NUMERICAL METHODS FOR FRACTIONAL DIFFUSION
Lecture 1: Integral Fractional Laplacian

Ricardo H. Nochetto

Department of Mathematics and
Institute for Physical Science and Technology
University of Maryland, USA

Nonlocal School on Fractional Equations, NSFE 2017
August 17-19, 2017

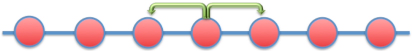
Department of Mathematics
Iowa State University

Outline

Outline

Local Jump Random Walk

- Consider a random walk of a particle along the real line.
- $h\mathbb{Z} = \{hz : z \in \mathbb{Z}\}$ — possible states of the particle.
- $u(x,t)$ — probability of the particle to be at $x \in h\mathbb{Z}$ at time $t \in \tau\mathbb{N}$.
- Local jump random walk: at each time step of size τ, the particle jumps to the left or right with probability $1/2$.



$$u(x,t+\tau) = \frac{1}{2}u(x+h,t) + \frac{1}{2}u(x-h,t)$$

If we consider $2\tau = h^2$, then we obtain

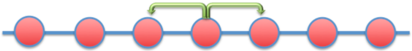
$$\frac{u(x,t+\tau)-u(x,t)}{\tau} = \frac{u(x+h,t)+u(x-h,t)-2u(x,t)}{h^2}$$

Letting $h, \tau \downarrow 0$ yields the heat equation

$$u_t - \Delta u = 0$$

Local Jump Random Walk

- Consider a random walk of a particle along the real line.
- $h\mathbb{Z} = \{hz : z \in \mathbb{Z}\}$ — possible states of the particle.
- $u(x,t)$ — probability of the particle to be at $x \in h\mathbb{Z}$ at time $t \in \tau\mathbb{N}$.
- Local jump random walk: at each time step of size τ, the particle jumps to the left or right with probability $1/2$.



$$u(x,t+\tau) = \frac{1}{2}u(x+h,t) + \frac{1}{2}u(x-h,t)$$

If we consider $2\tau = h^2$, then we obtain

$$\frac{u(x,t+\tau) - u(x,t)}{\tau} = \frac{u(x+h,t) + u(x-h,t) - 2u(x,t)}{h^2}$$

Letting $h, \tau \downarrow 0$ yields the heat equation

$$u_t - \Delta u = 0$$

Long Jump Random Walk

- The probability that the particle jumps from the point $hk \in h\mathbb{Z}$ to the point $hm \in h\mathbb{Z}$ is $\mathcal{K}(k-m) = \mathcal{K}(m-k)$:

$$u(x, t+\tau) = \sum_{k \in \mathbb{Z}} \mathcal{K}(k) u(x+hk, t).$$

- **No-time memory:** Since $\sum_{k\in\mathbb{Z}} \mathcal{K}(k) = 1$, this yields

$$u(x, t+\tau) - u(x,t) = \sum_{k \in \mathbb{Z}} \mathcal{K}(k) \left(u(x+hk, t) - u(x,t) \right)$$

- If $\mathcal{K}(y) \sim |y|^{-(1+2s)}$ with $s \in (0,1)$ and $\tau = h^{2s}$, then $\frac{\mathcal{K}(k)}{\tau} = h\mathcal{K}(kh)$. Letting $h, \tau \downarrow 0$ yields the fractional heat equation

$$\partial_t u = \int_{\mathbb{R}} \frac{u(x+y,t) - u(x,t)}{|y|^{1+2s}} \, \mathrm{d}y \quad \Leftrightarrow \quad \partial_t u + (-\Delta)^s u = 0.$$

- **Long-range time memory:** $\partial_t u \quad \Rightarrow \quad \partial_t^\gamma u \quad (0 < \gamma < 1)$

$$\partial_t^\gamma u + (-\Delta)^s u = 0.$$

Long Jump Random Walk

- The probability that the particle jumps from the point $hk \in h\mathbb{Z}$ to the point $hm \in h\mathbb{Z}$ is $\mathcal{K}(k-m) = \mathcal{K}(m-k)$:

$$u(x, t+\tau) = \sum_{k \in \mathbb{Z}} \mathcal{K}(k) u(x + hk, t).$$

- **No-time memory:** Since $\sum_{k \in \mathbb{Z}} \mathcal{K}(k) = 1$, this yields

$$u(x, t+\tau) - u(x,t) = \sum_{k \in \mathbb{Z}} \mathcal{K}(k) \left(u(x+hk, t) - u(x,t) \right)$$

- If $\mathcal{K}(y) \sim |y|^{-(1+2s)}$ with $s \in (0,1)$ and $\tau = h^{2s}$, then $\frac{\mathcal{K}(k)}{\tau} = h\mathcal{K}(kh)$. Letting $h, \tau \downarrow 0$ yields the fractional heat equation

$$\partial_t u = \int_{\mathbb{R}} \frac{u(x+y,t) - u(x,t)}{|y|^{1+2s}} \, \mathrm{d}y \quad \Leftrightarrow \quad \partial_t u + (-\Delta)^s u = 0.$$

