1)\alpha+1\big)}\\
&= Lh^{\alpha}\lim_{j\to\infty}\frac{\Gamma(j\alpha+1)}{\Gamma(j\alpha+\alpha+1)}\\
&= Lh^{\alpha}\lim_{j\to\infty}\frac{\Gamma(j\alpha+\alpha+1-\alpha)}{\Gamma(j\alpha+\alpha+1)}.
\end{aligned}$$

Let $z = j\alpha + \alpha$, so

$$\lim_{j\to\infty} \frac{\dfrac{L^j b_0 h^{(j+1)\alpha}}{\Gamma\big((j+1)\alpha+1\big)}}{\dfrac{L^{j-1} b_0 h^{j\alpha}}{\Gamma(j\alpha+1)}} = Lh^\alpha \lim_{z\to\infty} \frac{\Gamma(z+1-\alpha)}{\Gamma(z+1)}.$$

Since $z > 0$, by Gautschi's inequality (2.4),

$$\begin{aligned} \lim_{z\to\infty} z^{-\alpha} \le \lim_{z\to\infty} \frac{\Gamma(z+1-\alpha)}{\Gamma(z+1)} &\le (z+1)^{-\alpha} \\ 0 \le \lim_{z\to\infty} \frac{\Gamma(z+1-\alpha)}{\Gamma(z+1)} &\le 0. \end{aligned}$$

Thus, $\lim_{z\to\infty} \dfrac{\Gamma(z+1-\alpha)}{\Gamma(z+1)} = 0$, so

$$\lim_{j\to\infty} \frac{\dfrac{L^j b_0 h^{(j+1)\alpha}}{\Gamma\big((j+1)\alpha+1\big)}}{\dfrac{L^{j-1} b_0 h^{j\alpha}}{\Gamma(j\alpha+1)}} = 0.$$

Thus, the series converges, so it is necessary that $\dfrac{L^{j-1} b_0 h^{j\alpha}}{\Gamma(j\alpha+1)} \to 0$ as $j \to \infty$. Thus, $\{u_n(x)\}$ is Cauchy in the uniform norm, so $\{u_n(x)\}$ converges uniformly to some $u(x)$.

This function u is continuous since each u_n is continuous and the convergence is uniform. We also can see that $u \in D$. From (5.7),

$$\begin{aligned} \lim_{n\to\infty} \left| x^{1-\alpha} u_n(x) - x^{1-\alpha} u_0(x) \right| &\le \lim_{n\to\infty} \frac{b_0 h}{\Gamma(\alpha+1)} \\ &= \frac{b_0 h}{\Gamma(\alpha+1)}, \end{aligned}$$

but passing the limit through gives that

$$\left| x^{1-\alpha} u(x) - x^{1-\alpha} u_0(x) \right| \le \frac{b_0 h}{\Gamma(\alpha+1)},$$

so $u \in D$. Additionally, we see that u satisfies (5.6):

$$\begin{aligned} \lim_{n\to\infty} u_n(x) &= \lim_{n\to\infty} \left[b_1 \frac{x^{\alpha-1}}{\Gamma(\alpha)} + \frac{1}{\Gamma(\alpha)} \int_0^x \frac{f\big(t, u_{n-1}(t)\big)}{(x-t)^{1-\alpha}}\, dt \right] \\ u(x) &= b_1 \frac{x^{\alpha-1}}{\Gamma(\alpha)} + \frac{1}{\Gamma(\alpha)} \lim_{n\to\infty} \int_0^x \frac{f\big(t, u_{n-1}(t)\big)}{(x-t)^{1-\alpha}}\, dt. \end{aligned}$$

Since the integrand is bounded above by a value independent of x, we can pass the limit through the integral:

$$\begin{aligned} u(x) &= b_1 \frac{x^{\alpha-1}}{\Gamma(\alpha)} + \frac{1}{\Gamma(\alpha)} \int_0^x \lim_{n\to\infty} \frac{f\big(t, u_{n-1}(t)\big)}{(x-t)^{1-\alpha}}\, dt \\ &= b_1 \frac{x^{\alpha-1}}{\Gamma(\alpha)} + \frac{1}{\Gamma(\alpha)} \int_0^x \frac{f\big(t, u(t)\big)}{(x-t)^{1-\alpha}}\, dt, \end{aligned}$$

where the limit can pass through f since f is continuous.

