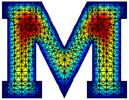
NUMERICAL METHODS FOR FRACTIONAL DIFFUSION
Lecture 2: Spectral Fractional Laplacian

Ricardo H. Nochetto

Department of Mathematics and
Institute for Physical Science and Technology
University of Maryland, USA



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Department of Mathematics
Iowa State University

Outline

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The Linear Elliptic Problem: Formulation

- **Domain:** Let Ω be a bounded domain with Lipschitz boundary $\partial\Omega$.

- **Operator:** Consider a second order, symmetric, elliptic differential operator:

$$\mathcal{L}u = -\text{div}\,(a\nabla u) + cu$$

- **PDE:** Let $s \in (0,1)$. Given $f : \Omega \to \mathbb{R}$, find u such that

$$\mathcal{L}^s u = f \quad \text{in } \Omega$$

  where $\mathcal{L}^s$ denotes the fractional power of $\mathcal{L}$ supplemented with Dirichlet boundary conditions (to be made precise).

- **Spectral definition of nonlocal operator $\mathcal{L}^s$:** Relation between spectral and integral definitions (Caffarelli and Stinga (2015)).

- **Goal:** design efficient PDE solution techniques for problems involving $\mathcal{L}^s$.

- **Simplication:** From now on $\mathcal{L} = -\Delta$. Our results hold for general operators!

Basic Spectral Theory

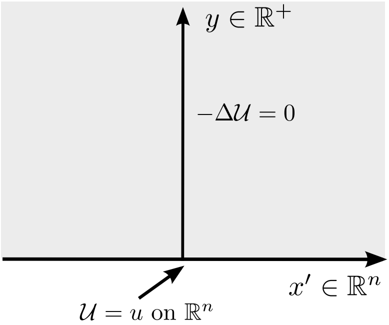
- **Operator:** $-\Delta : H^2(\Omega) \cap H_0^1(\Omega) \subset L^2(\Omega) \to L^2(\Omega)$ is symmetric, closed and unbounded and its inverse is compact.

- **Spectral decomposition:** The eigenpairs $\{\lambda_k, \varphi_k\}_{k=1}^\infty$ satisfy $\lambda_k \geq \lambda_0 > 0$
$$-\Delta\varphi_k = \lambda_k\varphi_k, \qquad \varphi_k|_{\partial\Omega} = 0,$$
and $\{\varphi_k\}_{k=1}^\infty$ form an orthonormal basis of $L^2(\Omega)$ and orthogonal basis of $H_0^1(\Omega)$.

- **Fractional Laplacian:** For u sufficiently smooth and $0 < s \leq 1$
$$u = \sum_{k=1}^\infty u_k\varphi_k \quad \longmapsto \quad (-\Delta)^s u := \sum_{k=1}^\infty u_k \lambda_k^s \varphi_k$$

- **Function spaces:** $(-\Delta)^s : \mathbb{H}^s(\Omega) \to \mathbb{H}^{-s}(\Omega)$, where
$$\mathbb{H}^s(\Omega) = \left\{ w = \sum_{k=1}^\infty w_k\varphi_k \ : \ \sum_{k=1}^\infty \lambda_k^s w_k^2 < \infty \right\} = \begin{cases} H^s(\Omega) & s \in (0, \frac{1}{2}) \\ H_{00}^{\frac{1}{2}}(\Omega) & s = \frac{1}{2} \\ H_0^s(\Omega) & s \in (\frac{1}{2}, 1). \end{cases}$$

The Dirichlet-to-Neumann Operator: $(-\Delta)^{1/2}$

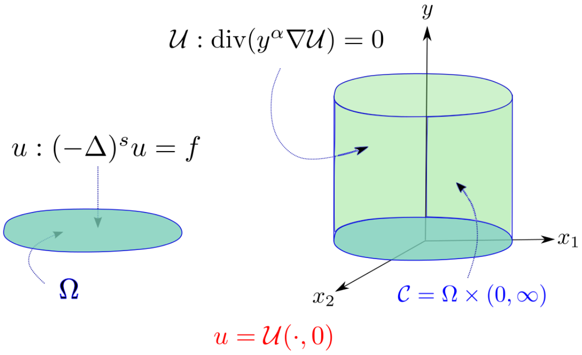


- $\mathrm{DtN} : u \mapsto -\partial_y \mathcal{U}(\cdot, 0)$ is such that

$$\mathrm{DtN}^2 \, u = \partial_y \left(\partial_y \mathcal{U}(\cdot, 0) \right) = -\Delta_{x'} \mathcal{U}(\cdot, 0) = -\Delta_{x'} u.$$

- DtN is positive, then $\mathrm{DtN} = (-\Delta_{x'})^{\frac{1}{2}}$ and $(-\Delta_{x'})^{\frac{1}{2}} u = \partial_\nu \mathcal{U}$.

The Dirichlet-to-Neumann Map: The Extension Problem for $0 < s < 1$



- **Parameters:** $s \in (0,1)$ and $\alpha = 1 - 2s \in (-1,1)$.
- **Neumann condition:** $\partial_{\nu^\alpha} \mathcal{U} = -\lim_{y \downarrow 0} y^\alpha \partial_y \mathcal{U} = d_s f$ on $\Omega \times \{0\}$.
- **Scaling constant:** $d_s = 2^\alpha \Gamma(1-s)/\Gamma(s)$.
- **Extension problem:**
  - $\Omega = \mathbb{R}^d$: Caffarelli, Silvestre (2007);
  - $\Omega \subset \mathbb{R}^d$ bounded and $\mathcal{U} = 0$ on $\partial_L \mathcal{C}$: Stinga, Torrea (2010–2012), Cabré, Tan (2010); Capella et al. (2011).

The α-harmonic extension

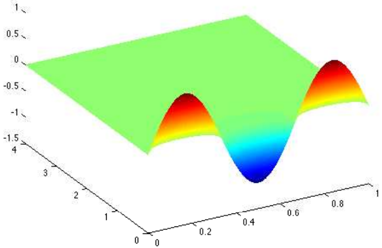
- Fractional powers of $-\Delta$ can be realized as a DtN operator:

$$\begin{cases} \mathrm{div}\,(y^\alpha \nabla \mathcal{U}) = 0 & \text{in } \mathcal{C} \\ \mathcal{U} = 0 & \text{on } \partial_L \mathcal{C} \\ \partial_{\nu^\alpha} \mathcal{U} = d_s f & \text{on } \Omega \times \{0\} \end{cases} \quad \Longleftrightarrow \quad (-\Delta)^s u = f \text{ in } \Omega$$

$$u = \mathcal{U}(\cdot, 0).$$

Here:

- $\mathcal{C} = \Omega \times (0, \infty)$
- $\alpha = 1 - 2s \in (-1, 1)$
- $\partial_{\nu^\alpha} \mathcal{U} = -\lim_{y \downarrow 0} y^\alpha \partial_y \mathcal{U} = d_s f$
- $d_s = 2^\alpha \Gamma(1-s)/\Gamma(s)$