- **Long-range time memory:** $\quad \partial_t u \quad \Rightarrow \quad \partial_t^\gamma u \quad (0 < \gamma < 1)$

$$\partial_t^\gamma u + (-\Delta)^s u = 0.$$

Applications of Nonlocal Operators and Fractional Diffusion

- Modeling anomalous diffusion (Metzler, Klafter 2000, 2004).
- Peridynamics (Silling 2000; Du, Gunzburger 2012; Lipton 2015).
- Modeling contaminant transport in porous media (Benson et al 2000; Seymour et al 2007).
- Finance (Carr et al. 2002; Matache, Schwab, von Petersdorff et al. 2004).
- Lévy processes (Bertoin 1996; Farkas, Reich, Schwab 2007).
- Nonlocal field theories (Eringen 1972, 2002).
- Materials science (Bates 2006).
- Image processing (Gilboa, Osher 2008).
  Based on our PDE approach $\rightarrow$ Gatto, Hesthaven (2014)
  Spectral method $\rightarrow$ Bartels, Antil (2017).
- Fractional Navier Stokes equation (Li et al 2012; Debbi 2014)

$$u_t + u \cdot \nabla u + (-\Delta)^s u + \nabla p = 0$$

- Fractional Cahn Hilliard equation (Segatti, 2014).

The domain Ω can be quite general!

Nonlocal Models: Historical Remarks

- **Nonlocal continuum physics:**
  - A.C. Eringen and D.G.B. Edelen, *On nonlocal elasticity*, International Journal of Engineering Science, 10 (1972), 233-248 (913 google scholar citations).
  - A.C. Eringen, *On differential equations of nonlocal elasticity and solutions of screw dislocation and surface waves*, J. Appl. Phys. 54, 4703 (1983). (1321 google scholar citations).
  - **A.C. Eringen, Nonlocal Continuum Field Theories, Springer (2002).** *Nonlocal continuum field theories are concerned with material bodies whose behavior at any interior point depends on the state of all other points in the body – rather than only on an effective field resulting from these points – in addition to its own state and the state of some calculable external field.*

- **Recent developments:**
  - **Peridynamics:** S.A. Silling, *Reformulation of elasticity theory for discontinuities and long-range forces*, Journal of the Mechanics and Physics of Solids (2000) (968 google scholar citations).
  - **Dirichlet-to-Neumann map:** L. Caffarelli and L. Silvestre, *An extension problem related to the fractional Laplacian*, Communications in Partial Differential Equations, (2007) (1078 google scholar citations).

Nonlocal Models: Historical Remarks

- **Nonlocal continuum physics:**
  - A.C. Eringen and D.G.B. Edelen, *On nonlocal elasticity*, International Journal of Engineering Science, 10 (1972), 233-248 (913 google scholar citations).
  - A.C. Eringen, *On differential equations of nonlocal elasticity and solutions of screw dislocation and surface waves*, J. Appl. Phys. 54, 4703 (1983). (1321 google scholar citations).
  - **A.C. Eringen, Nonlocal Continuum Field Theories, Springer (2002).** *Nonlocal continuum field theories are concerned with material bodies whose behavior at any interior point depends on the state of all other points in the body – rather than only on an effective field resulting from these points – in addition to its own state and the state of some calculable external field.*

- **Recent developments:**
  - **Peridynamics:** S.A. Silling, *Reformulation of elasticity theory for discontinuities and long-range forces*, Journal of the Mechanics and Physics of Solids (2000) (968 google scholar citations).
  - **Dirichlet-to-Neumann map:** L. Caffarelli and L. Silvestre, *An extension problem related to the fractional Laplacian*, Communications in Partial Differential Equations, (2007) (1078 google scholar citations).

Outline

Definition in $\mathbb{R}^d$ for $d \geq 1$

Let $s \in (0,1)$ and $u : \mathbb{R}^d \to \mathbb{R}$ be smooth enough (belongs to Schwartz class $\mathscr{S}$).

- **Fourier transform:**

$$\mathscr{F}\left((-\Delta)^s u\right)(\xi) = |\xi|^{2s} \mathscr{F}(u)$$

- **Integral representation:**

$$(-\Delta)^s u(x) = C(d,s) \text{ P.V.} \int_{\mathbb{R}^d} \frac{u(x) - u(x')}{|x - x'|^{d+2s}} \, dx',$$

  where $C(d,s) = \frac{2^{2s} s \Gamma(s+\frac{d}{2})}{\pi^{d/2} \Gamma(1-s)}$ is a normalization constant involving the Gamma-function Γ.

- **Pointwise limits** $s \to 0, 1$**:** there holds

$$\lim_{s \to 0} (-\Delta)^s u = u,$$
$$\lim_{s \to 1} (-\Delta)^s u = -\Delta u.$$

Integral Definition for Bounded Domain $\Omega \subset \mathbb{R}^d$

Let $\Omega \subset \mathbb{R}^d$ be open, with smooth boundary, and let $f : \Omega \to \mathbb{R}$ be smooth.

- **Boundary value problem:**

$$\begin{cases} (-\Delta)^s u = f & \text{in } \Omega, \\ u = 0 & \text{in } \Omega^c. \end{cases}$$

- **Integral representation:**

$$(-\Delta)^s u(x) = C(d,s) \text{ P.V.} \int_{\mathbb{R}^d} \frac{u(x) - u(x')}{|x - x'|^{d+2s}} \, dx' = f(x) \quad x \in \Omega$$

- **Boundary condition:** it is imposed in $\Omega^c = \mathbb{R}^d \setminus \Omega$

$$u = 0 \quad \text{in } \Omega^c.$$

- **Probabilistic interpretation:** It is the same as over $\mathbb{R}^d$ except that particles are killed upon reaching Ω^c.

