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ORIGINAL PAPER

A new nonlocal calculus framework. Helmholtz decompositions, properties, and convergence for nonlocal operators in the limit of the vanishing horizon

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Abstract

We introduce a new nonlocal calculus framework which parallels (and includes as a limiting case) the differential setting. The integral operators introduced have convolution structures and converge as the horizon of interaction shrinks to zero to the classical gradient, divergence, curl, and Laplacian. Moreover, a Helmholtz-type decomposition holds on the entire $\mathbb{R}^n$, so general vector fields can be decomposed into (nonlocal) divergence-free and curl-free components. We also identify the kernels of the nonlocal operators and prove additional properties towards building a nonlocal framework suitable for analysis of integro-differential systems.

Keywords Nonlocal calculus · Horizon of interaction · Kernels of operators · Helmholtz decomposition

Mathematics Subject Classification 26A33 · 35S30 · 41A35 · 45A05 · 45P05 · 46F12 · 46N20 · 47G10

1 Introduction and motivation

In recent years nonlocal models have been successfully introduced in a variety of applications such as dynamic fracture [1, 2], nonlocal diffusion [3], flocking and swarms [4], and image processing [5]. Thus, the development of a nonlocal calculus theory, together with the study of

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nonlocal operators has become the focus of many theoretical investigations. In this work we formulate new integral operators with kernels (usually chosen weakly singular) that measure long range interactions between particles. Besides establishing foundational results for the operators (such as identifying their kernels, their adjoints, and essential properties), we also prove a Helmholtz-type decomposition.

In the classical (differential) setting the Helmholtz decomposition states that we can decompose a three dimensional vector field as a sum of an irrotational function and a solenoidal component. As a consequence, such decompositions allow us to prescribe (under sufficient smoothness and vanishing conditions) the divergence and curl of a vector function. That is, for given φ, a scalar field over $\mathbb{R}^3$, and $\mathbf{a}$, a divergence-free vector field, one can find a function $\mathbf{F}$ such that $\nabla \times \mathbf{F} = \mathbf{a}$ and $\nabla \cdot \mathbf{F} = \varphi$.

As one of the fundamental theorems of vector calculus, the Helmholtz decomposition is of great importance in many areas of mathematics. It is particularly useful in areas such as electrostatics, fluid dynamics, and image processing. In electrostatics, Maxwell's equations in the static case require one to prescribe the curl and divergence of the magnetic flux density and the electric field [6]. In fluid dynamics there are many applications such as the solenoidal projection, which is used to numerically solve incompressible Navier–Stokes equations [7]. In image processing Helmholtz decompositions have seen use, for example, in detecting singularities in fingerprint images [8]. See the survey paper [9] for additional details and references.

Because of the widespread application of Helmholtz decompositions, it has become of particular interest to develop similar decompositions in the nonlocal framework. This paper is preceeded by a long list of studies on nonlocal calculus and nonlocal operators, which include fractional differential operators [10–14]. While these papers set up a framework for nonlocal operators with many similarities to its classical counterpart, they have lacked Helmholtz decompositions. Recently, in [13] such a decomposition was obtained for two-point functions on bounded domains, under different choices for boundary conditions. A nonlocal Helmholtz decomposition was obtained in [15] for functions in the periodic setting.

Integral operators allow input functions with no smoothness, when the interaction kernel is only integrable. Such flexibility allows the development of models where discontinuities and rough features are permitted without any smoothing or approximating procedures. The theory of peridynamics [1], in particular, has been successful in capturing dynamic fracture and deformations in a variety of materials. Mathematically, a new paradigm allows the study of well-posedness for systems in "rough" topologies, for which new mathematical tools need to be developed.

