Quasistatic evolution with unstable nonlocal forces

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Nonlocal school on Fractional Equations NSFE 2022

Supported by ARO Grant W911NF1910245

This is joint work with Debdeep Bhattacharya
Preprint: arXiv:2204.04571v2

Outline of talk

- Motivation and context
- Mathematical description of quasistatic evolution
- Existence theory using fixed point methods
- The stability tensor
- Quasistatic evolutions exist in the neighborhood of local minima
- Necessary conditions for existence of quasistatic evolution

Motivation

- Nonlocal forces can be used for modeling extreme displacements
- Nonlocal dynamic damage models. Peridynamics: Silling, Journal Mechanics and Physics of Solids 2000
- In this talk we consider extreme deformations in the absence of inertia: The Quasistatic case
- This corresponds to the load on a body changing so slowly that the acceleration of material element is negligible

Overview-Extreme Displacements

- Extreme displacements are in response to forces that damage the material
- Strength domain of a material is set of strains where the force must increase for increasing strain
- If the strain is outside the strength domain the material experiences damage and the force decrease is coincident with increasing strain
- Extreme displacements correspond to strains outside the strength domain of the material

Extreme deformation nonlocal dynamics

$$\rho\ddot{\boldsymbol{u}}(t,\boldsymbol{x}) = \int_{H_\epsilon(\boldsymbol{x})} \boldsymbol{f}(\boldsymbol{u}(t,\boldsymbol{x}),\boldsymbol{u}(t,\boldsymbol{y}))\, d\boldsymbol{y} + \boldsymbol{b}(\boldsymbol{x},\boldsymbol{t}). \qquad (1)$$

$\boldsymbol{u}(t,\boldsymbol{x})$ is displacement at position $\boldsymbol{x}$ at time t, $\boldsymbol{b}(\boldsymbol{x},t)$ is body force at $(\boldsymbol{x},t)$, $\boldsymbol{f}(\boldsymbol{u}(t,\boldsymbol{x}),\boldsymbol{u}(t,\boldsymbol{y}))$ = material force acting on $\boldsymbol{x}$ due to displacement at $\boldsymbol{y}$,
$\mathcal{H}_\epsilon(x) = \{y \in \mathbb{R}^d : |y-x| < \epsilon\}$, $d = 2$ or 3.

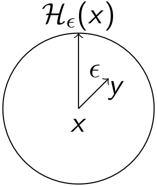
Quasistatic nonlocal dynamics

Effects of inertia negligible.

$$0 = \int_{H_\epsilon(\boldsymbol{x})} \boldsymbol{f}(\boldsymbol{u}(t,\boldsymbol{x}), \boldsymbol{u}(t,\boldsymbol{y}))\, d\boldsymbol{y} + \boldsymbol{b}(\boldsymbol{x},\boldsymbol{t}). \tag{2}$$

$\boldsymbol{u}(t,\boldsymbol{x})$ is displacement at position $\boldsymbol{x}$ at time t, $\boldsymbol{b}(\boldsymbol{x},t)$ is body force at $(\boldsymbol{x},t)$. Time becomes a load parameter for body force $\boldsymbol{b}(\boldsymbol{x},t)$, $t \in [0,T]$.

Range of nonlocal interaction



- ϵ = length scale of nonlocal interaction
- Interaction between x and y only within a ϵ-neighborhood of x
- $\mathcal{H}_\epsilon(x) = \{y \in \mathbb{R}^3 : |y - x| < \epsilon\}$

The domain Ω and horizon about $x \in \Omega$

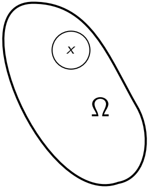


Figure: **Domain and horizon.**

Displacement and strain

The displacement at x denoted by $u(x)$, and the strain between x and y

$$\mathcal{S} = S(\boldsymbol{y}, \boldsymbol{x}, t; \boldsymbol{u})) = \frac{u(t,y) - u(t,x)}{|y - x|} \cdot e, \text{ where } e = \frac{y-x}{|y-x|} \quad (3)$$