Function Spaces

- **Fractional Sobolev space in $\mathbb{R}^d$:**

$$H^s(\mathbb{R}^d) = \left\{ w \in L^2(\mathbb{R}^d) \colon |w|_{H^s(\mathbb{R}^d)} < \infty \right\}$$

with

$$\langle u, w \rangle := \int\int_{\mathbb{R}^d \times \mathbb{R}^d} \frac{(u(x) - u(x'))(w(x) - w(x'))}{|x - x'|^{d+2s}} dx' dx,$$

$$|w|_{H^s(\mathbb{R}^d)} := \langle w, w \rangle^{\frac{1}{2}}, \quad \|w\|_{H^s(\mathbb{R}^d)} := \left(|w|^2_{H^s(\mathbb{R}^d)} + \|w\|^2_{L^2(\mathbb{R}^d)} \right)^{\frac{1}{2}}.$$

- **Fractional Sobolev space in Ω:**

$$\mathbb{H}^s(\Omega) := \left\{ w|_\Omega : w \in H^s(\mathbb{R}^d),\ w|_{\mathbb{R}^d \setminus \Omega} = 0 \right\}, \quad \|w\|_{\mathbb{H}^s(\Omega)} := \|w\|_{H^s(\mathbb{R}^d)}.$$

- **Poincaré inequality in $\mathbb{H}^s(\Omega)$:**

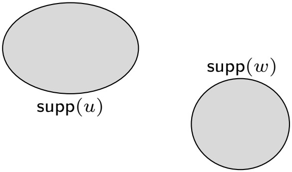
$$\|w\|_{L^2(\Omega)} \le c(\Omega, d, s) |w|_{H^s(\mathbb{R}^d)} \quad \forall\, w \in \mathbb{H}^s(\Omega)$$

and therefore $|\cdot|_{H^s(\mathbb{R}^d)}$ is a norm in $\mathbb{H}^s(\Omega)$.

- **Dual space:** $\mathbb{H}^{-s}(\Omega) = \mathbb{H}^s(\Omega)^*$.

Non-locality

- The H^s-seminorms **are not** additive with respect to domain partitions.
- Functions with disjoint supports may have a non-zero inner product: if $u, v > 0$ on its supports



$$\langle u, v\rangle = \iint_{\mathrm{supp}(u)\times\mathrm{supp}(w)} \frac{-2u(x)w(x')}{|x-x'|^{n+2s}}\,dx\,dx' < 0.$$

- Computation of integrals on unbounded domains $\Omega \times \Omega^c$ ($\Omega^c = \mathbb{R}^d \setminus \Omega$):

$$\int_\Omega \int_{\Omega^c} \frac{u(x)w(x)}{|x-x'|^{d+2s}} dx dx'.$$

Variational Formulation

- **Bilinear form in $\mathbb{H}^s(\Omega)$:**

$$[\![u,w]\!] := \frac{C(d,s)}{2} \underbrace{\int\int_{\mathbb{R}^d\times\mathbb{R}^d} \frac{(u(x)-u(x'))(w(x)-w(x'))}{|x-x'|^{d+2s}}\,dx'\,dx}_{=\langle u,w\rangle}$$

This form is symmetric, continuous and coercive, and equivalent to the inner product $\langle\cdot,\cdot\rangle$ in $\mathbb{H}^s(\Omega)$; recall Poincaré inequality

$$\|w\|_{L^2(\Omega)} \le c(\Omega,n,s)|w|_{H^s(\mathbb{R}^d)} \quad \forall w\in\mathbb{H}^s(\Omega).$$

- **Variational formulation:** for any $f\in\mathbb{H}^{-s}(\Omega)$, consider

$$u\in\mathbb{H}^s(\Omega): \quad [\![u,w]\!] = (f,w) \quad \forall w\in H^s(\Omega),$$

where $(\cdot,\cdot)$ stands for the duality pairing. Existence, uniqueness, and stability follows from Lax-Milgram.

Outline

Motivation

Integral Definition

Regularity of Solutions

A Priori Error Analysis

A Posteriori Error Analysis

Extensions

Open Problems

Sobolev Regularity of Solutions (Grubb (2015))

- **Theorem** (Vishik & Eskin (1965), Grubb (2015)). If $f \in H^r(\Omega)$ for some $r \geq 0$ and $\partial\Omega \in C^\infty$, then for all $\varepsilon > 0$

$$u \in \begin{cases} H^{2s+r}(\Omega) & \text{if } s+r < 1/2, \\ H^{s+1/2-\varepsilon}(\Omega) & \text{if } s+r \geq 1/2. \end{cases}$$

The Dirichlet boundary conditions preclude further gain of regularity.

- **Example:** If $\Omega = B(0,r)$ and $f \equiv 1$, then the solution u is given by

$$u(x) = C(r^2 - |x|^2)_+^s,$$

which **does not belong to** $H^{s+1/2}(\Omega)$. The regularity above is sharp!