Finally, we prove uniqueness of the solution. By way of contradiction, suppose that $u^{(1)}, u^{(2)}$ are distinct solutions to (5.5), and let $\frac{Lh^\alpha}{\Gamma(\alpha+1)} < 1$ for sufficiently small h. Taking the absolute value of the difference of the solutions gives that

$$\begin{aligned}
\left|u^{(1)}(x) - u^{(2)}(x)\right| &= \left|b_1 \frac{x^{\alpha-1}}{\Gamma(\alpha)} + \frac{1}{\Gamma(\alpha)} \int_0^x \frac{f\big(t, u^{(1)}(t)\big)}{(x-t)^{1-\alpha}}\, dt - b_1 \frac{x^{\alpha-1}}{\Gamma(\alpha)} - \frac{1}{\Gamma(\alpha)} \int_0^x \frac{f\big(t, u^{(2)}(t)\big)}{(x-t)^{1-\alpha}}\, dt\right| \\
&\leq \frac{1}{\Gamma(\alpha)} \int_0^x \left|\frac{f\big(t, u_1(t)\big) - f\big(t, u_0(t)\big)}{(x-t)^{1-\alpha}}\right| dt \\
&= \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{1-\alpha} \left|f\big(t, u^{(1)}(t)\big) - f\big(t, u^{(2)}(t)\big)\right| dt \\
&\leq \frac{L}{\Gamma(\alpha)} \int_0^x (x-t)^{1-\alpha} \left|u^{(1)}(t) - u^{(2)}(t)\right| dt.
\end{aligned}$$

Since the difference of $u^{(1)}, u^{(2)}$ is continuous on $[0, h]$, then by the extreme value theorem, there is a maximum value δ at $x = \varepsilon$, so

$$\begin{aligned}
\left|u^{(1)}(\varepsilon) - u^{(2)}(\varepsilon)\right| &\leq \frac{L}{\Gamma(\alpha)} \int_0^\varepsilon (\varepsilon - t)^{1-\alpha} \delta\, dt \\
\delta &\leq \frac{L\delta}{\Gamma(\alpha)} \frac{\varepsilon^\alpha}{\alpha} \\
&\leq \frac{L\delta h^\alpha}{\Gamma(\alpha+1)}.
\end{aligned}$$

Rearranging gives that $\frac{Lh^\alpha}{\Gamma(\alpha+1)} \geq 1$, a contradiction. ■

5.2. **Solving the initial value problem.**

We can now return to the question of finding the function that satisfies the initial value problem (5.2). Without loss of generality, let $C = 1$. We also note that by the previous proof, we achieve the following:

$$\begin{aligned}
u_0(x) &= \frac{x^{\alpha-1}}{\Gamma(\alpha)} \\
u_n(x) &= \frac{x^{\alpha-1}}{\Gamma(\alpha)} + \frac{\lambda}{\Gamma(\alpha)} \int_0^x \frac{u_{n-1}(t)}{(x-t)^{1-\alpha}}\, dt.
\end{aligned} \tag{5.10}$$

From (5.10), we calculate $u_1(x)$:

$$\begin{aligned}
u_1(x) &= \frac{x^{\alpha-1}}{\Gamma(\alpha)} + \frac{\lambda}{\Gamma(\alpha)} \int_0^x \frac{u_0(t)}{(x-t)^{1-\alpha}}\, dt \\
&= \frac{x^{\alpha-1}}{\Gamma(\alpha)} + \frac{\lambda}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} \frac{t^{\alpha-1}}{\Gamma(\alpha)}\, dt \\
&= \frac{x^{\alpha-1}}{\Gamma(\alpha)} + \frac{\lambda}{\Gamma^2(\alpha)} \int_0^x t^{\alpha-1}(x-t)^{\alpha-1}\, dt \\
&= \frac{x^{\alpha-1}}{\Gamma(\alpha)} + \frac{\lambda x^{\alpha-1}}{\Gamma^2(\alpha)} \int_0^x t^{\alpha-1}\Big(1 - \frac{t}{x}\Big)^{\alpha-1}\, dt.
\end{aligned}$$