The nonlocal force density $\boldsymbol{f}$ is given in terms of the nonlocal potential $\mathcal{W}(S)$ by

$$\boldsymbol{f}(\boldsymbol{y},\boldsymbol{x},\boldsymbol{u}) = 2\partial_S \mathcal{W}(S(\boldsymbol{y},\boldsymbol{x},\boldsymbol{u}))\boldsymbol{e}_{\boldsymbol{y}-\boldsymbol{x}}, \tag{4}$$

where

$$\mathcal{W}(S(\boldsymbol{y},\boldsymbol{x},\boldsymbol{u})) = \frac{J^\epsilon(|\boldsymbol{y}-\boldsymbol{x}|)}{\epsilon^{d+1}\omega_d\,|\boldsymbol{y}-\boldsymbol{x}|} g(\sqrt{|\boldsymbol{y}-\boldsymbol{x}|}S(\boldsymbol{y},\boldsymbol{x},\boldsymbol{u})). \tag{5}$$

Here, $J^\epsilon(r) = J(\frac{r}{\epsilon})$, where J is a non-negative bounded function supported on $[0,1]$. J is called the *influence function* as it determines the influence of the bond force of peridynamic neighbors $\boldsymbol{y}$ on the center $\boldsymbol{x}$ of $H_\epsilon(\boldsymbol{x})$. The volume of unit ball in $\mathbb{R}^d$ is denoted by ω_d.

Quasistatic nonlocal evolution with unstable forces

The quasistatic evolution (2) is expressed by

$$\mathcal{L}[\boldsymbol{u}](\boldsymbol{x},t)=\boldsymbol{b}(\boldsymbol{x},t),\ \ \boldsymbol{x}\in\Omega,\ \ t\in[0,T]$$

where the integral operator $\mathcal{L}$ is defined as

$$\mathcal{L}[\boldsymbol{u}](\boldsymbol{x},t)=-\int_{H_\epsilon(\boldsymbol{x})\cap\Omega} 2\frac{J^\epsilon(|\boldsymbol{y}-\boldsymbol{x}|)}{\epsilon^{d+1}\omega_d\sqrt{|\boldsymbol{y}-\boldsymbol{x}|}}g'\left(\sqrt{|\boldsymbol{y}-\boldsymbol{x}|}S(\boldsymbol{y},\boldsymbol{x},\boldsymbol{u})\right)\boldsymbol{e}_{\boldsymbol{y}-\boldsymbol{x}}d\boldsymbol{y}. \quad (6)$$

Force - strain law for extreme deformations. Constituitive relation characterized by g'

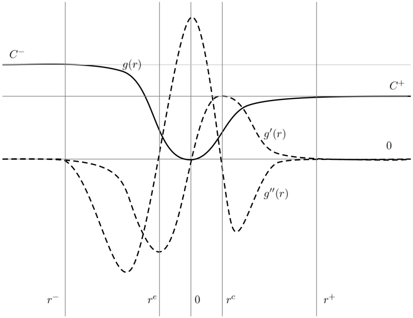


Figure: The potential function $g(r)$ and derivatives $g'(r)$ and $g''(r)$ for tensile force. Here C^+ and C^- are the asymptotic values of g. The derivative of the force potential goes smoothly to zero at r^+ and r^-. Additionally g''' is bounded.

Constitutive law continued: Strength Domain

Definition

Strength Domain. For $\mathbf{x} \in \Omega$ and $\mathbf{y} \in \Omega \cap H_\epsilon(\mathbf{x})$ the strength domain of the material is given by all displacements with strain inside the interval

$$r^e < \sqrt{|\mathbf{y}-\mathbf{x}|}S(\mathbf{y},\mathbf{x},\mathbf{u}) < r^c, \tag{7}$$

or

$$\frac{r^e}{\sqrt{|\mathbf{y}-\mathbf{x}|}} < S(\mathbf{y},\mathbf{x},\mathbf{u}) < \frac{r^c}{\sqrt{|\mathbf{y}-\mathbf{x}|}}. \tag{8}$$