- **Boundary behavior** (Grubb (2015)). If $\partial\Omega \in C^\infty$ then

$$u(x) \approx \text{dist}(x, \partial\Omega)^s + v(x)$$

with v smooth.

Sobolev Regularity of Solutions (Grubb (2015))

- **Theorem** (Vishik & Eskin (1965), Grubb (2015)). If $f \in H^r(\Omega)$ for some $r \geq 0$ and $\partial\Omega \in C^\infty$, then for all $\varepsilon > 0$

$$u \in \begin{cases} H^{2s+r}(\Omega) & \text{if } s + r < 1/2, \\ H^{s+1/2-\varepsilon}(\Omega) & \text{if } s + r \geq 1/2. \end{cases}$$

  The Dirichlet boundary conditions preclude further gain of regularity.

- **Example:** If $\Omega = B(0, r)$ and $f \equiv 1$, then the solution u is given by

$$u(x) = C(r^2 - |x|^2)_+^s,$$

  which **does not belong to** $H^{s+1/2}(\Omega)$. The regularity above is sharp!

- **Boundary behavior** (Grubb (2015)). If $\partial\Omega \in C^\infty$ then

$$u(x) \approx \operatorname{dist}(x, \partial\Omega)^s + v(x)$$

  with v smooth.

Hölder Regularity of Solutions (Ros-Oton & Serra (2014))

- **Hölder regularity:** If $f \in L^\infty(\Omega)$, then $u \in C^s(\mathbb{R}^d)$ and

$$\|u\|_{C^s(\mathbb{R}^d)} \le C(\Omega, s)\|f\|_{L^\infty(\Omega)}.$$

Furthermore, defining $\delta(x) := \mathrm{dist}(x, \partial\Omega)$, the function u/δ^s can be continuously extended to $\overline{\Omega}$.

- **Boundary behavior:** if $1/2 < s < 1$ and $f \in C^\beta(\Omega)$ ($\beta < 2 - 2s$), then there exist constants $C_1, C_2 > 0$ such that

$$\sup_{x,x' \in \Omega} \delta(x,y)^{\beta+s} \frac{|Du(x) - Du(x')|}{|x - x'|^{\beta+2s-1}} \le C_1, \qquad \sup_{x \in \Omega} \delta(x)^{1-s}|Du(x)| \le C_2,$$

where $\delta(x,x') = \min\{\delta(x), \delta(x')\}$.

Weighted Fractional Sobolev Regularity (Acosta & Borthagaray (2017))

- **Definition of space $H^{1+\theta}_{\alpha}(\Omega)$:** Let $\alpha \geq 0$ and $\theta \in (0,1)$.

$$\|v\|^2_{H^{1+\theta}_{\alpha}(\Omega)} := \|v\|^2_{H^1(\Omega)} + \iint_{\Omega\times\Omega} \frac{|Dv(x) - Dv(y)|^2}{|x-y|^{n+2\theta}} \delta(x,y)^{2\alpha} dx\, dy$$

- **Weighted estimates:** Let $1/2 < s < 1$, $f \in C^{1-s}(\Omega)$, and $\varepsilon > 0$ small. Then, the solution u of $(-\Delta)^s u = f$ which vanishes in Ω^c belongs to $H^{1+s-2\varepsilon}_{1/2-\varepsilon}(\Omega)$ and satisfies the estimate

$$\|u\|_{H^{1+s-2\varepsilon}_{1/2-\varepsilon}(\Omega)} \leq \frac{C(\Omega,s)}{\varepsilon} \|f\|_{C^{1-s}(\Omega)}.$$

(This is based on results by Ros-Oton and Serra (2014)).

- **Weighted Fractional Poincaré inequality:** Let $0 < \alpha < \ell < 1$ and S be star-shaped w.r.t. a ball. Then, there exists $C > 0$ s.t. for all $v \in L^2(S)$ satisfying $\int_S v = 0$,

$$\|v\|_{L^2(S)} \leq C \text{diam(S)}^{\ell-\alpha} |v|_{H^{\ell}_{\alpha}(S)}.$$

(This is based on results by Hurri-Syrjänen and Vähäkangas (2013)).

Outline

Formulation and Best Approximation

- **Mesh:** Let $\mathcal{T}$ be a shape-regular (with constant σ) and quasi-uniform mesh of Ω of size h.

- **Finite element space:** Let

$$\mathbb{U}(\mathcal{T}) = \{v \in C^0(\overline{\Omega})\colon v\big|_T \in \mathcal{P}_1 \ \forall T \in \mathcal{T}\}.$$

- **Discrete problem:** Find $U \in \mathbb{U}(\mathcal{T})$ such that

$$[\![U, W]\!] = (f, W) \quad \forall\, W \in \mathbb{U}(\mathcal{T}).$$

- **Best approximation:** Since we project over $\mathbb{U}(\mathcal{T})$ with respect to the energy norm $|\cdot|_{\mathbb{H}^s(\Omega)}$ induced by $[\![\cdot,\cdot]\!]$, we get

$$|u - U|_{\mathbb{H}^s(\Omega)} = \min_{W \in \mathbb{U}(\mathcal{T})} |u - W|_{\mathbb{H}^s(\Omega)}.$$