Again, we let $r = \dfrac{t}{x}$, so proper substitution gives that

$$\begin{aligned} u_1(x) &= \frac{x^{\alpha-1}}{\Gamma(\alpha)} + \frac{\lambda x^{\alpha-1}}{\Gamma^2(\alpha)} \int_0^1 x^{\alpha} r^{\alpha-1}(1-r)^{\alpha-1}\, dr \\ &= \frac{x^{\alpha-1}}{\Gamma(\alpha)} + \frac{\lambda x^{2\alpha-1}}{\Gamma^2(\alpha)} \int_0^1 r^{\alpha-1}(1-r)^{\alpha-1}\, dr \\ &= \frac{x^{\alpha-1}}{\Gamma(\alpha)} + \frac{\lambda x^{2\alpha-1}}{\Gamma^2(\alpha)} \frac{\Gamma(\alpha)\Gamma(\alpha)}{\Gamma(2\alpha)} \\ &= \frac{x^{\alpha-1}}{\Gamma(\alpha)} + \frac{\lambda x^{2\alpha-1}}{\Gamma(2\alpha)}. \end{aligned}$$

By a similar calculation, we find $u_2(x)$:

$$\begin{aligned} u_2(x) &= \frac{x^{\alpha-1}}{\Gamma(\alpha)} + \frac{\lambda}{\Gamma(\alpha)} \int_0^x \frac{u_1(t)}{(x-t)^{1-\alpha}}\, dt \\ &= \frac{x^{\alpha-1}}{\Gamma(\alpha)} + \frac{\lambda}{\Gamma(\alpha)} \int_0^x \frac{\frac{t^{\alpha-1}}{\Gamma(\alpha)} + \frac{\lambda t^{2\alpha-1}}{\Gamma(2\alpha)}}{(x-t)^{1-\alpha}}\, dt \\ &= \frac{x^{\alpha-1}}{\Gamma(\alpha)} + \frac{\lambda}{\Gamma(\alpha)} \int_0^x (x-t)^{1-\alpha} \frac{t^{\alpha-1}}{\Gamma(\alpha)}\, dt + \frac{\lambda}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} \frac{\lambda t^{2\alpha-1}}{\Gamma(2\alpha)}\, dt \\ &= u_1(x) + \frac{\lambda^2}{\Gamma(\alpha)\Gamma(2\alpha)} \int_0^x t^{2\alpha-1}(x-t)^{\alpha-1}\, dt \\ &= u_1(x) + \frac{\lambda^2 x^{\alpha-1}}{\Gamma(\alpha)\Gamma(2\alpha)} \int_0^x t^{2\alpha-1}\Big(1 - \frac{t}{x}\Big)^{\alpha-1}\, dt. \end{aligned}$$

Once again, let $r = \dfrac{t}{x}$, so

$$\begin{aligned} u_2(x) &= u_1(x) + \frac{\lambda^2 x^{\alpha-1}}{\Gamma(\alpha)\Gamma(2\alpha)} \int_0^1 x^{2\alpha} r^{2\alpha-1}(1-r)^{\alpha-1}\, dr \\ &= u_1(x) + \frac{\lambda^2 x^{3\alpha-1}}{\Gamma(\alpha)\Gamma(2\alpha)} \int_0^1 r^{2\alpha-1}(1-r)^{\alpha-1}\, dr \\ &= u_1(x) + \frac{\lambda^2 x^{3\alpha-1}}{\Gamma(\alpha)\Gamma(2\alpha)} \frac{\Gamma(2\alpha)\Gamma(\alpha)}{\Gamma(3\alpha)} \\ &= \frac{x^{\alpha-1}}{\Gamma(\alpha)} + \frac{\lambda x^{2\alpha-1}}{\Gamma(2\alpha)} + \frac{\lambda^2 x^{3\alpha-1}}{\Gamma(3\alpha)}. \end{aligned}$$

Inductively, we see that

$$\begin{aligned} u_n(x) &= \sum_{k=0}^{n} \frac{\lambda^k x^{(k+1)\alpha-1}}{\Gamma\big((k+1)\alpha\big)} \\ &= x^{1-\alpha} \sum_{k=0}^{n} \frac{\lambda^k x^{\alpha k}}{\Gamma(\alpha k+\alpha)} \\ &= x^{1-\alpha} \sum_{k=0}^{n} \frac{(\lambda x^{\alpha})^k}{\Gamma(\alpha k+\alpha)}. \end{aligned}$$

Finally, taking $n \to \infty$ ultimately gives that

$$u(x) = x^{1-\alpha} \sum_{k=0}^{\infty} \frac{(\lambda x^{\alpha})^k}{\Gamma(\alpha k + \alpha)} = x^{1-\alpha} E_{\alpha,\alpha}(\lambda x^{\alpha}), \tag{5.11}$$

where $E_{\alpha,\alpha}$ is a special case of the Mittag-Leffler function from (5.3). Therefore, we can state our findings precisely:

Theorem 5.5. *The unique solution to the initial value problem*

$$\begin{cases} D^{\alpha} u(x) = \lambda u(x) & x > 0, \quad 0 < \alpha < 1 \\ D^{\alpha-1} u(0) = C \end{cases}$$

is the function $u(x) = Cx^{1-\alpha} E_{\alpha,\alpha}(\lambda x^{\alpha})$.