This is the set of strains where the magnitude of force increases with increasing strain. Material failure occurs for strains outside this interval where the force becomes unstable. Moreover $g''(\sqrt{|\mathbf{y}-\mathbf{x}|}S(\mathbf{y},\mathbf{x},\mathbf{u})) > 0$ in the strength domain. The strength domain is a convex set. A displacement is said to lie strictly inside the strength domain if $\sqrt{|\mathbf{y}-\mathbf{x}|}S(\mathbf{y},\mathbf{x},\mathbf{u})$ lies within a closed interval inside (r^e, r^c).

Constant strain at length scale ϵ and $\epsilon << 1$: Recover linear elasticity

$$\mathcal{L}[\boldsymbol{u}](\boldsymbol{x}, t) = -\mathrm{div}\mathbb{C}\mathcal{E}(\boldsymbol{u}(\boldsymbol{x}, \mathrm{t})),$$

and the quasistatic evolution is described by classical theory

$$-\mathrm{div}\mathbb{C}\mathcal{E}(\boldsymbol{u}(\boldsymbol{x}, \mathrm{t})) = \boldsymbol{b}(\boldsymbol{x}, \mathrm{t}),\ \boldsymbol{x} \in \Omega,\ \mathrm{t} \in [0, \mathrm{T}],$$

Where $\mathcal{E}(\boldsymbol{u}(\boldsymbol{x}, t)) = (\nabla \boldsymbol{u} + \nabla \boldsymbol{u}^T)/2$ and $\mathbb{C}$ is time independent, coercive, and bounded. Solution $\boldsymbol{u}(\boldsymbol{x}, t)$ is a global energy minimizer for every $t \in [0, T]$.

We identify the subspace of $L^\infty(\Omega;\mathbb{R}^d)$ in which we find solutions to the load control problems. First let Π denote the space of rigid motions, i.e.,

$$\Pi = \{Q\boldsymbol{x} + \boldsymbol{c} : Q^T = -Q\}Q \in \mathbb{R}^{d\times d}, \boldsymbol{x}, \boldsymbol{c} \in \mathbb{R}^d\}. \tag{9}$$

Direct use Lemma 2 of (Du, Gunzburger, Lehoucq, and Zhou 2013) shows that the rigid rotations comprise the null space of the strain operator:

Proposition

$S(\boldsymbol{y},\boldsymbol{x},\boldsymbol{u}) = 0$ *for all* $\boldsymbol{x} \in \Omega$ *and* $\boldsymbol{y} \in H_\epsilon(\boldsymbol{x})$ *if and only if* $\boldsymbol{u} \in \Pi$.

From this proposition it follows that $\mathcal{L}(\boldsymbol{w}) = 0$ if $\boldsymbol{w} \in \Pi$.

The function space $\mathcal{V}$ used for quasistatic evolutions under load control is given by;

Definition

The space $\mathcal{V} \subset L^\infty(\Omega, \mathbb{R}^d)$ is defined by

$$\mathcal{V} = \{\boldsymbol{u} \in L^\infty(\Omega; \mathbb{R}^d);\ \int_D \boldsymbol{u} \cdot \boldsymbol{w}\, d\boldsymbol{x} = 0,\ \text{for all } \boldsymbol{w} \in \Pi\}. \tag{10}$$

And

Lemma

For $\boldsymbol{u} \in \mathcal{V}$ one has $\mathcal{L}(\boldsymbol{u}) \in \mathcal{V}$.

The derivative of $\mathcal{L}$ at $\boldsymbol{u}$ is denoted by $D(\mathcal{L})[\boldsymbol{u}]$ and is a linear operator on $L^\infty(\Omega, \mathbb{R}^d)$. $D(\mathcal{L})[\boldsymbol{u}]$ is continuously Fréchet differentiable.