Formulation and Best Approximation

- **Mesh:** Let $\mathcal{T}$ be a shape-regular (with constant σ) and quasi-uniform mesh of Ω of size h.
- **Finite element space:** Let
$$\mathbb{U}(\mathcal{T}) = \{v \in C^0(\overline{\Omega}) \colon v\big|_T \in \mathcal{P}_1 \ \forall T \in \mathcal{T}\}.$$
- **Discrete problem:** Find $U \in \mathbb{U}(\mathcal{T})$ such that
$$[\![U, W]\!] = (f, W) \quad \forall\, W \in \mathbb{U}(\mathcal{T}).$$
- **Best approximation:** Since we project over $\mathbb{U}(\mathcal{T})$ with respect to the energy norm $|\cdot|_{\mathbb{H}^s(\Omega)}$ induced by $[\![\cdot,\cdot]\!]$, we get
$$|u - U|_{\mathbb{H}^s(\Omega)} = \min_{W \in \mathbb{U}(\mathcal{T})} |u - W|_{\mathbb{H}^s(\Omega)}.$$

Interpolation estimates in $\mathbb{H}^s(\Omega)$

- **Localized estimates in $H^s(\Omega)$** (Faermann (2002)):

$$|w|^2_{H^s(\Omega)} \le \sum_{K\in\mathcal{T}} \left[\int_K \int_{S_K} \frac{|w(x)-w(x')|^2}{|x-x'|^{d+2s}} \, \mathrm{d}x' \, \mathrm{d}x + \frac{C(d,\sigma)}{s h_K^{2s}} \|w\|^2_{L^2(K)} \right],$$

  where S_K is the patch associated with $K \in \mathcal{T}$ and σ is the shape regularity constant of $\mathcal{T}$.

- **Quasi-interpolation** (P. Ciarlet Jr (2013)): If $\Pi_{\mathcal{T}}$ is Scott-Zhang operator,

$$\int_K \int_{S_K} \frac{|(w-\Pi_{\mathcal{T}} w)(x) - (w-\Pi_{\mathcal{T}} w)(x')|^2}{|x-x'|^{d+2s}} \, \mathrm{d}x' \, \mathrm{d}x \lesssim h_K^{2\ell-2s} |w|^2_{H^\ell(S_K)},$$

  where the hidden constant depends on d, σ, ℓ and blows up as $s \uparrow 1$.

- **Error estimates for quasi-uniform meshes** (Acosta-Borthagaray (2017))

$$|u-U|_{\mathbb{H}^s(\Omega)} \le C(s,\sigma) h^{\frac{1}{2}} |\ln h| \, \|f\|_{H^{1/2-s}(\Omega)}.$$

Example: $u(x) = C(r^2 - |x|^2)^s_+$ with $\Omega = B(0,1) \subset \mathbb{R}^2$, $f = 1$

s	0.1	0.3	0.5	0.7	0.9
Order	0.497	0.498	0.501	0.504	0.532

Rate is quasi-optimal! Q: Is it possible to improve the order of convergence?

Interpolation estimates in $\mathbb{H}^s(\Omega)$

- **Localized estimates in $H^s(\Omega)$** (Faermann (2002)):

$$|w|^2_{H^s(\Omega)} \le \sum_{K\in\mathcal{T}} \left[\int_K \int_{S_K} \frac{|w(x)-w(x')|^2}{|x-x'|^{d+2s}} \, \mathrm{d}x' \, \mathrm{d}x + \frac{C(d,\sigma)}{s h_K^{2s}} \|w\|^2_{L^2(K)} \right],$$

  where S_K is the patch associated with $K\in\mathcal{T}$ and σ is the shape regularity constant of $\mathcal{T}$.

- **Quasi-interpolation** (P. Ciarlet Jr (2013)): If $\Pi_{\mathcal{T}}$ is Scott-Zhang operator,

$$\int_K \int_{S_K} \frac{|(w-\Pi_{\mathcal{T}}w)(x)-(w-\Pi_{\mathcal{T}}w)(x')|^2}{|x-x'|^{d+2s}} \, \mathrm{d}x' \, \mathrm{d}x \lesssim h_K^{2\ell-2s} |w|^2_{H^\ell(S_K)},$$

  where the hidden constant depends on d, σ, ℓ and blows up as $s\uparrow 1$.

- **Error estimates for quasi-uniform meshes** (Acosta-Borthagaray (2017))

$$|u-U|_{\mathbb{H}^s(\Omega)} \le C(s,\sigma) h^{\frac12} |\ln h| \, \|f\|_{H^{1/2-s}(\Omega)}.$$

  Example: $u(x) = C(r^2-|x|^2)^s_+$ with $\Omega = B(0,1)\subset\mathbb{R}^2$, $f=1$

s	0.1	0.3	0.5	0.7	0.9
Order	0.497	0.498	0.501	0.504	0.532

  Rate is quasi-optimal! Q: Is it possible to improve the order of convergence?