Here we introduce a new nonlocal framework, different than the one in [12], with nonlocal counterparts for the gradient, curl, divergence, and Laplacian operators. Distribution-valued kernels localize these integral operators to yield their classical counterparts. Moreover, we are showing that in this nonlocal framework we have that the only curl-free functions are gradients, while the divergence-free ones can be expressed as curls. These last properties, in particular, were missing from the calculus framework of [12, 16], thus precluding certain developments in nonlocal calculus. Further results, such as integration by parts theorems (equivalent to identifying the adjoint operators) give added versatility to the framework. The one-dimensional gradient that we define here has appeared previously in [17], under the name of nonlocal derivative. The author studied weak convergence of this operator (on $\mathbb{R}$), as the horizon vanishes, together with associated systems (nonlocal equivalents of initial value problems). Here, the focus is developing a new theory of vectorial calculus in the nonlocal frameworks.

The nonlocal operators are constructed using convolutions, thus they are well-suited for investigations using Fourier transforms. Such an argument enables us to show that every vector-valued function defined on the entire space can be decomposed in two components: one that can be expressed as a nonlocal gradient and one that can be expressed as a nonlocal curl, establishing a Helmholtz-type theorem for this setting.

Of particular importance for nonlocal theories is showing connections to the local setting. Besides establishing similar structures and properties for the nonlocal spaces and operators, one aims to show that nonlocal operators (and solutions) converge to the differential counterparts as the horizon of interaction (which measures the support of the operator's kernel) vanishes. For the nonlocal calculus of [12], it has been shown that nonlocal Laplacians (bond-based and state-based), as well as nonlocal biharmonic operators converge to their differential versions [18–20]. Similarly, we show here that all nonlocal operators (of first and second order) applied to sufficiently smooth functions approximate the output of differential operators applied to the same functions at quadratic rates with respect to the radius of interaction (this is similar for the existing nonlocal calculus framework).

1.1 Significance of this paper

Below we summarize the most noteworthy results and contributions of the paper:

1. *New framework for nonlocal vector calculus* We introduce nonlocal operators (counterparts of the classical gradient, divergence, curl, and Laplacian) with similar structures and properties to the differential ones. These operators are kernel-dependent—a feature which makes them easily adaptable to a variety of physical applications. In fact, they can be further generalized, by allowing kernels to be space, time, or even solution dependent. For integrable kernels, the operators are not smoothing, which makes them good candidates for systems where discontinous solutions are permissible.
2. *Parallel structure and properties of nonlocal framework* In the calculus framework introduced we prove a collection of results that show a similar structure of the nonlocal setting to the classical one. In particular, we identify distribution-valued kernels that transform the nonlocal operators into classical gradient, curl, and divergence. This setting differs from the one provided in [21], where connections to the classical framework were also provided, however, through different choices for the kernel due to the different structures of the operators. Thus, the nonlocal Laplacian in [21] is a singly nonlocal operator, while here the nonlocal Laplacian is a doubly nonlocal operator (as defined in Definition 6 below).
3. *Helmholtz-Hodge decompositions* By exploiting the convolution structure of the operators and using the Fourier transform we obtain a nonlocal Helmholtz decomposition for one-point vector fields. As these decompositions have been shown to have far-reaching applicability in the classical framework, it is expected that their implementation in nonlocal models will be highly advantageous as well.
4. *Convergence to classical counterparts* Besides the twin-like structures of nonlocal and local calculus frameworks, we establish additional connections. By localizing the kernels of the integral operators we obtain their differential counterparts. This framework allows us to study convergence of the nonlocal gradients to classical counterparts (two-point formulations of the gradients, being algebraic differences, do not allow that). Since in our nonlocal framework the Laplacian is still decomposed as the divergence of the gradient, by using the convergence for each of the components (the gradient and the divergence), we obtain convergence of the new nonlocal Laplacian to its classical counterpart.