Theorem (Differentiability)

The linear transform $D(\mathcal{L})[\boldsymbol{u}] : \mathcal{V} \to \mathcal{V}$ is the Fréchet derivative of $\mathcal{L}$ and exists with respect to the $L^\infty(\Omega, \mathbb{R}^d)$ norm for all $\boldsymbol{u} \in \mathcal{V}$, i.e.,

$$\lim_{\|\Delta \boldsymbol{u}\|_\infty \to 0} \frac{\|\mathcal{L}[\boldsymbol{u} + \Delta \boldsymbol{u}](\boldsymbol{x}) - \mathcal{L}[\boldsymbol{u}](\boldsymbol{x}) - D(\mathcal{L})[\boldsymbol{u}]\Delta \boldsymbol{u}\|_\infty}{\|\Delta \boldsymbol{u}\|_\infty} = 0. \quad (11)$$

Moreover it is Lipshitz continuous in $\boldsymbol{u}$, i.e., for $\boldsymbol{\delta}, \Delta \boldsymbol{u} \in \mathcal{V}$ there is a constant C independent of $\boldsymbol{\delta}$ such that

$$\frac{\|D(\mathcal{L})[\boldsymbol{u} + \boldsymbol{\delta}]\Delta \boldsymbol{u} - D(\mathcal{L})[\boldsymbol{u}]\Delta \boldsymbol{u}\|_\infty}{\|\Delta \boldsymbol{u}\|_\infty} \leq C\|\boldsymbol{\delta}\|_\infty. \quad (12)$$

The derivative is a bounded linear map on $L^\infty(\Omega,\mathbb{R}^d)$. There is a fixed positive constant $C>0$ independent of $\Delta\boldsymbol{u}$ in $L^\infty(\Omega,\mathbb{R}^d)$ such that

$$\|D(\mathcal{L})[\boldsymbol{u}]\Delta\boldsymbol{u}\|_\infty \leq C\|\Delta\boldsymbol{u}\|_{L^\infty(\Omega;\mathbb{R}^d)}, \tag{13}$$

The operator norm of $D(\mathcal{L})[\boldsymbol{u}]$ is written $|||D(\mathcal{L})[\boldsymbol{u}]|||$ and defined by

$$|||D(\mathcal{L})[\boldsymbol{u}]||| = \sup_{\Delta\boldsymbol{u}\in L^\infty(\Omega,\mathbb{R}^d)} \frac{\|D(\mathcal{L})[\boldsymbol{u}]\Delta\boldsymbol{u}\|_\infty}{\|\Delta\boldsymbol{u}\|_\infty}. \tag{14}$$

The inverse map when it exists is written $D(\mathcal{L})[\boldsymbol{u}]^{-1}$.

Ball in L^∞

A ball of radius R centered at an element $\hat{\boldsymbol{u}}$ of $L^\infty(\Omega), \mathbb{R}^d)$ is denoted by,

$$B(\hat{\boldsymbol{u}}, R) = \{\boldsymbol{u} : \|\boldsymbol{u} - \hat{\boldsymbol{u}}\|_\infty \leq R\}\,. \tag{15}$$

Local Exsistence

Theorem (Load control)

Let $\boldsymbol{u}_0, \boldsymbol{b}_0 \in \mathcal{V}$ be such that $\mathcal{L}[\boldsymbol{u}_0] = \boldsymbol{b}_0$ and assume $D(\mathcal{L})[\boldsymbol{u}_0]^{-1}$ exists and is bounded on on $\mathcal{V}$, set

$$R = \frac{C}{|||D(\mathcal{L})[\boldsymbol{u}_0]^{-1}|||}. \tag{16}$$

Then for any given load path such that $\boldsymbol{b}(t) : [0, T] \to \mathcal{V}$ is continuous and $\boldsymbol{b}(t) \in B\left(\boldsymbol{b}_0, \frac{R}{|||D(\mathcal{L})[\boldsymbol{u}_0]^{-1}|||}\right)$, there exists a unique continuous solution path $\boldsymbol{u}(t) \in \mathcal{V}$ lying inside $B(\boldsymbol{u}_0, R)$ such that

$$\mathcal{L}[\boldsymbol{u}(t)] = \boldsymbol{b}(t), \ \text{for } t \in [0, T].$$