- **Integral representation:** (A. Bonito and J. Pasiack, 2014)

$$(-\Delta)^{-s} = \frac{\sin(\pi s)}{\pi} \int_0^\infty t^{-s} (tI - \Delta)^{-1} dt.$$

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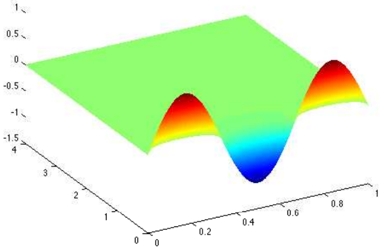

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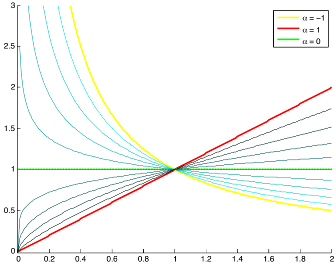
Weak Formulation

- **Space:**

$$\mathring{H}^1_L(y^\alpha, \mathcal{C}) = \left\{ w \in L^2(y^\alpha, \mathcal{C}) : \nabla w \in L^2(y^\alpha, \mathcal{C}),\ w|_{\partial_L \mathcal{C}} = 0 \right\}.$$

- **Weight:** y^α with $\alpha = 1 - 2s \in (-1, 1)$



The weight y^α is degenerate ($\alpha > 0$) or singular($\alpha < 0$)!

- **Weak formulation:** seek $\mathcal{U} \in \mathring{H}^1_L(y^\alpha, \mathcal{C})$ such that

$$\int_{\mathcal{C}} y^\alpha \nabla \mathcal{U} \cdot \nabla \phi = d_s \langle f, \mathrm{tr}_\Omega \phi \rangle_{\mathbb{H}^{-s}(\Omega), \mathbb{H}^s(\Omega)}, \qquad \forall \phi \in \mathring{H}^1_L(y^\alpha, \mathcal{C}).$$

Muckenhoupt Weights

- **Key property:** There is a constant C such that for every $a, b \in \mathbb{R}$, with $a > b$,

$$\frac{1}{b-a}\int_a^b |y|^\alpha \, \mathrm{d}y \cdot \frac{1}{b-a}\int_a^b |y|^{-\alpha} \, \mathrm{d}y \leq C.$$

This means y^α belongs to the Muckenhoupt class A_2.

- **Important consequences:**
  - The Hardy-Littlewood maximal operator is continuous on $L^2(y^\alpha, \mathcal{C})$.
  - Singular integral operators are continuous on $L^2(y^\alpha, \mathcal{C})$.
  - $L^2(y^\alpha, \mathcal{C}) \hookrightarrow L^1_{loc}(\mathcal{C})$.
  - $H^1(y^\alpha, \mathcal{C})$ is Hilbert and $\mathcal{C}_b^\infty(\mathcal{C})$ is dense.
  - Traces on $\partial_L \mathcal{C}$ are well defined.

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Weighted Sobolev Spaces

- **Weighted Poincaré inequality:** There is a constant C, s.t.

$$\int_{\mathcal{C}} y^\alpha |w|^2 \le C \int_{\mathcal{C}} y^\alpha |\nabla w|^2 \quad \forall w \in \overset{\circ}{H}{}^1_L(y^\alpha, \mathcal{C}).$$

- **Surjective trace operator:** $\quad \operatorname{tr}_\Omega : \overset{\circ}{H}{}^1_L(y^\alpha, \mathcal{C}) \to \mathbb{H}^s(\Omega).$

- **Existence and uniqueness:** Lax-Milgram applies for every $f \in \mathbb{H}^{-s}(\Omega)$. Also

$$\|\mathcal{U}\|_{\overset{\circ}{H}{}^1_L(y^\alpha, \mathcal{C})} = \|u\|_{\mathbb{H}^s(\Omega)} = \sqrt{d_s}\|f\|_{\mathbb{H}^{-s}(\Omega)}.$$

- **Regularity:**
  - Anisotropic regularity
  - Singular behavior in extended variable y.

Outline

Spectral Representation of $\mathcal{U}$ (N, Otárola, Salgado (2015))

- **Spectral representation:** $\mathcal{U}(x,y) = \sum_{k=1}^{\infty} u_k \varphi_k(x)\psi_k(y)$ with $u_k = \lambda_k^{-s} f_k$.

- **2-point boundary value problem:** the function ψ_k satisfies

$$\psi_k'' + \frac{\alpha}{y}\psi_k' = \lambda_k \psi_k, \quad \text{in } (0,\infty); \qquad \psi_k(0) = 1, \quad \lim_{y\to\infty} \psi_k(y) = 0,$$

whence for $s \neq \frac{1}{2}$

$$\psi_k(y) = c_s (\sqrt{\lambda_k} y)^s K_s(\sqrt{\lambda_k} y),$$

where $c_s = 2^{1-s}/\Gamma(s)$ and K_s denotes the modified Bessel function of the second kind. For $s = \frac{1}{2}$, we have $\psi_k(y) = \exp(-\sqrt{\lambda_k} y)$.

- **Asymptotic behavior:** function ψ_k satisfies as $y \to 0$

$$\psi_k'(y) \approx y^{-\alpha}, \qquad \psi_k''(y) \approx y^{-\alpha-1},$$

and $\psi_k(y) \approx \left(\sqrt{\lambda_k} y\right)^{s-\frac{1}{2}} e^{-\sqrt{\lambda_k} y}$ as $y \to \infty$.

Global Sobolev Regularity (N, Otárola, Salgado (2015))

- **Compatible data:** Let $f \in \mathbb{H}^{1-s}(\Omega)$, which means that f has a vanishing trace for $s < \frac{1}{2}$.

- **Space regularity:**

$$\|\Delta_x \mathcal{U}\|^2_{L^2(y^\alpha,\mathcal{C})} + \|\partial_y \nabla_x \mathcal{U}\|^2_{L^2(y^\alpha,\mathcal{C})} = d_s \|f\|^2_{\mathbb{H}^{1-s}(\Omega)}$$

- **Regularity in extended variable y:** If $s \neq \frac{1}{2}$ and $\beta > 2\alpha + 1$ then

$$\|\partial_{yy} \mathcal{U}\|_{L^2(y^\beta,\mathcal{C})} \lesssim \|f\|_{L^2(\Omega)}.$$

  If $s = \frac{1}{2}$, then

$$\|\mathcal{U}\|_{H^2(\mathcal{C})} \lesssim \|f\|_{\mathbb{H}^{1/2}(\Omega)}.$$

- **Elliptic pick-up regularity:** If Ω convex, then

$$\|w\|_{H^2(\Omega)} \lesssim \|\Delta_x w\|_{L^2(\Omega)} \quad \forall w \in H^2(\Omega) \cap H^1_0(\Omega).$$