1.2 Paper organization

In Sect. 2 we set up the notation and general assumptions to prepare the introduction of the nonlocal operators in Sect. 3, together with some tools that we will be using in the sequel. The nonlocal Helmholtz decomposition is then proven in Sect. 4, where existence of the components, as well as uniqueness are shown. In Sect. 5, we develop the nonlocal calculus framework by identifying the kernels and adjoints for the nonlocal gradient, divergence, curl, and Laplacian. Following that, in Sect. 6, connections to the classical framework are demonstrated, where we first recover the classical Helmholtz decomposition and differential operators with a special choice for the interaction kernel; moreover, convergence of the nonlocal operators to their classical counterparts, as the horizon of interaction converges to zero, is shown under a simplifying assumption for the kernel. With the nonlocal framework established, we move to a discussion of the nonlocal Poisson problem on $\mathbb{R}^n$ in Sect. 7. Namely, we prove well-posedness results and provide some examples, which highlight the dependence of the Poisson problem on the selection of the interaction kernel. The paper concludes in Sect. 8 with a discussion of the significance of the results together with future areas of investigation.

2 Setup and notation

The framework and the results of the paper use the Fourier transform extensively. Thus the operators introduced can have their domains on subsets of L^p spaces, or more generally, on the space of tempered distributions $\mathcal{S}'(\mathbb{R}^n)$, the dual of the space of fast decaying functions $\mathcal{S}(\mathbb{R}^n)$, which consists of the smooth functions that, along with their derivatives, decay faster than any polynomial. Before providing a more precise definition, for vectors $\mathbf{v}$ we will use bolded notation and its components will be notated as $v_1, \ldots, v_n$. Now, $f \in C^\infty(\mathbb{R}^n)$ is in $\mathcal{S}$ if

$$\sup_{x\in\mathbb{R}^n} |\mathbf{x}^{\boldsymbol{\beta}} D^{\boldsymbol{\gamma}} f| < \infty$$

holds for all multi-indices $\boldsymbol{\beta}, \boldsymbol{\gamma} \in \mathbb{N}^n$. This space is just one example of a useful function space to consider with these operators. We will often take $\mathcal{S}(\mathbb{R}^n)$ to be the domain of the nonlocal operators, but it is worth noting that most of the proofs work identically with different domains. The main concern will be that the Fourier transform is defined on these spaces.

Throughout the paper, we use $\boldsymbol{\alpha}$ to denote the vector kernel of the nonlocal operators. When we use the nonlocal curl, $\boldsymbol{\alpha}$ is assumed to be in $\mathbb{R}^3$, instead of the general $\mathbb{R}^n$. All L^p spaces will be considered on $\mathbb{R}^n$, as such we will often drop the domain in the notation. For the L^2 inner product of two functions $f, g \in L^2$ we use the notation (f, g). The adjoint of an operator T, in L^2 will be denoted T^*.

For the Fourier transform of a scalar function, u, we will take the formulation given by:

$$\mathcal{F}[u] := \widehat{u}(\boldsymbol{\xi}) := \int_{\mathbb{R}^n} u(\mathbf{x}) e^{-i\boldsymbol{\xi}\cdot\mathbf{x}}\, d\mathbf{x}.$$

This definition gives the inverse Fourier transform

$$u(\mathbf{x}) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \widehat{u}(\boldsymbol{\xi}) e^{i\boldsymbol{\xi}\cdot\mathbf{x}}\, d\boldsymbol{\xi}.$$

Recall that $\mathcal{F} : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ and $\mathcal{F} : L^1(\mathbb{R}^n) \to C(\mathbb{R}^n)$. The following are two standard identities for the Fourier transform. If f, g have a Fourier transform, then

$$\mathcal{F}(f * g) = \mathcal{F}(f)\mathcal{F}(g). \tag{1}$$

If $\boldsymbol{\beta}$ is a multi-index, then

$$\mathcal{F}[D^{\boldsymbol{\beta}} u(\mathbf{x})] = (i\boldsymbol{\xi})^{\boldsymbol{\beta}}\, \widehat{u}(\boldsymbol{\xi}). \tag{2}$$