We now show that $D(\mathcal{L})[\boldsymbol{u}_0]^{-1}$ exists and is bounded on $\mathcal{V}$ for fields $\boldsymbol{u}_0$ inside the strength domain. This claim applies to a large class of specimen shapes characterized by the interior cone condition. This class of specimen geometries was introduced for the elastic equilibrium problem using convex nonlocal forces treated in (Du, Gunzburger, Lehoucq, and Zhou 2013), (Mengesha and Du, 2014). The condition is given in the following definition (Adams 2003):

Definition

The interior cone condition for Ω states that there exists a positive constant angle $\theta > 0$ such that any $\boldsymbol{x} \in \Omega$ contains a spherical cone $C_{\lambda,\theta}(\boldsymbol{x}, \boldsymbol{e}_{\boldsymbol{x}})$ with its apex at $\boldsymbol{x}$, radius λ aperture angle 2θ bisected by an axis in the direction of a unit vector $\boldsymbol{e}_{\boldsymbol{x}}$.

The condition rules out domains with cusps.

We define the stability tensor.

Definition

Stability tensor

$$\mathbb{A}[\boldsymbol{u}](\boldsymbol{x}) = \int_{H_\epsilon(\boldsymbol{x})\cap\Omega} \frac{J^\epsilon(|\boldsymbol{y}-\boldsymbol{x}|)}{\epsilon^{d+1}\omega_d|\boldsymbol{y}-\boldsymbol{x}|} g''\left(\sqrt{|\boldsymbol{y}-\boldsymbol{x}|}S\left(\boldsymbol{y},\boldsymbol{x},\boldsymbol{u}\right)\right) \boldsymbol{e}_{\boldsymbol{y}-\boldsymbol{x}} \otimes \boldsymbol{e}_{\boldsymbol{y}-\boldsymbol{x}} d\boldsymbol{y}. \tag{17}$$

We write $\mathbb{A}[\boldsymbol{u}] - \gamma\mathbb{I} > 0$ when for all $\boldsymbol{x} \in \Omega$ and all $\boldsymbol{v} \in \mathbb{R}^d$, $\mathbb{A}[\boldsymbol{u}](\boldsymbol{x})\boldsymbol{v}\cdot\boldsymbol{v} - \gamma|\boldsymbol{v}|^2 > 0$.

Invertability

When the domain satisfies the interior cone condition and one has a deformation $\boldsymbol{u}_0$ inside the strength domain we have;

Theorem (Invertibility)

Assume $\boldsymbol{u}_0$ *lies strictly inside the strength domain, i.e., for all* $\boldsymbol{x} \in \Omega$ *and* $\boldsymbol{y} \in \Omega \cap H_\epsilon(\boldsymbol{x})$ *the strain* $\sqrt{\boldsymbol{x}-\boldsymbol{y}}S(\boldsymbol{y},\boldsymbol{x},\boldsymbol{u}_0)$ *lies inside a closed interval contained inside the interval* (r^e, r^c)*, and if* Ω *satisfies the interior cone condition then* $Ker\{D(\mathcal{L})[\boldsymbol{u}]\} = \{\boldsymbol{0}\}$ *and there is a* $\gamma > 0$ *for which* $\mathbb{A}[\boldsymbol{u}_0] - \gamma\mathbb{I} > 0$ *so* $D(\mathcal{L})[\boldsymbol{u}_0]^{-1}$ *is well defined on* $\mathcal{V}$ *and* $|||D(\mathcal{L})[\boldsymbol{u}_0]^{-1}||| < \infty$*.*

Critical points of the peridynamic energy

We introduce the peridynamic potential energy.