  Under this assumption, we further have

$$\|D^2_x \mathcal{U}\|_{L^2(y^\alpha,\mathcal{C})} \lesssim \|f\|_{\mathbb{H}^{1-s}(\Omega)}.$$

Boundary Regularity (Caffarelli, Stinga (2016))

- **Case $s \neq \frac{1}{2}$:** If $\mathrm{dist}(x, \partial\Omega)$ is the distance to $\partial\Omega$, then there exist functions v 'smooth' such that for all $x \in \Omega$

$$
\begin{aligned}
u(x) &\approx \mathrm{dist}(x,\partial\Omega)^{2s} + v(x) \qquad && 0 < s < \frac{1}{2} \\
u(x) &\approx \mathrm{dist}(x,\partial\Omega) + v(x) \qquad && \frac{1}{2} < s < 1.
\end{aligned}
$$

- **Case $s = \frac{1}{2}$:** This is an exceptional case (Costabel, Dauge (1993))

$$
u(x) \approx \mathrm{dist}(x,\partial\Omega)\big|\log \mathrm{dist}(x,\partial\Omega)\big| + v(x)
$$

Analytic Regularity (Banjai, Melenk, N, Otárola, Salgado, Schwab (2017))

- **Behavior of $\psi(z) = c_s z^s K_s(z)$ near $z = 0$:**

$$\left| \frac{\mathrm{d}^\ell}{\mathrm{d}z^\ell} \psi(z) \right| \leq C d_s \ell! z^{2s-\ell},$$

  where $d_s = 2^{1-2s}\Gamma(1-s)/\Gamma(s)$.

- **Behavior of $\psi(z)$ for z large:**

$$\left| \frac{\mathrm{d}^\ell}{\mathrm{d}z^\ell} \psi(z) \right| \leq C_{\epsilon,s} \ell! \epsilon^{-\ell} z^{s-\ell-\frac{1}{2}} e^{-(1-\epsilon)z}$$

- **Global regularity of $\mathcal{U}$:** If $0 \leq \tilde{\nu} < s$ and $0 \leq \nu < 1+s$, then there exists $\kappa > 1$ such that

$$\begin{aligned}
\|\partial_y^{\ell+1}\mathcal{U}\|_{L^2(\omega_{\alpha+2\ell-2\tilde{\nu},\gamma},\mathcal{C})} &\lesssim \kappa^{\ell+1}(\ell+1)!\,\|f\|_{\mathbb{H}^{-s+\tilde{\nu}}(\Omega)}, \\
\|\nabla_{x'}\partial_y^{\ell+1}\mathcal{U}\|_{L^2(\omega_{\alpha+2(\ell+1)-2\nu,\gamma},\mathcal{C})} &\lesssim \kappa^{\ell+1}(\ell+1)!\,\|f\|_{\mathbb{H}^{-s+\nu}(\Omega)}, \\
\|\mathcal{L}_{x'}\partial_y^{\ell+1}\mathcal{U}\|_{L^2(\omega_{\alpha+2(\ell+1)-2\nu,\gamma},\mathcal{C})} &\lesssim \kappa^{\ell+1}(\ell+1)!\,\|f\|_{\mathbb{H}^{1-s+\nu}(\Omega)},
\end{aligned}$$

  with weight $\omega_{\beta,\gamma}(y) = y^\beta e^{\gamma y}, 0 \leq \gamma < 2\sqrt{\lambda_1}$.

Domain Truncation: $\mathcal{C} \to \mathcal{C}_{\mathcal{Y}}$ (N, Otárola, Salgado (2015))

- **Unbounded domain:** $\mathcal{C} := \Omega \times (0, \infty)$

- **Theorem (exponential decay).** For every $\mathcal{Y} > 0$

$$\|\mathcal{U}\|_{\mathring{H}_L^1(y^\alpha, \Omega \times (\mathcal{Y}, \infty))} \lesssim e^{-\sqrt{\lambda_1}\mathcal{Y}/2} \|f\|_{\mathbb{H}^{-s}(\Omega)}.$$

- **Truncated domain:** $\mathcal{C}_{\mathcal{Y}} := \Omega \times (0, \mathcal{Y})$. Let $\mathcal{V}$ solve

$$\begin{cases} \operatorname{div}\,(y^\alpha \nabla \mathcal{V}) = 0 & \text{in } \mathcal{C}_{\mathcal{Y}} = \Omega \times (0, \mathcal{Y}), \\ \mathcal{V} = 0 & \text{on } \partial_L \mathcal{C}_{\mathcal{Y}} \cup \Omega \times \{\mathcal{Y}\}, \\ \partial_{\nu^\alpha} \mathcal{V} = d_s f & \text{on } \Omega \times \{0\}. \end{cases}$$

- **Theorem (exponential convergence).** For all $\mathcal{Y} > 0$,

$$\|\mathcal{U} - \mathcal{V}\|_{\mathring{H}_L^1(y^\alpha, \mathcal{C}_{\mathcal{Y}})} \lesssim e^{-\sqrt{\lambda_1}\mathcal{Y}/4} \|f\|_{\mathbb{H}^{-s}(\Omega)}.$$

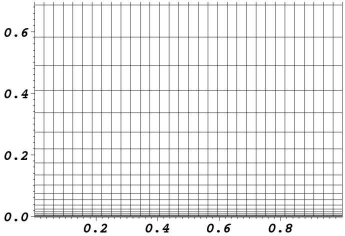
Outline

Finite Element Method: Anisotropic Mesh

- $\mathscr{T}_\Omega = \{K\}$: conforming and shape regular partition of Ω (simplices or cubes)
- $\mathscr{T}_\mathcal{Y} = \{T\}$: partition of $\mathcal{C}_\mathcal{Y}$ into cells of the form

$$T = K \times I, \quad K \in \mathscr{T}_\Omega, \quad I = (a, b).$$

- **Anisotropic meshes:** $\mathcal{U}_{yy} \approx y^{-\alpha-1}$ as $y \downarrow 0 \quad \Rightarrow \quad$ anisotropic elements



Shape regularity condition does NOT hold!

- **Geometric mesh condition:** if $T = K \times I$ and $T' = K' \times I'$ are neighbors

$$\frac{|I|}{|I'|} \simeq 1.$$

The Finite Element Method

- **Discrete spaces:** If $\Gamma_D = \partial_L \mathcal{C} \cup \Omega \times \{\mathcal{Y}\}$ is the Dirichlet boundary, then

$$\mathbb{V}(\mathscr{T}_{\mathcal{Y}}) = \left\{ W \in \mathcal{C}^0(\overline{\mathcal{C}_{\mathcal{Y}}}) : \; W|_T \in \mathcal{P}_1(K) \otimes \mathbb{P}_1(I), \;\; W|_{\Gamma_D} = 0 \right\}$$

Here $\mathcal{P}_1 = \mathbb{P}_1$ if K is a simplex and $\mathcal{P}_1 = \mathbb{Q}_1$ if K is a “brick”.