Remark 5.6. In other words, the function whose fractional derivative of order $0 < \alpha < 1$ is equal to itself is $u(x) = Cx^{1-\alpha} E_{\alpha,\alpha}(\lambda x^{\alpha})$. If $\alpha = 1$, then

$$\begin{aligned} u(x) &= Cx^0 E_{1,1}(\lambda x^1) \\ &= CE_{1,1}(\lambda x) \\ &= Ce^{\lambda x}, \end{aligned}$$

as expected from (5.1).

Remark 5.7. We may be tempted to extend our domain of our solution to (5.2) from $x > 0$ to $x \in \mathbb{R}$ as in the solution for (5.1). However, this allowance would create roots of negative arguments due to the factor $x^{1-\alpha}$. While this may not be a problem for certain choices of α, in general, we will keep the domain $x > 0$.

Remark 5.8. As in previous sections, we take $0 < \alpha < 1$, but this process can be generalized to $n - 1 < \alpha < n$ for any positive integer n. This would heavily impact our solution domain, and the solution that we would get is a further generalization of (5.11). See Chapter 8 of [3] for a deeper explanation.

6. The tautochrone problem

We now move to discuss a physical application of fractional derivatives. Suppose we have a point-mass with mass m and some curve $y = y(x)$ with an endpoint on the origin. Suppose that the mass is released at rest from a starting point (x, y) on the curve, and let (u, v) be some intermediate point along the curve. Define $T(y)$ to be the total time of descent of the mass from the initial height y along the curve. We neglect any loss of energy due to friction. *Abel's mechanical problem* states that given $T(y)$, the goal is to find the equation of the curve $y = y(x)$.

Let s be the arc length of the curve $y(x)$. Suppose that the curve rises from left to right, so the mass will fall in the direction opposite that of $\frac{ds}{dt}$. Since no energy is lost in the system, the total energy of the mass is conserved between the points (x, y) and (u, v). If we consider the kinetic energy (E_K) and potential energy (E_P) of the mass at the two locations, then we have

$$\begin{aligned} E_{K(u,v)} + E_{P(u,v)} &= E_{K(x,y)} + E_{P(x,y)} \\ \frac{1}{2}m\Big(\frac{ds}{dt}\Big)^2 + mgv &= mgy, \end{aligned}$$

so we rearrange to get

$$-\frac{ds}{dt} = \sqrt{2g(y-v)}.$$

Since the mass is descending in the direction opposite of the rate of change of the arc length, we solved for its opposite instead. We can rearrange this last equality to give

$$dt = -\frac{ds}{\sqrt{2g(y-v)}}.$$

Next, we consider s to be a function of v, so differentiating with respect to v gives that $ds = s'(v)\,dv$. Substituting into the previous equality gives that

$$dt = -\frac{s'(v)\,dv}{\sqrt{2g(y-v)}}.$$

We now introduce $T(y)$, which is precisely the integral of both sides of the equation from $v = y$ to $v = 0$:

$$\begin{aligned} T(y) &= \int_y^0 -\frac{s'(v)\,dv}{\sqrt{2g(y-v)}} \\ &= \frac{1}{\sqrt{2g}} \int_0^y \frac{s'(v)\,dv}{\sqrt{y-v}}. \end{aligned} \tag{6.1}$$