$$PD[\boldsymbol{u}] = \int_{\Omega} \int_{H_\epsilon(\boldsymbol{x})\cap\Omega} |\boldsymbol{y}-\boldsymbol{x}|\,\mathcal{W}(S(\boldsymbol{y},\boldsymbol{x},\boldsymbol{u}))d\boldsymbol{y}d\boldsymbol{x}.$$

The total energy of the system is given by

$$E[\boldsymbol{u}] = PD[\boldsymbol{u}] - \int_{\Omega} \boldsymbol{u}\cdot\boldsymbol{b}_0\ d\boldsymbol{x}.$$

A critical point $\boldsymbol{u}$ for the total energy is given by a solution to the Euler Lagrange equation

$$\mathcal{L}[\boldsymbol{u}] = \boldsymbol{b}.$$

Theorem

Assume $\boldsymbol{u}_0 \in B(\boldsymbol{u}_0, R)$ lies strictly inside the strength domain and is a critical point of the energy $E[\boldsymbol{u}]$ for the choice $\boldsymbol{b} = \boldsymbol{b}_0$. Then

$$E[\boldsymbol{u}] \geq E[\boldsymbol{u}_0], \tag{18}$$

over all displacements $\boldsymbol{u}$ in the strength domain within the ball $B(\boldsymbol{u}_0, R)$.

Location of nonlocal quasistatic evolutions in the neighbornood of energy minimizers

We discover that a nonlocal quasistatic evolution exists within a neighborhood of a local energy minimizer over displacements inside the strength the domain.

Theorem

Assume $\boldsymbol{u}_0$ lies strictly inside the strength domain and is a critical point of the energy $E[\boldsymbol{u}]$ for the choice $\boldsymbol{b} = \boldsymbol{b}_0$, Under the hypothesis of Theorem 8 we can choose R given by (16)such that for any continuous load path $\boldsymbol{b}(t) : [0, T] \to \mathcal{V}$ with $\boldsymbol{b}(t) \in B\left(\boldsymbol{b}_0, \frac{R}{2|||D(\mathcal{L})[\boldsymbol{u}_0]^{-1}|||}\right)$, there exists a unique continuous solution path $\boldsymbol{u}(t) \in \mathcal{V}$ lying inside $B(\boldsymbol{u}_0, R)$ such that

$$\mathcal{L}[\boldsymbol{u}(t)] = \boldsymbol{b}(t), \text{ for } t \in [0, T]. \tag{19}$$

Here $\boldsymbol{u}_0$ is a critical point and minimizer of the energy $E[\boldsymbol{u}]$ for the choice $\boldsymbol{b} = \boldsymbol{b}_0$,

$$E[\boldsymbol{u}] \geq E[\boldsymbol{u}_0],$$

over all displacements $\boldsymbol{u}$ belonging to the strength domain inside the ball $B(\boldsymbol{u}_0, R)$.

The positivity requirement for the stability tensor is necessary for invertabilty.

Theorem (Necessity condition for invertibility)

Given $\boldsymbol{u}_0 \in \mathcal{V}$ *and suppose* $Ker\{\mathbb{A}[\boldsymbol{u}_0](\boldsymbol{x})\} \neq 0$ *on a set* $\mathcal{F} \subset \Omega$ *of nonzero Lebesgue measure. Then* $D(\mathcal{L})[\boldsymbol{u}_0]^{-1}$ *is not defined on* $\mathcal{V}$.

Theorem (Energy balance)

On choosing R smaller if necessary then the following holds: If $\boldsymbol{b}(t)$ is differentiable with respect to t and the derivatives are continuous in t then the derivative $\frac{\partial \boldsymbol{u}(t,\boldsymbol{x})}{\partial t}$ belongs to $L^\infty(D,\mathbb{R}^d)$ and is continuous in t and related to the loading rate $\boldsymbol{b}_t$ by

$$D(\mathcal{L})[\boldsymbol{u}(t)]^{-1}\left(\frac{\partial \boldsymbol{b}(t)}{\partial t}\right) = \frac{\partial \boldsymbol{u}(t)}{\partial t}, \tag{20}$$

and we have energy balance for $0 < t < T$, given by

$$PD[\boldsymbol{u}(t)] - \int_\Omega \boldsymbol{u}(t)\cdot\boldsymbol{b}(t)\,d\boldsymbol{x} = PD[\boldsymbol{u}_0] - \int_\Omega \boldsymbol{u}_0\cdot\boldsymbol{b}_0\,d\boldsymbol{x}$$

$$-\int_0^t\int_\Omega \boldsymbol{u}(\tau)\cdot\boldsymbol{b}_t(\tau)\,d\boldsymbol{x}\,d\tau,$$