- **Galerkin method** for the extension: Find $V \in \mathbb{V}(\mathscr{T}_{\mathcal{Y}})$ such that

$$\int_{\mathcal{C}_{\mathcal{Y}}} y^\alpha \nabla V \nabla W = d_s \langle f, \operatorname{tr}_\Omega W \rangle_{\mathbb{H}^{-s}(\Omega) \times \mathbb{H}^s(\Omega)}, \quad \forall W \in \mathbb{V}(\mathscr{T}_{\mathcal{Y}}).$$

Define the solution U as the trace of V:

$$U := \operatorname{tr}_\Omega V \in \mathbb{U}(\mathscr{T}_\Omega) = \operatorname{tr}_\Omega \mathbb{V}(\mathscr{T}_{\mathcal{Y}}).$$

- **Quasi-best approximation:** Projection Theorem implies

$$\|\mathcal{V} - V\|_{\overset{\circ}{H}{}^1_L(y^\alpha, \mathcal{C}_{\mathcal{Y}})} = \inf_{W \in \mathbb{V}(\mathscr{T}_{\mathcal{Y}})} \|\mathcal{V} - W\|_{\overset{\circ}{H}{}^1_L(y^\alpha, \mathcal{C}_{\mathcal{Y}})}$$

and reduces the a priori error analysis to a question of approximation theory in weighted spaces. Usually we set $W = \Pi v \in \mathbb{V}(\mathscr{T}_{\mathcal{Y}})$ where Π is a suitable interpolation operator.

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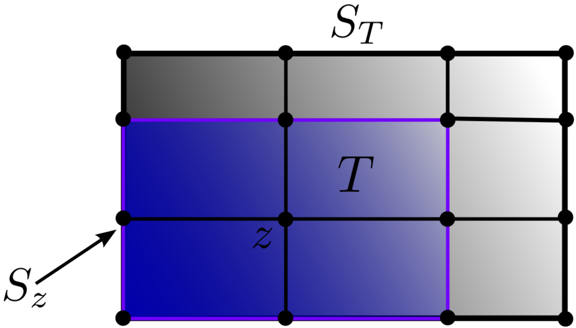
Polynomial Approximation in Weighted Spaces

- **Weight:** Consider $\omega \in A_p(\mathbb{R}^N)$ and $\phi \in L^p(\omega, D)$, with $D \subset \mathbb{R}^N$.
- **Stars and patches:** Given a node z of the mesh, we define



- **The averaged Taylor polynomial:** Given $m \in \mathbb{N}$, we define

$$Q_z^m \phi(y) = \int \sum_{|\alpha| \leq m} \frac{1}{\alpha!} D^\alpha \phi(x)(y-x)^\alpha \psi_z(x) \, \mathrm{d}x.$$

The Averaged Taylor Polynomial $Q_z^m \phi$

- **Definition:** Given $m \in \mathbb{N}$, we define

$$Q_z^m \phi(y) = \int \sum_{|\alpha| \leq m} \frac{1}{\alpha!} D^\alpha \phi(x)(y-x)^\alpha \psi_z(x) \, \mathrm{d}x.$$

- **Key property:** We have

$$|v(x) - Q_z^m v(x)| \lesssim \int \frac{D^m v(y)}{|x-y|^{n-m}} \, \mathrm{d}y.$$

- **Error analysis:**
  - By Poincaré

$$\|v - Q_z^0 v\|_{L^p(\omega, S_z)} \leq \operatorname{diam}(S_z) \|\nabla v\|_{L^p(\omega, S_z)}.$$

  - Induction: apply Poincaré inductively on $0 \leq k \leq m$.

$$\|v - Q_z^m v\|_{W_p^k(\omega, S_z)} \leq \operatorname{diam}(S_z)^{m-k} \|v\|_{W_p^m(\omega, S_z)}.$$

The Quasi-Interpolation Operator

- **Averaged interpolation operator** Π**:** *á la* Durán, Lombardi, 2005 (Sobolev 1950; Dupont, Scott 1980)

$$\Pi\phi(z) = Q_z^m \phi(z) \quad \text{for all nodes } z.$$

- **Properties:**
  - This is defined for all polynomials of degree m and any element shape (simplices or rectangles).
  - We do not go back to the reference element — This is important for anisotropic estimates.
  - Tensor product meshes $\mathscr{T}_{\mathcal{Y}}$: $T = K \times I \in \mathscr{T}_{\mathcal{Y}}$ where
    - $K \in \mathscr{T}_\Omega$ shape-regular and
    - I satisfies $|I|/|I'| \lesssim 1$ for all adjacent intervals I'.

Error Estimates on Rectangles

- **Theorem.** If $\omega \in A_p(\mathbb{R}^N)$, and $\phi \in W^1_p(\omega, S_R)$

$$\|\phi - \Pi\phi\|_{L^p(\omega,R)} \lesssim \sum_{i=1}^{N} h_R^i \|\partial_i \phi\|_{L^p(\omega,S_R)}.$$

If $\phi \in W^2_p(\omega, S_R)$

$$\|\partial_j(\phi - \Pi\phi)\|_{L^p(\omega,R)} \lesssim \sum_{i=1}^{N} h_R^i \|\partial_i \partial_j \phi\|_{L^p(\omega,S_R)},$$

$$\|\phi - \Pi\phi\|_{L^p(\omega,R)} \lesssim \sum_{i,j=1}^{N} h_R^i h_R^j \|\partial_i \partial_j \phi\|_{L^p(\omega,S_R)}.$$

- **Directional estimates:** note products of the form $h_R^i h_R^j \|\partial_i \partial_j \phi\|_{L^p(\omega,S_R)}$.
- **Extensions:** simplicial elements, different metrics, and applications.

Regularity of the Extension $\mathcal{U}$

- **Regularity in y:** Uses separation of variables and properties of Bessel functions to obtain

$$\mathcal{U}_{yy} \approx y^{-\alpha-1} \quad \text{as } y \downarrow 0 \qquad \Longrightarrow \qquad \mathcal{U} \notin H^2(y^\alpha, \mathcal{C}).$$

- **Theorem (anisotropic regularity of the extension).** If $f \in \mathbb{H}^{1-s}(\Omega)$ and Ω is $C^{1,1}$ or a convex polygon

$$\|\Delta_{x'}\mathcal{U}\|^2_{L^2(y^\alpha,\mathcal{C})} + \|\partial_y \nabla_{x'}\mathcal{U}\|^2_{L^2(y^\alpha,\mathcal{C})} = d_s \|f\|^2_{\mathbb{H}^{1-s}(\Omega)}.$$

If $\beta > 1 + 2\alpha$, then

$$\|\partial_{yy}\mathcal{U}\|_{L^2(y^\beta,\mathcal{C})} \lesssim \|f\|_{L^2(\Omega)}.$$

A Priori Error Estimates: Quasiuniform Meshes

- **Quasiuniform meshes:** $h \approx h_T \approx h_K \approx h_I$ for all $T = K \times I \in \mathscr{T}_{\mathcal{Y}}$.