Error Estimates in Graded Meshes (Acosta & Borthagaray (2017))

- **Weighted fractional Poincaré inequality:** If S is star-shaped with respect to a ball, d_S is the diameter of S, and $\overline{w} = ⨍_S w$ for $w \in H^2_\kappa(S)$, then

$$\|w - \overline{w}\|_{L^2(S)} \lesssim d_S^{s-\kappa} |w|_{H^s_\kappa(S)},$$

- **Weighted quasi-interpolation:**

$$\int_T \int_{S_T} \frac{|(v - \Pi_h v)(x) - (v - \Pi_h v)(x')|^2}{|x - x'|^{n+2s}} dx' dx \le C h_T^{1-2\varepsilon} |v|^2_{H^{1+s-2\varepsilon}_{1/2-\varepsilon}(S_T)}.$$

- **Energy error estimate:** Let $d = 2$ and $\mathcal{T}$ be a graded mesh satisfying

$$h_K \le C(\sigma) \begin{cases} h^2, & K \cap \partial\Omega \ne \emptyset, \\ h \operatorname{dist}(K, \partial\Omega)^{1/2}, & K \cap \partial\Omega = \emptyset, \end{cases}$$

whence $\#\mathcal{T} \approx h^{-2} |\log h|$. If $1/2 < s < 1$, then

$$\|u - U\|_{\mathbb{H}^s(\Omega)} \lesssim (\#\mathcal{T})^{-\frac{1}{2}} |\log(\#\mathcal{T})| \, \|f\|_{C^{1-s}(\overline{\Omega})}.$$

- **Improvement:** This also reads $\|u - U\|_{\mathbb{H}^s(\Omega)} \lesssim h|\log h| \, \|f\|_{C^{1-s}(\overline{\Omega})}$ and is thus first order.

Error Estimates in Graded Meshes (Acosta & Borthagaray (2017))

- **Weighted fractional Poincaré inequality:** If S is star-shaped with respect to a ball, d_S is the diameter of S, and $\overline{w} = \fint_S w$ for $w \in H^2_\kappa(S)$, then

$$\|w - \overline{w}\|_{L^2(S)} \lesssim d_S^{s-\kappa} |w|_{H^s_\kappa(S)},$$

- **Weighted quasi-interpolation:**

$$\int_T \int_{S_T} \frac{|(v - \Pi_h v)(x) - (v - \Pi_h v)(x')|^2}{|x - x'|^{n+2s}} dx' dx \leq C h_T^{1-2\varepsilon} |v|^2_{H^{1+s-2\varepsilon}_{1/2-\varepsilon}(S_T)}.$$

- **Energy error estimate:** Let $d = 2$ and $\mathcal{T}$ be a graded mesh satisfying

$$h_K \leq C(\sigma) \begin{cases} h^2, & K \cap \partial\Omega \neq \emptyset, \\ h \operatorname{dist}(K, \partial\Omega)^{1/2}, & K \cap \partial\Omega = \emptyset, \end{cases}$$

  whence $\#\mathcal{T} \approx h^{-2} |\log h|$. If $1/2 < s < 1$, then

$$\|u - U\|_{\mathbb{H}^s(\Omega)} \lesssim (\#\mathcal{T})^{-\frac{1}{2}} |\log(\#\mathcal{T})| \, \|f\|_{C^{1-s}(\overline{\Omega})}.$$

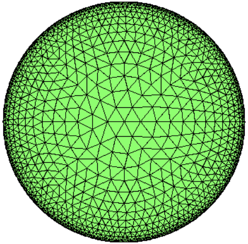
- **Improvement:** This also reads $\|u - U\|_{\mathbb{H}^s(\Omega)} \lesssim h |\log h| \, \|f\|_{C^{1-s}(\overline{\Omega})}$ and is thus first order.

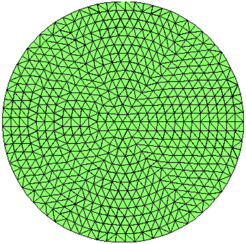
Numerical Experiment (Acosta & Borthagaray (2017))

Exact solution: $u(x) = C(r^2 - |x|^2)_+^s$ with $\Omega = B(0,1) \subset \mathbb{R}^2$, $f = 1$.

Experiment with either uniform or graded $\mathcal{T}$: let $h_K \approx h \,\mathrm{dist}(K, \partial\Omega)^{1/2}$

Value of s	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
Uniform $\mathcal{T}$	0.497	0.496	0.498	0.500	0.501	0.505	0.504	0.503	0.532
Graded $\mathcal{T}$	1.066	1.040	1.019	1.002	1.066	1.051	0.990	0.985	0.977





Optimality: First order accuracy $\|u - U\|_{\mathbb{H}^s(\Omega)} \lesssim h|\log h|$ seems optimal.

Implementation in 2d (Acosta, Bersetche & Borthagaray (2017))

- **Basis functions:** $\{\phi_i\}_{i=1}^{I} \quad \Rightarrow \quad \text{span } \{\phi_i\}_{i=1}^{I} = \mathbb{U}(\mathcal{T})$.