Proof of Theorem 1: Frechet derivitave of $\mathcal{L}[\boldsymbol{u}]$.

Proof of Theorem 1 follows from fundamental theorem of calculus applied to g' and g'' and then to g'' and g'''.

Proof of Theorem 2: $\mathcal{L}[\boldsymbol{u}] = b$ as fixed point.

Set

$$F_{\boldsymbol{u}_0}(\boldsymbol{u}-\boldsymbol{u}_0) = -\mathcal{L}[(\boldsymbol{u}-\boldsymbol{u}_0)+\boldsymbol{u}_0] + \mathcal{L}[\boldsymbol{u}_0] - D(-\mathcal{L})[\boldsymbol{u}_0](\boldsymbol{u}-\boldsymbol{u}_0) \tag{21}$$

$$= -\mathcal{L}[\boldsymbol{u}] + \mathcal{L}[\boldsymbol{u}_0] - D(-\mathcal{L})[\boldsymbol{u}_0](\boldsymbol{u}-\boldsymbol{u}_0)$$

$T_{\boldsymbol{b}-\boldsymbol{b}_0}(\boldsymbol{u}-\boldsymbol{u}_0)$ is defined by

$$T_{\boldsymbol{b}-\boldsymbol{b}_0}(\boldsymbol{u}-\boldsymbol{u}_0) = -D(-\mathcal{L})[\boldsymbol{u}_0]^{-1}\left(F_{\boldsymbol{u}_0}(\boldsymbol{u}-\boldsymbol{u}_0) + (\boldsymbol{b}-\boldsymbol{b}_0)\right). \tag{22}$$

On $B(\boldsymbol{u}_0, R)$ for sufficiently small R have a fixed point $T_{\boldsymbol{b}-\boldsymbol{b}_0}(\boldsymbol{u}-\boldsymbol{u}_0) = \boldsymbol{u}-\boldsymbol{u}_0$. This condition gives $\mathcal{L}[\boldsymbol{u}] = b$.

Choice of R

$$R = \frac{C}{|||D(\mathcal{L})[\boldsymbol{u}_0]^{-1}|||}.$$

Proof of Theorem 3: Invertability of $D(\mathcal{L})[\boldsymbol{u}]$.

We proceed in two steps.

Step 1: We show that $D(\mathcal{L})[\boldsymbol{u}]$ is invertable on the L^2 closure of $\mathcal{V}$ denoted by $\overline{\mathcal{V}}^2$. This is proven using the symmetry of $D(\mathcal{L})[\boldsymbol{u}]$ as an operator with respect to the L^2 inner product on $\overline{\mathcal{V}}^2$ and assuming $Ker\{D(\mathcal{L})[\boldsymbol{u}]\} = \{\mathbf{0}\}$ and the hypothesis $\mathbb{A}[\boldsymbol{u}_0] - \gamma\mathbb{I} > 0$ for some γ. We use the hypothesis $\mathbb{A}[\boldsymbol{u}_0] - \gamma\mathbb{I} > 0$ again to show $D(\mathcal{L})[\boldsymbol{u}]$ is invertable on $\mathcal{V}$.

Proof of Theorem 3: Invertability of $D(\mathcal{L})[\boldsymbol{u}]$ continued.