- **Theorem (a priori error estimates):** The following estimate holds for all $\epsilon > 0$

$$\begin{aligned}\|\nabla(\mathcal{V} - V)\|_{L^2(y^\alpha, \mathcal{C}_{\mathcal{Y}})} &\lesssim h_K \|\partial_y \nabla_{x'} v\|_{L^2(y^\alpha, \mathcal{C})} + h_I^{s-\epsilon} \|\partial_{yy} v\|_{L^2(y^\beta, \mathcal{C})} \\ &\lesssim h^{s-\epsilon} \|f\|_{\mathbb{H}^{1-s}(\Omega)}.\end{aligned}$$

- **A priori error estimates for trace:**

$$\|u - U\|_{\mathbb{H}^s(\Omega)} \lesssim h^{s-\epsilon} \|f\|_{\mathbb{H}^{1-s}(\Omega)}.$$

  - Combine interpolation error estimate with truncation error estimate
  - This is suboptimal in terms of order because $u \in \mathbb{H}^{1+s}(\Omega)$ (order $s - \epsilon$ instead of $1 - \epsilon$)
  - Is it sharp for quasi-uniform meshes?

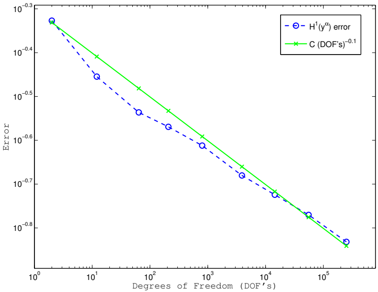
Numerical Experiment: Quasiuniform Mesh

- **Domain and exact solution:** Let $\Omega = (0,1)$ and $f = \pi^{2s}\sin(\pi x)$, then

$$\mathcal{U} = \frac{2^{1-s}\pi^s}{\Gamma(s)} \sin(\pi x') y^s K_s(\pi y)$$

where K_s is a Bessel funciton of second kind.

- **Experiment for $s = 0.2$:** The energy error behaves like $\mathrm{DOFs}^{-0.1} \approx h^{0.2}$, as predicted! Note that DOFs is measured in $\mathbb{R}^2$.



A Priori Error Estimates: Graded Meshes (N, Otárola, Salgado (2015))

- **Principle of error equilibration:** We use a graded mesh on $(0, \mathscr{Y})$

$$y_j = \mathscr{Y}\left(\tfrac{j}{M}\right)^{\gamma}, \quad j = \overline{0, M}, \quad \gamma > 1$$

$\mathcal{U}_{yy} \approx y^{-\alpha-1} \implies$ energy equidistribution for $\gamma > 3/(1-\alpha)$.

- **Theorem (a priori error estimates).** If $f \in \mathbb{H}^{1-s}(\Omega)$ and $\mathscr{Y} \approx |\log N|$,

$$\|u - U\|_{\mathbb{H}^s(\Omega)} = \|\nabla(\mathcal{U} - V)\|_{L^2(y^\alpha, \mathcal{C})} \lesssim |\log N|^s N^{-\frac{1}{d+1}} \|f\|_{\mathbb{H}^{1-s}(\Omega)},$$

or equivalently in terms of meshsize $h \approx N^{-1/(d+1)}$ in Ω

$$\|u - U\|_{\mathbb{H}^s(\Omega)} \lesssim h|\log h|^s \|u\|_{\mathbb{H}^{1+s}(\Omega)}.$$

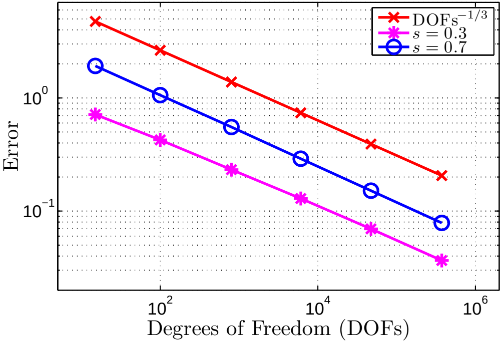
- **Optimality:**
  - This is near optimal in terms of regularity $u \in \mathbb{H}^{1+s}(\Omega)$ and decay rate (almost linear in h);
  - This is suboptimal in terms of total degrees of freedom (dofs) N which scales like $N \approx N_\Omega^{1+\frac{1}{d}} \gg N_\Omega$ w.r.t. dofs N_Ω in Ω.

Experimental Rates for a Circle and $s = 0.3$, $s = 0.7$

- **Domain and exact forcing:** Set $\Omega = D(0,1) \subset \mathbb{R}^2$ and

$$f = j_{1,1}^{2s} J_1(j_{1,1} r)(A_{1,1}\cos(\theta) + B_{1,1}\sin(\theta)).$$

  where J_1 is the 1-st Bessel function of the first kind.
- **Experimental rates of convergence:** With graded meshes we get



- **Optimality:** The experimental convergence rate $-1/3$ is optimal !

Outline

Diagonalization (w. Banjai, Melenk, Otárola, Salgado, and Schwab (2017))

- **Discretization in y:** Let $\mathcal{G}^M$ be an arbitrary mesh in $(0,\mathcal{Y})$ with $M = \#\mathcal{G}^M$ and let $\mathbb{V}_M^{\boldsymbol{r}}(\mathcal{C}_\mathcal{Y}) = H_0^1(\Omega) \otimes S^{\boldsymbol{r}}(0,\mathcal{Y};\mathcal{G}^M)$ be a space of polynomial degree $\boldsymbol{r}$.

- **Semidiscrete solution:** $\mathcal{U}_M \in \mathbb{V}_M^{\boldsymbol{r}}(\mathcal{C}_\mathcal{Y})$ satisfies

$$\int_{\mathcal{C}_\mathcal{Y}} y^\alpha \nabla \mathcal{U}_M \nabla \phi = d_s \langle f, \mathrm{tr}\phi \rangle \quad \forall\, \phi \in \mathbb{V}_M^{\boldsymbol{r}}(\mathcal{C}_\mathcal{Y}).$$

- **Eigenvalue problem:** Let $\mathcal{M} = \dim S^{\boldsymbol{r}}(0,\mathcal{Y};\mathcal{G}^M)$ and $(\mu_i, v_i)_{i=1}^{\mathcal{M}}$ be (normalized) eigenpairs of

$$\mu \int_{y=0}^{\mathcal{Y}} y^\alpha v'(y) w'(y)\, dy = \int_{y=0}^{\mathcal{Y}} y^\alpha v(y) w(y)\, dy \qquad \forall w \in S^{\boldsymbol{r}}(0,\mathcal{Y};\mathcal{G}^M).$$

- **Representation:** If $\mathcal{U}_M(x',y) = \sum_{j=1}^{\mathcal{M}} U_j(x') v_j(y)$ with $U_j \in H_0^1(\Omega)$, then

$$a_{\mu_i,\Omega}(U_i, V) = d_s v_i(0) \langle f, V \rangle \qquad \forall V \in H_0^1(\Omega),$$

where $a_{\mu_i,\Omega}$ is the singularly perturbed bilinear form

$$a_{\mu_i,\Omega}(U,V) := \mu_i \int_\Omega \nabla U \nabla V \,\mathrm{d}x' + \int_\Omega UV \,\mathrm{d}x$$

Tensor Product Discretization

- **Ritz projections:** $\Pi_i u \in S_0^q(\mathscr{T}_\Omega)$ satisfies
$$a_{\mu_i,\Omega}(u - \Pi_i u, v) = 0 \quad \forall\, v \in S_0^q(\mathscr{T}_\Omega);$$
$S_0^q(\mathscr{T}_\Omega) \subset H_0^1(\Omega)$ is the subspace of pw polynomials of degree $\leq q$ over $\mathscr{T}_\Omega$.