- **Matrix formulation:** If $K = (K_{ij})_{ij=1}^{I}$ with

$$K_{i,j} = [\![\phi_i, \phi_j]\!] = \frac{C(d,s)}{2} \iint_Q \frac{(\phi_i(x) - \phi_i(x'))(\phi_j(x) - \phi_j(x'))}{|x - x'|^{2+2s}} \, \mathrm{d}x' \, \mathrm{d}x.$$

and $\mathbf{U} = (U_i)_{i=1}^{I}$, $\mathbf{F} = (\langle f, \phi_i \rangle)_{i=1}^{I}$ satisfy $U = \sum_{i=1}^{I} U_i \phi_i \in \mathbb{U}(\mathcal{T})$, then

$$K\mathbf{U} = \mathbf{F}.$$

- **Computation:** We have $K_{i,j} = \frac{C(d,s)}{2} \sum_{\ell=1}^{I} \left(\sum_{m=1}^{I} I_{\ell,m}^{i,j} + 2J_{\ell}^{i,j} \right)$ with

$$I_{\ell,m}^{i,j} := \int_{K_\ell} \int_{K_m} \frac{(\phi_i(x) - \phi_i(x'))(\phi_j(x) - \phi_j(x'))}{|x - x'|^{2+2s}} \, \mathrm{d}x' \, \mathrm{d}x,$$

$$J_{\ell}^{i,j} := \int_{K_\ell} \int_{B^c} \frac{\phi_i(x)\phi_j(x)}{|x - x'|^{2+2s}} \, \mathrm{d}x' \, \mathrm{d}x.$$

- **Computational difficulties**
  - Non-integrable singularities
  - Unbounded domains

Implementation Details

- **Case $\overline{K}_\ell \cap \overline{K}_m = \emptyset$:** Integrand of $I^{i,j}_{\ell,m}$ is regular and can be integrated approximately using high order quadrature.

- **Case $\overline{K}_\ell \cap \overline{K}_m \neq \emptyset$:** Integrand of $I^{i,j}_{\ell,m}$ is singular and require techniques similar to BEM (Sauter & Schwab's book (2011)).
    - Map affinely to reference element $\overline{K}$;
    - Split $4d$ integration domain into subsimplices and use Duffy transformations (1982) to map to $4d$ unit cubes;
    - Exploit that Jacobians of Duffy maps are regularizing and split integrals into singular but explicitly integrable part and numerically tractable part.

- **Unbounded domain:** Write

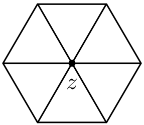
$$J^{i,j}_\ell = \int_{K_\ell} \phi_i(x)\phi_j(x)\varrho(x)\ \mathrm{d}x, \qquad \varrho(x) := \int_{B^c} \frac{1}{|x-x'|^{2+2s}}\ \mathrm{d}x',$$

and compute accurately $\rho(x)$ for $x \in \Omega$ using the radial structure and that $\operatorname{dist}(x,x') > \operatorname{dist}(\Omega, B^c) > 0$ for $x' \in B^c$.

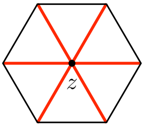
Outline

A Posteriori Error Analysis (N, von Petersdoff and C. Zhang (2010))

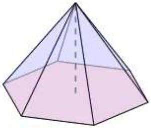
- **Local structure:** Local patch (star) and basis Function:



(a) Local patch ω_z



(b) Skeleton γ_z



(c) Basis function ψ_z

- **Partition of Unity:** $\sum_{z\in\mathcal{P}} \phi_z = 1$
- **Error-Residual:** If $R = f - (-\Delta)^s U$ is the residual, then

$$[\![u-U,w]\!] = (R,w) = \sum_{z\in\mathcal{P}} (R, w\phi_z) = \sum_{z\in\mathcal{P}} (R,(w-\bar{w}_z)\phi_z)$$

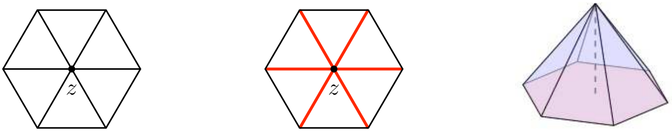
$$= \sum_{z\in\mathcal{P}} (R-\bar{R}_z,(w-\bar{w}_z)\phi_z) = \sum_{z\in\mathcal{P}} ((R-\bar{R}_z)\phi_z, w-\bar{w}_z)$$

where $\bar{w}_z, \bar{R}_z \in \mathbb{R}$ are weighted mean-values:

$\bar{w}_z = (w,\phi_z)\,/\,(1,\phi_z)$ if $z \in \mathcal{P}\cap\Omega$ and $\bar{w}_z = 0$ if $z\in\mathcal{P}\cap\Gamma$.

A Posteriori Error Analysis (N, von Petersdoff and C. Zhang (2010))

- **Local structure:** Local patch (star) and basis Function:



(a) Local patch ω_z (b) Skeleton γ_z (c) Basis function ψ_z

- **Partition of Unity:** $\sum_{z\in\mathcal{P}} \phi_z = 1$
- **Error-Residual:** If $R = f - (-\Delta)^s U$ is the residual, then

$$[\![u - U, w]\!] = (R, w) = \sum_{z\in\mathcal{P}} (R, w\phi_z) = \sum_{z\in\mathcal{P}} (R, (w - \bar{w}_z)\phi_z)$$
$$= \sum_{z\in\mathcal{P}} \big(R - \bar{R}_z, (w - \bar{w}_z)\phi_z\big) = \sum_{z\in\mathcal{P}} \big((R - \bar{R}_z)\phi_z, w - \bar{w}_z\big)$$

where $\bar{w}_z, \bar{R}_z \in \mathbb{R}$ are weighted mean-values:

$\bar{w}_z = (w, \phi_z) \,/\, (1, \phi_z)$ if $z \in \mathcal{P} \cap \Omega$ and $\bar{w}_z = 0$ if $z \in \mathcal{P} \cap \Gamma$.