Step 2: We show that the hypothesis given by $Ker\{D(\mathcal{L})[\boldsymbol{u}]\} = \{\mathbf{0}\}$ and $\mathbb{A}[\boldsymbol{u}_0] - \gamma\mathbb{I} > 0$ hold when $S(\boldsymbol{y}, \boldsymbol{x}, \boldsymbol{u})$ lies in the strength domain for all $\boldsymbol{x} \in \Omega$ and $\boldsymbol{y} \in \Omega \cap H_\epsilon(\boldsymbol{x})$ and Ω satisfies the interior cone condition.

Proof of Theorem 4: Stationary points in strength domain are minima.

Proof of Theorem 4: Stationary points inside strength domain are minima of the peridynamic energy over all deformations with strain inside the strength domain. Here the minimum property follows from the convexity of the strength domain.

Proof of Theorem 5: Quasi-static evolutions exist in the neighborhood of minimizers.

Evolutions can begin at stationary points $\boldsymbol{u}_0$ inside strength domain which are minima of the peridynamic energy $E[\boldsymbol{u}]$ for $\boldsymbol{b} = \boldsymbol{b}_0$ over all $\boldsymbol{u}$ with strains in the strength domain.

Proof of Theorem 6: $\mathbb{A}[\boldsymbol{u}_0] - \gamma\mathbb{I} > 0$ is necessary for invertability of $D(\mathcal{L})[\boldsymbol{u}_0]$ on $\mathcal{V}$.

To prove the theorem we construct a sequence of nonzero elements $\{\boldsymbol{w}_n\} \in \mathcal{V}$ to show that $\|D(\mathcal{L})[\boldsymbol{u}_0]\boldsymbol{w}_n\|_\infty \to 0$ but that $\|\boldsymbol{w}_n\|_\infty = 1$. This statement holds for all subsequences of $\boldsymbol{w}_n$. Hence the sequence is a singular sequence.
And from Weyl's criterion zero lies in the essential spectrum of $D(\mathcal{L})[\boldsymbol{u}_0]$ and the inverse does not exist on $\mathcal{V} \setminus \{\mathbf{0}\}$.

Proof of Theorem 7: Energy balance.

To prove theorem 7 we apply the Lipschitz continuity of $D(\mathcal{L})[\boldsymbol{u}]$ with respect to $\boldsymbol{u}$on $\mathcal{V}$ to see that there is C independent of $\boldsymbol{u}-\boldsymbol{u}_0 \in \mathcal{V}$ such that

$$|||D(\mathcal{L})[\boldsymbol{u}] - D(\mathcal{L})[\boldsymbol{u}_0]||| \leq C||\boldsymbol{u}-\boldsymbol{u}_0||_\infty \tag{23}$$

Take $\boldsymbol{u} \in B(\boldsymbol{u}_0, R)$ such that $CR < |||D(\mathcal{L})[\boldsymbol{u}_0]^{-1}|||^{-1}$

Proof of Theorem 7: Energy balance cont.

For this choice of R we get existence of $D(\mathcal{L})[\boldsymbol{u}]^{-1}$.
A simple argument gives

$$D(\mathcal{L})[\boldsymbol{u}(t)]^{-1}\left(\frac{\partial \boldsymbol{b}(t)}{\partial t}\right) = \frac{\partial \boldsymbol{u}(t)}{\partial t}, \tag{24}$$

and $\boldsymbol{u}_t$ exists and is continuous in t. Energy balance follows on multiplying $\mathcal{L}[\boldsymbol{u}(t)] = \boldsymbol{b}(t)$ by $\boldsymbol{u}_t(t)$ and integration over D.

Conclusion

- Nonlocal models for damage modeling require force strain laws with unstable forces
- Fixed point methods deliver local existence of quasistatic evolution
- Stability matrix has appeared earlier in both dynamic and static problems (Silling, Weckner, Ascari, Bobaru 2010), (Lipton 2016), (Lipton Lehoucq, Jha 2019), (Du, Gunzburger, Lehoucq, and Zhou 2013), (Mengesha and Du, 2014).
- Existence of quasistatic evolution in neighborhood of local minimizers

Thank you!