Finally, we also have two standard identities that relate convolutions with the L^2 inner product and norm. If $f, g \in L^2$, $h \in L^1$, and $h^-(x) := h(-x)$ then

$$(f, h * g) = (f * h^-, g). \tag{3}$$

Also, known as Young's inequality for convolutions, we have, if $f \in L^p(\mathbb{R}^n)$ and $g \in L^q(\mathbb{R}^n)$ with $\dfrac{1}{p} + \dfrac{1}{q} = \dfrac{1}{r} + 1$ and $p, q, r \in [1, \infty]$, then

$$\|f * g\|_r \leqslant \|f\|_p \|g\|_q. \tag{4}$$

3 Nonlocal operators and preliminary results

3.1 Definitions

We begin by introducing two convolution operators (dot and cross convolutions) which will be used to define the nonlocal divergence and curl, and subsequently, the nonlocal Laplacian. Note that, as in previous nonlocal frameworks, each of these operators is kernel dependent. Here, the kernel is a vector valued function that captures point-interactions.

Definition 1 For $\mathbf{f}, \mathbf{g}$ vector valued functions define *the dot convolution operator* by:

$$\mathbf{f} *. \mathbf{g} := \int_{\mathbb{R}^n} \mathbf{f}(\boldsymbol{\tau}) \cdot \mathbf{g}(\mathbf{t} - \boldsymbol{\tau})\, d\boldsymbol{\tau}.$$

All properties (such as commutativity, associativity) that hold for normal convolution will transfer to the dot convolution.

Definition 2 We define *the cross convolution operator* for functions $\mathbf{f}, \mathbf{g}$ as

$$\mathbf{f} *_{\times} \mathbf{g} := \begin{pmatrix} f_2 * g_3 - g_2 * f_3 \\ g_1 * f_3 - f_1 * g_3 \\ f_1 * g_2 - g_1 * f_2 \end{pmatrix}.$$

Definition 3 *The nonlocal gradient* of a scalar function, u, with respect to a vector kernel $\boldsymbol{\alpha}$, is

$$\mathcal{G}_{\boldsymbol{\alpha}}[u] := -\boldsymbol{\alpha} * u,$$

where the convolution is evaluated componentwise.

Definition 4 *The nonlocal divergence* with kernel $\boldsymbol{\alpha}$ of a vector function $\mathbf{v}$ is defined as

$$\mathcal{D}_{\boldsymbol{\alpha}}[\mathbf{v}] := -\boldsymbol{\alpha} *. \mathbf{v}.$$

Definition 5 We define *the nonlocal curl* with kernel $\boldsymbol{\alpha}$ of a function $\mathbf{v}$ to be

$$\mathcal{C}_{\boldsymbol{\alpha}}[\mathbf{v}] := -\boldsymbol{\alpha} *_{\times} \mathbf{v}.$$

Definition 6 *The nonlocal Laplacian* of u, is given by the scalar function

$$\mathcal{L}_{\boldsymbol{\alpha}}[u] := \mathcal{D}_{\boldsymbol{\alpha}}[\mathcal{G}_{\boldsymbol{\alpha}}[u]] = \alpha_1 * \alpha_1 * u + \cdots + \alpha_n * \alpha_n * u.$$

Remark 1 The dimension n, for the functions in the above definitions could be arbitrary; restrictions on the dimension are placed only on the range of functions involved in cross-products.

Remark 2 We note here that the doubly nonlocal Laplacian introduced above is different than the singly nonlocal operator defined in [21], but also different than the state-based doubly nonlocal Laplacian defined in [22] which is motivated by the theory of state-based peridynamics. This work shows once again the flexibility of nonlocal frameworks which allow a wide-range of options to capture behavior modeled in the classical differential setting by one operator (here, the Laplacian).