- **Discrete solution:** Let $U_{h,M} \in S_0^q(\mathscr{T}_\Omega) \otimes S^{\boldsymbol{r}}(0,\mathcal{Y};\mathcal{G}^M)$ satisfy
$$\int_{\mathcal{C}_\mathcal{Y}} y^\alpha \nabla U_{h,M} \nabla V = d_s \langle f, \mathrm{tr} V\rangle \quad \forall\, V \in S_0^q(\mathscr{T}_\Omega) \otimes S^{\boldsymbol{r}}(0,\mathcal{Y};\mathcal{G}^M)$$
and note that it can be represented as follows
$$U_{h,M}(x',y) = \sum_{i=1}^{\mathcal{M}} \Pi_i U_i(x') v_i(y).$$

- **Parallelization:** This corresponds to solving $\mathcal{M}$ decoupled elliptic problems with the singularly perturbed bilinear form $a_{\mu_i,\Omega}$ for $1 \leq i \leq \mathcal{M}$.

- **Exponential convergence:** Let $f \in \mathbb{H}^{\nu - s}(\Omega)$ for $0 < \nu < s$. If $\mathcal{Y} \approx M$, the mesh $\mathcal{G}^M$ is geometric towards $y = 0$, and the polynomial degree $\boldsymbol{r}$ grows linearly from $y = 0$, then there exists $b > 0$ such that
$$\|\nabla(\mathcal{U} - \mathcal{U}_M)\|_{L^2(y^\alpha,\mathcal{C})} \lesssim e^{-bM} \|f\|_{\mathbb{H}^{-s+\nu}(\Omega)}.$$

Tensor P_1-FEM (w. Banjai, Melenk, Otárola, Salgado, Schwab (2017))

- **Data regularity:** $f \in L^2(\Omega)$ and $\Omega \subset \mathbb{R}^2$ polygonal with corners $\mathbf{c}$.

- **Solution regularity in weighted spaces:** The solution to $-\Delta w = f$ in Ω and $w = 0$ on $\partial\Omega$ satisfies

$$\|w\|_{H^2_\beta(\Omega)} \lesssim \|f\|_{L^2(\Omega)}$$

with weight $\prod_{\mathbf{c}} |x - \mathbf{c}|^{2\beta}$.

- **Graded mesh in Ω:** Let $\mathscr{T}_\Omega$ be graded towards the reentrant corners so that, if $N = \#\mathscr{T}_\Omega$ and $h = N^{-1/2}$, for any $w \in S^1_0(\mathscr{T}_\Omega)$

$$N\|w - \Pi w\|^2_{L^2(\Omega)} \lesssim \|w\|^2_{H^1(\Omega)}, \quad N^2\|w - \Pi w\|^2_{L^2(\Omega)} \lesssim \|w\|^2_{H^2_\beta(\Omega)}.$$

- **Error estimates:** If $\mathcal{G}^M_\eta$ is a suitable graded radical mesh $\left\{y_i = \left(\frac{i}{M}\right)^\eta \mathcal{Y}\right\}_{i=0}^M$, with $\eta s > 1$ and $M \approx N^{\frac{1}{2}} = (\#\mathscr{T}_\Omega)^{\frac{1}{2}}$, the discrete solution $U_{h,M}$ satisfies

$$\|u - \operatorname{tr} U_{h,M}\|_{\mathbb{H}^s(\Omega)} \leq h\|f\|_{\mathbb{H}^{1-s}(\Omega)}$$

and

$$\dim \mathbb{V}^{1,1}_{h,M}(\mathscr{T}_\Omega, \mathcal{G}^M) \approx h^{-3}\log|\log h| \approx N_\Omega^{1+\frac{1}{2}} \log\log N_\Omega.$$

Sparse Grid FEM (w. Banjai, Melenk, Otárola, Salgado, Schwab (2017))

- **Complexity of tensor product:** quantity $N_\Omega^{1+\frac{1}{2}}$ is suboptimal.
- **Sparse grid space:** Let

$$\mathbb{V}_L^{1,1}(\mathcal{C}_\mathcal{Y}) = \sum_{\ell,\ell' \geq 0,\, \ell+\ell' \leq L} S_0^1(\mathscr{T}_\Omega^\ell) \otimes S^1(0,\mathcal{Y};\mathcal{G}_\eta^{2^{\ell'}}),$$

  where $\mathscr{T}_\Omega^\ell$ and $\mathcal{G}_\eta^{2^{\ell'}}$ are nested meshes of levels ℓ and ℓ' graded towards corners $\mathbf{c}$ of Ω and $y=0$, respectively.
- **Error estimate:** Let $1<\nu<1+s$, $\eta(\nu-1)\geq 1$, and $\mathcal{Y} \approx |\log h_L|$. If $f \in \mathbb{H}^{\nu-s}(\Omega)$, then $\mathcal{U}_L \in \mathbb{V}_L^{1,1}(\mathcal{C}_\mathcal{Y})$ satisfies

$$\|\mathcal{U}-\mathcal{U}_L\|_{L^2(y^\alpha,\mathcal{C})} \lesssim h_L |\log h_L|\, \|f\|_{\mathbb{H}^{\nu-s}(\Omega)}$$

$$\dim \mathbb{V}_L^{1,1}(\mathcal{C}_\mathcal{Y}) \lesssim N_\Omega \log\log N_\Omega.$$

- **Complexity of sparse grids:** this is quasi-optimal in terms of N_Ω.