Residual Estimation

- **Localized upper bound of dual norm.** Assume that $G = \sum_{z\in\mathcal{P}} g_z$ and $g_z \in \mathbb{H}^s(\omega_z)^*$ vanish in $\Omega\backslash\omega_z$. Then for $s \in [0,1]$

$$\|G\|^2_{\mathbb{H}^{-s}(\Omega)} \le (d+1) \sum_{z\in\mathcal{P}} \|g_z\|^2_{\mathbb{H}^s(\omega_z)^*}.$$

- **Upper bound of local dual norm.** Let $g_z \in L^p(\omega_z)$ satisfy $\int_{\omega_z} g_z = 0$ for $z \in \mathcal{P}$ such that $\partial\omega_z \cap \Gamma$ has measure 0. If $s \in [0,1]$ and $1 \le p < \infty$ satisfy $\dfrac{1}{p} < \dfrac{s}{d} + \dfrac{1}{2}$, then

$$\|g_z\|_{\mathbb{H}^s(\omega_z)^*} \lesssim h_z^{s+d(1/2-1/p)} \|g_z\|_{L^p(\omega_z)}.$$

- **Upper bound.** Let $f \in L^p(\Omega)$ and $p > 1, 0 < s < 1$ satisfy the restriction $2s - 1 < \frac{1}{p} < \frac{s}{d} + \frac{1}{2}$. Then

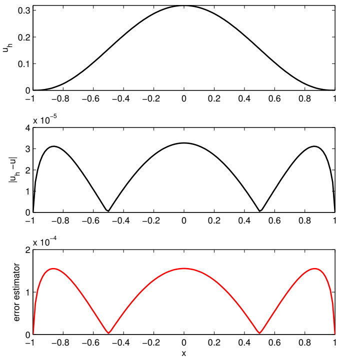
$$\|u - U\|^2_{\mathbb{H}^s(\Omega)} \lesssim \mathcal{E}^2 := \sum_{z\in\mathcal{P}} h_z^{2(s+\frac{d}{2}-\frac{d}{p})} \|(R - R_z)\,\phi_z\|^2_{L^p(\omega_z)}$$

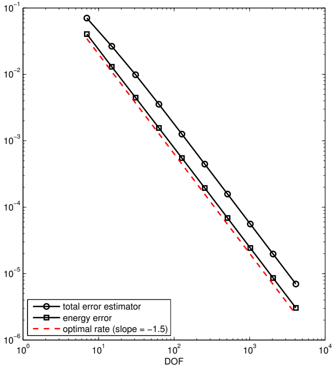
Implementation Issues

- **Data for Examples:** $d = 1, \Omega = (-1, 1)$, $s \leq 3/4$, $p = 2$.
- **Graded mesh towards $x = \pm 1$:** solution behaves as $|x \pm 1|^s$ so the grading to restore optimal rate is $|x_j \pm 1|^s = (j/M)^\beta$ with $\beta > 4 - 2s$.
- **Stiffness matrix:**
  - $d = 1$**:** Expressed analytically in terms of a fourth antiderivative of the kernel function $K(x)$ for $d = 1$. This circumvents quadrature.
  - $d > 1$**:** Requires special quadrature (Sauter, Schwab, von Petersdorff).
- **Residual:** Singularities of $R = f - \mathcal{A}U$ at nodes x_j are of the form $|x - x_j|^{1-2s}$ except for $s = 1/2$, in which case it is logarithm. We compute L^2-norms of R using special quadrature consisting of graded partitioning of intervals and Gauss rules (Schwab, von Petersdorff).

Experimental Order of Convergence for $s = 1/2$ and Smooth Solution

DOF	$\|u - U\|_{\mathbb{H}^s(\Omega)}$	$\mathcal{E}$	Effectivity
15	1.3021e-002	6.2052e-002	4.7655
31	4.4597e-003	2.2014e-002	4.9362
63	1.5618e-003	7.7849e-003	4.9846
127	5.5069e-004	2.7527e-003	4.9986
255	1.9455e-004	9.7327e-004	5.0027
EOC	1.501	1.500	–





Outline

Extensions and Applications

- **Eigenvalue problems** (Borthagaray, Del Pezzo, and Martínez (2016))
- **Time-dependent problems** (Acosta, Bersetche, and Borthagaray (2017))
- **Non-homogeneous Dirichlet conditions** (Acosta, Borthagaray, and Heuer (2017)): mixed method and Lagrange multiplier (fractional flux).
- **Non-local models for interface problems** (Borthagaray and P. Ciarlet Jr. (2017)).

Outline

Open Problems

- **Computations in $3d$:** implementation of fractional Laplacian; regularity and numerical analysis are valid for $d > 2$.
- **High-order methods:** hp-FEM with suitable mesh refinement near boundary might yield exponential convergence rates.
- **Efficiency:** Compression techniques and fast multilevel solvers (Ainsworth and Glusa (2017)).
- **Quadrature:** Error analysis of effect of quadrature close to singularities of kernel (Sauter and Schwab (2011)).
- **A posteriori error analysis:** implementation for $d > 1$ of residual-type estimators; alternative approaches.
- **Nonlinear problems:** obstacle (elliptic and parabolic), fractional minimal surfaces, fractional phase transitions, fractional fully-nonlinear problems.