Remark 3 The nonlocal Laplacian is the divergence of the gradient, and as such it would model diffusion according to a nonlocal version of Fourier's law given by

$$\mathbf{q} = -k\mathcal{G}_{\boldsymbol{\alpha}}[u], \tag{5}$$

where for heat conduction applications, $\mathbf{q}$ denotes the flux density, k is the thermal conductivity, and u is temperature. Different models for heat diffusion have been introduced before in [23] in the nonlocal calculus framework of [21]. Note that with the previous nonlocal Laplacian of [21] the flux is given by a two-point operator (indicating the mass moving from location x to location y), on which the nonlocal divergence is acting. In our formulation, the nonlocal diffusion operator is the result of applying the nonlocal divergence on the one-point flux operator. Here, the flux models, just as in the classical case, the amount passing through a location x.

3.2 Identities

The nonlocal operators introduced above satisfy similar identities to their classical counterparts, which we show below. We will take

Proposition 1 (Fourier identity) *Suppose* $\mathbf{f}, \mathbf{g} \in \mathcal{S}(\mathbb{R}^n)$ *are vector valued functions. Then*

$$\mathcal{F}[\mathbf{f} *_{\times} \mathbf{g}] = \widehat{\mathbf{f}} \times \widehat{\mathbf{g}}. \tag{6}$$

Proof Expand the cross product and repeatedly apply (1). That is,

$$\mathcal{F}^{-1}[\widehat{\mathbf{f}} \times \widehat{\mathbf{g}}] = \mathcal{F}^{-1}\left[\begin{pmatrix} \widehat{f_2}\widehat{g_3} - \widehat{g_2}\widehat{f_3} \\ \widehat{g_1}\widehat{f_3} - \widehat{f_1}\widehat{g_3} \\ \widehat{f_1}\widehat{g_2} - \widehat{g_1}\widehat{f_2} \end{pmatrix}\right] = \begin{pmatrix} f_2 * g_3 - g_2 * f_3 \\ g_1 * f_3 - f_1 * g_3 \\ f_1 * g_2 - g_1 * f_2 \end{pmatrix} = \mathbf{f} *_{\times} \mathbf{g}.$$

□

Proposition 2 (Curl of the curl) *Given some vector function* $\mathbf{f} \in \mathcal{S}(\mathbb{R}^n)$ *and a vector kernel* $\boldsymbol{\alpha} \in \mathcal{S}(\mathbb{R}^n)$, *we have*

$$\mathcal{C}_{\boldsymbol{\alpha}}[\mathcal{C}_{\boldsymbol{\alpha}}[\mathbf{f}]] = \mathcal{G}_{\boldsymbol{\alpha}}[\mathcal{D}_{\boldsymbol{\alpha}}[\mathbf{f}]] - \mathcal{L}_{\boldsymbol{\alpha}}[\mathbf{f}], \tag{7}$$

where we take the Laplacian of a vector function componentwise.

Proof First, recall the cross product identity

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}. \tag{8}$$

Using this and Proposition 1 yields

$$\mathcal{F}[\mathcal{C}_{\boldsymbol{\alpha}}[\mathcal{C}_{\boldsymbol{\alpha}}[\mathbf{f}]]] = \mathcal{F}\left[\boldsymbol{\alpha} *_{\times} (\boldsymbol{\alpha} *_{\times} \mathbf{f})\right] = \widehat{\boldsymbol{\alpha}} \times (\widehat{\boldsymbol{\alpha}} \times \widehat{\mathbf{f}}) = (\widehat{\boldsymbol{\alpha}} \cdot \widehat{\mathbf{f}})\widehat{\boldsymbol{\alpha}} - (\widehat{\boldsymbol{\alpha}} \cdot \widehat{\boldsymbol{\alpha}})\widehat{\mathbf{f}},$$