On the other hand, we know that the arc length is defined by

$$s = s(y) = \int_0^y \sqrt{1 + \Big(\frac{dx}{d\bar{y}}\Big)^2}\, d\bar{y},$$

so by the fundamental theorem of calculus,

$$s'(y) = \sqrt{1 + \Big(\frac{dx}{dy}\Big)^2}.$$

Since we have completely determined $s'(y)$, it now makes sense to define a function f by $f(y) \coloneqq s'(y)$. Plugging back into (6.1) gives

$$T(y) = \frac{1}{\sqrt{2g}} \int_0^y \frac{f(v)\,dv}{\sqrt{y-v}}.$$

We now come to state a special case of Abel's problem, which is the *tautochrone problem*. If $T(y) = T$ is held constant, then the goal is to find the shape of the curve. Let it be clear that in this case, no matter where the mass is released on the curve, it will reach the origin in the same time T. Such a curve is called a *tautochrone*. We now note that the resulting equation resembles the Riemann-Liouville fractional integral of order $\frac{1}{2}$ of f with the exception of some constants, so the equation takes the simple form of

$$T = \frac{\sqrt{\pi}}{\sqrt{2g}} I^{\frac{1}{2}} f(y). \tag{6.2}$$

We could solve this problem with Laplace transforms if we wanted, as is done in [4], but we have the machinery of fractional differentiation at our disposal. First, rearrange (6.2) as follows:

$$I^{\frac{1}{2}} f(y) = \frac{T\sqrt{2g}}{\sqrt{\pi}}. \tag{6.3}$$

We must differentiate both sides of the equation to the order of $\frac{1}{2}$ for the operators to cancel, but we must choose the correct definition for the fractional derivative. For insight, we look at the right-hand side of (6.3).

If we were to apply the Caputo derivative to this constant, the first order derivative of the constant will turn the integrand into 0, which is not what we want. However, by (4.6) in Theorem 4.14, we have that the Riemann-Liouville derivative applied to the Riemann-Liouville integral of a function is the function itself, so we apply $D_R^{\frac{1}{2}}$ to both sides of (6.3):

$$\begin{aligned} D_R^{\frac{1}{2}} I^{\frac{1}{2}} f(y) &= D_R^{\frac{1}{2}} \left(\frac{T\sqrt{2g}}{\sqrt{\pi}} \right) \\ f(y) &= \left(\frac{T\sqrt{2g}}{\sqrt{\pi}} \right) D_R^{\frac{1}{2}}(1). \end{aligned}$$

We can apply the result from the Riemann-Liouville derivative calculation in Example 4.11:

$$\begin{aligned} f(y) &= \frac{T\sqrt{2g}}{\sqrt{\pi}} \frac{y^{-\frac{1}{2}}}{\sqrt{\pi}} \\ &= \frac{T\sqrt{2g}}{\pi y^{\frac{1}{2}}} \\ &= \sqrt{\frac{b}{y}}, \end{aligned}$$

where $b \coloneqq \frac{2gT^2}{\pi^2}$ for convenience. As we continue on, recalling our definition of f allows us to immediately say that

$$\sqrt{\frac{b}{y}} = \sqrt{1 + \left(\frac{dx}{dy}\right)^2}$$

$$\frac{b}{y} = 1 + \left(\frac{dx}{dy}\right)^2,$$

which is a separable ordinary differential equation. Separating variables and integrating gives that

$$x = \int \sqrt{\frac{b-y}{y}}\, dy.$$

Let $y = b\sin^2\phi$, so substituting gives that

$$\begin{aligned} x &= \int \sqrt{\frac{b - b\sin^2\phi}{b\sin^2\phi}}\, 2b\sin\phi\cos\phi\, d\phi \\ &= b\int 2\cos^2\phi\, d\phi \\ &= b\int (1+\cos 2\phi)\, d\phi \\ &= \frac{b}{2}(2\phi + \sin 2\phi) + C. \end{aligned}$$

Plugging in the initial condition that requires the tautochrone to pass through the origin gives that $C = 0$. For the sake of simplification, by the power reduction formula for sine, we have that $y = \frac{b}{2}(1 - \cos 2\phi)$. Therefore, the tautochrone is modeled by the parametric equations

$$\begin{cases} x(\theta) = a(\theta + \sin\theta) \\ y(\theta) = a(1 - \cos\theta), \end{cases}$$

where $a \coloneqq \frac{b}{2} = \frac{gT^2}{\pi^2}$ and $\theta \coloneqq 2\phi$. To conclude,

Theorem 6.1. *The tautochrone curve is modeled by the following parametric equations:*

$$\begin{cases} x(\theta) = \frac{gT^2}{\pi^2}(\theta + \sin\theta) \\ y(\theta) = \frac{gT^2}{\pi^2}(1 - \cos\theta), \end{cases}$$

where T is the time constant and g is the acceleration due to gravity.

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