hp-FEM in y and P_1-FEM in Ω

- **Graded geometric mesh:** Let $\mathcal{G}_\sigma^M = \{\mathcal{Y}\sigma^{M-i}\}_{i=1}^M$ with $\sigma < 1$.
- **Data regularity:** $f \in \mathbb{H}^{1-s}(\Omega)$ and $\Omega \subset \mathbb{R}^2$ polygonal with corners $\mathbf{c}$.
- **FE space:** $\mathbb{V}_{h,M}^{1,\boldsymbol{r}}(\mathscr{T}_\Omega, \mathcal{G}_\sigma^M)$ is the space of pw polynomials of degree 1 over $\mathscr{T}_\Omega$ and pw polynomials of degree $\boldsymbol{r}$ growing linearly from 0 over $\mathcal{G}_\sigma^M$.
- **Error estimates:** Let $\mathscr{T}_\Omega$ be a suitably graded mesh towards the reentrant corners $\mathbf{c}$. If $\mathcal{Y} \approx |\log h|$ and $U_{h,M} \in \mathbb{V}_{h,M}^{1,\boldsymbol{r}}(\mathscr{T}_\Omega, \mathcal{G}_\sigma^M)$ is the Galerkin solution, then

$$\|\nabla(\mathcal{U} - U_{h,M})\|_{L^2(y^\alpha, \mathcal{C})} \lesssim h\|f\|_{\mathbb{H}^{1-s}(\Omega)}$$

$$\dim \mathbb{V}_{h,M}^{1,\boldsymbol{r}}(\mathscr{T}_\Omega, \mathcal{G}_\sigma^M) \approx h^{-2}|\log h|^2 \approx N_\Omega |\log N_\Omega|$$

- **Complexity:** This is quasi-optimal in terms of N_Ω. It extends Vexler et al (2017) to nonconvex domains.

hp-FEM in y and Ω: Exponential Rate of Convergence

- **Data regularity:** The domain $\Omega \subset \mathbb{R}^2$ and f are analytic.

- **Graded mesh in Ω:** The mesh $\mathscr{T}_\Omega$ is inisotropic and graded towards $\partial\Omega$ so that it resolves the smallest scale $\mu_{\mathcal{M}}$ of the singularly perturbed problems originating from the diagonalization.

- **Graded mesh in y:** Let $\mathcal{G}_\sigma^M = \left\{\mathcal{Y}\sigma^{M-i}\right\}_{i=1}^M$ with $\sigma < 1$.

- **Error estimate:** If $\mathcal{Y} \approx M$, $\boldsymbol{r}$ grows linearly from $y = 0$, then the Galerkin solution $U_{h,M} \in S_0^q(\mathscr{T}_\Omega) \otimes S^{\boldsymbol{r}}(\mathcal{G}_\sigma^M)$ and the total number $N_{\Omega,\mathcal{Y}}$ of degrees of freedom satisfy

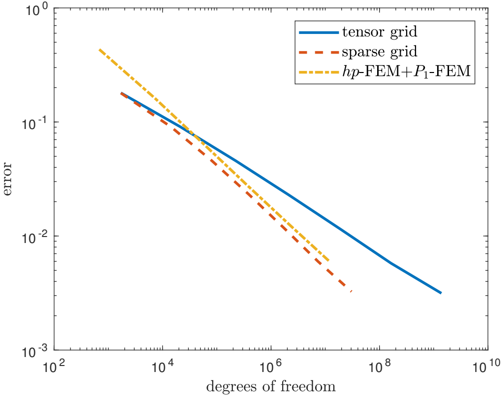
$$\|\nabla(\mathcal{U} - U_{h,M})\|_{L^2(y^\alpha,\mathcal{C})} \lesssim M^2 e^{-bq} + e^{-bM}$$
$$N_{\Omega,\mathcal{Y}} \approx q^2 M^3.$$

- **Exponential rate of convergence:** If $q \approx M$, then

$$\|\nabla(\mathcal{U} - U_{h,M})\|_{L^2(y^\alpha,\mathcal{C})} \lesssim e^{-b' N_{\Omega,\mathcal{Y}}^{1/5}}.$$

Complexity: Performance of Different FEMs

- **Data:** Ω L-shaped domain in $\mathbb{R}^2$; $f = 1$; $s = 3/4$.
- **Error:** It is always measured in the energy space $\mathbb{H}^s(\Omega)$.



- **Conclusions:** Both sparse grid FEM and hp-FEM reduced substantially the dofs relative to tensor FEM and deliver quasi-optimal complexity.

Outline

Isotropic A Posteriori Error Indicators (w Chen, Otárola and Salgado (2015))

- **Residual error indicator**: If we were to integrate by parts the discrete problem over an element $T \in \mathscr{T}_{\mathcal{Y}}$, we would get

$$\int_T y^\alpha \nabla V \nabla W = \int_{\partial T} y^\alpha W \nabla V \cdot \boldsymbol{\nu} - \int_T \operatorname{div}\,(y^\alpha \nabla V) W$$

  Since $\alpha \in (-1, 1)$, the boundary integral is meaningless for $y = 0$.

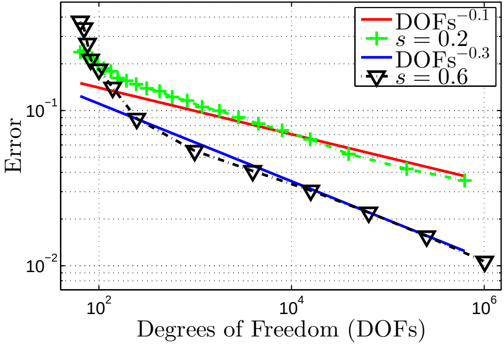
- **Alternative error indicators:** Residual indicators are not the only possibility:
  - Local problems on stars: $\mathcal{E}_z^2 = \int_{S_z} y^\alpha |\nabla Z|^2$ (Z solution of a BVP in S_z).
  - Zienkiewicz-Zhu estimators
  - Hypercircle estimators

- **Local problems on stars:** We prove for all nodes $z \in \mathcal{N}$

$$\mathcal{E}_z^2 \lesssim \|\nabla(v - V)\|^2_{L^2(y^\alpha, S_z)} \lesssim \mathcal{E}_z^2 + \operatorname{osc}(y^\alpha, V, f, S_z)^2$$

Numerical Experiment with Isotropic Refinement

- **Domain and solution:** Set $\mathcal{C}_\mathcal{Y} = (0,1) \times (0,4)$ and $u = \sin(\pi x)$
- **Experimental convergence rates:**



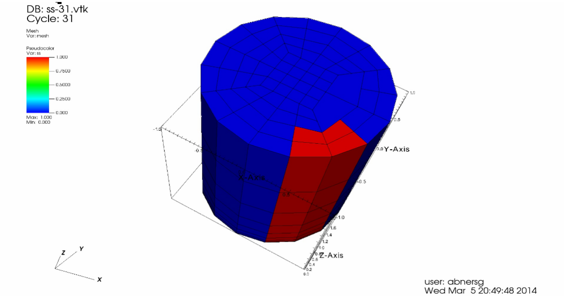
- The error decays like $(\#\mathscr{T}_\mathcal{Y})^{-(1-|\alpha|)/4}$ as in uniform/isotropic refinement!
- **Adaptivity:** does it help?

Anisotropic Error Estimation (w Chen, Otárola and Salgado (2015))

- **Anisotropic a posteriori error estimator:** we need to distinguish the behavior on the extended variable y from the rest.