Now expand this and take the inverse Fourier transform of both sides:

$$\begin{aligned}
\mathcal{C}_{\boldsymbol{\alpha}}[\mathcal{C}_{\boldsymbol{\alpha}}[\mathbf{f}]] &= \mathcal{F}^{-1}\left[(\widehat{\boldsymbol{\alpha}} \cdot \widehat{\mathbf{f}})\widehat{\boldsymbol{\alpha}}\right] - \mathcal{F}^{-1}\left[(\widehat{\boldsymbol{\alpha}} \cdot \widehat{\boldsymbol{\alpha}})\widehat{\mathbf{f}}\right] \\
&= \mathcal{F}^{-1}\begin{pmatrix} (\widehat{\alpha}_1 \widehat{f_1} + \widehat{\alpha}_2 \widehat{f_2} + \widehat{\alpha}_3 \widehat{f_3})\widehat{\alpha}_1 \\ (\widehat{\alpha}_1 \widehat{f_1} + \widehat{\alpha}_2 \widehat{f_2} + \widehat{\alpha}_3 \widehat{f_3})\widehat{\alpha}_2 \\ (\widehat{\alpha}_1 \widehat{f_1} + \widehat{\alpha}_2 \widehat{f_2} + \widehat{\alpha}_3 \widehat{f_3})\widehat{\alpha}_3 \end{pmatrix} \\
&\quad -\mathcal{F}^{-1}\begin{pmatrix} (\widehat{\alpha}_1\widehat{\alpha}_1 + \widehat{\alpha}_2\widehat{\alpha}_2 + \widehat{\alpha}_3\widehat{\alpha}_3)\widehat{f_1} \\ (\widehat{\alpha}_1\widehat{\alpha}_1 + \widehat{\alpha}_2\widehat{\alpha}_2 + \widehat{\alpha}_3\widehat{\alpha}_3)\widehat{f_2} \\ (\widehat{\alpha}_1\widehat{\alpha}_1 + \widehat{\alpha}_2\widehat{\alpha}_2 + \widehat{\alpha}_3\widehat{\alpha}_3)\widehat{f_3} \end{pmatrix}.
\end{aligned}$$

Applying (1), we see

$$\begin{aligned}
\mathcal{C}_{\boldsymbol{\alpha}}[\mathcal{C}_{\boldsymbol{\alpha}}[\mathbf{f}]] &= \begin{pmatrix} (\alpha_1 * f_1 + \alpha_2 * f_2 + \alpha_3 * f_3) * \alpha_1 \\ (\alpha_1 * f_1 + \alpha_2 * f_2 + \alpha_3 * f_3) * \alpha_2 \\ (\alpha_1 * f_1 + \alpha_2 * f_2 + \alpha_3 * f_3) * \alpha_3 \end{pmatrix} \\
&\quad - \begin{pmatrix} (\alpha_1 * \alpha_1 + \alpha_2 * \alpha_2 + \alpha_3 * \alpha_3) * f_1 \\ (\alpha_1 * \alpha_1 + \alpha_2 * \alpha_2 + \alpha_3 * \alpha_3) * f_2 \\ (\alpha_1 * \alpha_1 + \alpha_2 * \alpha_2 + \alpha_3 * \alpha_3) * f_3 \end{pmatrix} \\
&= -\boldsymbol{\alpha} * \mathcal{D}_{\boldsymbol{\alpha}}[\mathbf{f}] - \mathcal{L}_{\boldsymbol{\alpha}}[\mathbf{f}] \\
&= \mathcal{G}_{\boldsymbol{\alpha}}[\mathcal{D}_{\boldsymbol{\alpha}}[\mathbf{f}]] - \mathcal{L}_{\boldsymbol{\alpha}}[\mathbf{f}],
\end{aligned}$$

as desired. □

Proposition 3 (Compositions of nonlocal operators) *Similar as in the classical framework, for* $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^3$ *and* $\Phi : \mathbb{R}^3 \to \mathbb{R}$ *we have*

$$\mathcal{D}_{\boldsymbol{\alpha}}[\mathcal{C}_{\boldsymbol{\alpha}}[\mathbf{f}]] = 0, \quad \mathcal{C}_{\boldsymbol{\alpha}}[\mathcal{G}_{\boldsymbol{\alpha}}[\Phi]] = 0.$$