- The theory of a posteriori error estimation (and adaptivity) on anisotropic discretizations is still in its infancy.

- **Cylindrical stars:** We propose an error estimator based on solving local problems on sets $\mathcal{C}_{z'} = S_{z'} \times (0, \mathcal{Y})$ as depicted in red in the figure:



An Ideal A Posteriori Error Estimator

- **Local space:** For $z' \in \Omega$ a node, let $\mathcal{C}_{z'} = S_{z'} \times (0, \mathcal{Y})$ and define

$$\mathcal{W}(\mathcal{C}_{z'}) := \left\{ w \in H^1(y^\alpha, \mathcal{C}_{z'}) : \quad w = 0 \text{ on } \partial\mathcal{C}_{z'} \setminus \Omega \times \{0\} \right\}.$$

- **Local star indicator:** The error indicator $\eta_{z'} \in \mathcal{W}(\mathcal{C}_{z'})$ is given by

$$\int_{\mathcal{C}_{z'}} y^\alpha \nabla \eta_{z'} \nabla w = d_s \langle f, tr_\Omega w \rangle_{\mathbb{H}^{-s}(\Omega) \times \mathbb{H}^s(\Omega)} - \int_{\mathcal{C}_{z'}} y^\alpha \nabla V \nabla w,$$

for all $w \in \mathcal{W}(\mathcal{C}_{z'})$.

- **Global error estimator:**

$$\mathcal{E}_{\mathscr{T}_\Omega} = \left(\sum_{z'} \mathcal{E}_{z'}^2 \right)^{1/2}, \quad \mathcal{E}_{z'} = \|\nabla \eta_{z'}\|_{L^2(y^\alpha, \mathcal{C}_{z'})}.$$

Anisotropic A Posteriori Error Analysis

- **Theorem (oscillation free lower bound):** For every node $z' \in \Omega$ we have

$$\mathcal{E}_{z'} \leq \|\nabla e\|_{L^2(y^\alpha, \mathcal{C}_{z'})}.$$

- **Data oscillation:** If $f_{z'|K} = \frac{1}{|K|}\int_K f$ for every element $K \subset S_{z'}$, then

$$\operatorname{osc}_{\mathscr{T}_\Omega}(f)^2 = \sum_{z'} \operatorname{osc}_{z'}(f)^2, \quad \operatorname{osc}_{z'}(f)^2 = d_s h_{z'}^{2s} \|f - f_{z'}\|^2_{L^2(S_{z'})}$$

- **Theorem (global upper bound):**

$$\|\nabla e\|^2_{L^2(y^\alpha, \mathcal{C}_\mathscr{Y})} \lesssim \mathcal{E}^2_{\mathscr{T}_\Omega} + \operatorname{osc}_{\mathscr{T}_\Omega}(f)^2.$$

- **Computable estimator:** Restrict $\mathcal{W}(\mathcal{C}_{z'})$ to a discrete subspace

$$\left\{W \in \mathcal{W}(\mathcal{C}_{z'}) : W|_T \in \mathcal{P}_2(K) \otimes \mathbb{P}_2(I), \forall T = K \times I\right\}$$

$\mathcal{P}_2(K) = \mathbb{Q}_2(K)$ for rectangles, $\mathcal{P}_2(K) = \mathbb{P}_2(K) \oplus \mathbb{B}_3(K)$ for simplices.

- **Open:** Rigorous upper and lower bounds for computable estimator.

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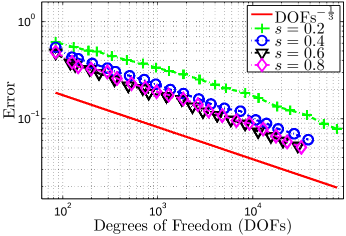
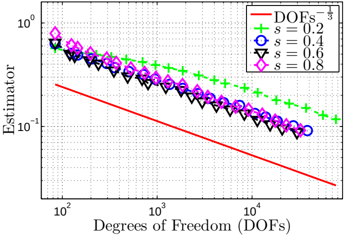
- **Open:** Rigorous upper and lower bounds for computable estimator.

L-Shaped Domain with Incompatible Data: Decay Rates

- **Domain:** Ω is the standard L-shaped domain in 2d.
- **Forcing:** $f = 1$. For $s < \frac{1}{2}$ the data is incompatible with the problem and creates a boundary layer.
- **Regularity:** The nature of the singularity of the solution is not known.
- **Experimental error and estimator:** error computed against a very fine discrete solution.

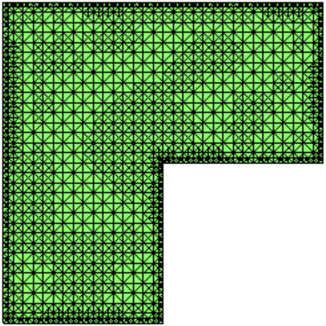




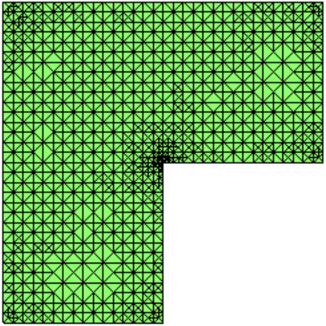
- **Optimal decay rate:** We get $DOF^{-1/3}$ for all s.

L-Shaped Domain with Incompatible Data: Meshes

- Meshes: For $s < 1/2$ the solution exhibits a boundary layer.



$s = 0.2$



$s = 0.8$

- **Question**: Is there any theory on anisotropic adaptive approximation? (Cohen Mirebeau 2010-2012) (Petrushev 2007-2009)?

Outline

The Spectral Fractional Laplacian

Regularity

A Priori Error Analysis

Tensor Product FEMs

A Posteriori Error Analysis

Conclusions and Open Problems

Conclusions and Open Problems

- **PDE approach:** Exploits the extension and converts the nonlocal problem into a local PDE problem in one higher dimension, which is rather flexible and applicable.
- **A priori error analysis:** Complete and quasi-optimal analysis for anisotropic meshes. Measuring complexity in terms of total degrees of freedom:
  - $P_1 - P_1$**-elements:** yields suboptimal complexity and linear rate for Ω convex and compatible data. Extension to non-convex domains.
  - **Sparse tensor** $P_1 - P_1$**-elements:** yields quasi-optimal complexity and linear rate for Ω polygonal with compatible data.
  - hp**-elements:** yields quasi-optimal complexity and exponential rate for analytic but incompatible data.
- **A posteriori error analysis:** Ideal estimator based cylindrical stars.
  - **Computable estimator:** Rigorous upper and lower bounds missing.
  - **Adaptivity:** Convergence and optimality is still open (issue is anisotropic meshes and lack of shape regularity).
- $3d$**-computations:**
  - **Extended variable:** virtual implementation of extended variable is open.
  - $3d$ hp**-FEM:** theory and implementation are open.