Proof For the divergence composed with the curl, note that

$$\begin{aligned}
\mathcal{D}_{\boldsymbol{\alpha}}[\mathcal{C}_{\boldsymbol{\alpha}}[\mathbf{f}]] &= \boldsymbol{\alpha} *. (\boldsymbol{\alpha} *_{\times} \mathbf{f}) \\
&= \alpha_1 * (\alpha_2 * f_3 - f_2 * \alpha_3) + \alpha_2 * (f_1 * \alpha_3 - \alpha_1 * f_3) \\
&\quad + \alpha_3 * (\alpha_1 * f_2 - f_1 * \alpha_2) \\
&= 0,
\end{aligned}$$

For the curl composed with the gradient, note that

$$\begin{aligned}
\mathcal{C}_{\boldsymbol{\alpha}}[\mathcal{G}_{\boldsymbol{\alpha}}[\Phi]] &= \boldsymbol{\alpha} *_{\times} (\boldsymbol{\alpha} * \Phi) \\
&= \begin{pmatrix} \alpha_2 * (\alpha_3 * \Phi) - (\alpha_2 * \Phi) * \alpha_3 \\ (\alpha_1 * \Phi) * \alpha_3 - \alpha_1 * (\alpha_3 * \Phi) \\ \alpha_1 * (\alpha_2 * \Phi) - (\alpha_1 * \Phi) * \alpha_2 \end{pmatrix} \\
&= 0.
\end{aligned}$$

□

Remark 4 In the above proposition, taking the functions to be in $\mathcal{S}$ would be appropriate, but here we do not even require the Fourier transform to be defined, so we may take much larger spaces and the compositions still yield zero.

Proposition 4 (Regularity) *For* $1 \leqslant p \leqslant \infty$ *assume that the kernel* $\boldsymbol{\alpha} \in L^1(\mathbb{R}^n)$*, and the functions* $u \in L^p(\mathbb{R}^n)$, $\mathbf{v} \in L^p(\mathbb{R}^n)$, $\mathbf{w} \in L^p(\mathbb{R}^3)$. *Then*

$$\mathcal{G}_{\boldsymbol{\alpha}}[u] \in L^p(\mathbb{R}^n),\ \mathcal{D}_{\boldsymbol{\alpha}}[\mathbf{v}] \in L^p(\mathbb{R}^n),\ \mathcal{C}_{\boldsymbol{\alpha}}[\mathbf{w}] \in L^p(\mathbb{R}^3),\ \mathcal{L}_{\boldsymbol{\alpha}}[u] \in L^p(\mathbb{R}^n).$$

Proof This is a trivial application of (4) in the case $q = 1$. □

Remark 5 Similar regularity results for $\boldsymbol{\alpha} \in L^q(\mathbb{R}^n)$, $1 \leqslant q \leqslant \infty$ can be obtained from (4). If $\boldsymbol{\alpha}$ has more regularity (e.g. it is C^1), then these operators endow that regularity on the output function by standard properties of convolutions.

Remark 6 Note that no regularity is gained nor lost when applying these nonlocal operators. Additionally, note that with this nonlocal Laplacian, we can take $\boldsymbol{\alpha} \in L^1(\mathbb{R}^n)$, whereas the nonlocal Laplacian of [3, 21] given by

$$\mathscr{L}_{\boldsymbol{\alpha}}(u) = u * |\boldsymbol{\alpha}|^2 - u$$

required $\boldsymbol{\alpha} \in L^2(\mathbb{R}^n)$ in order to yield the same level of integrability for the output.

Proposition 5 (Integration by parts) *Suppose that* $\boldsymbol{\alpha} \in L^1(\mathbb{R}^n)$ *is antisymmetric and* $b \in L^2(\mathbb{R}^n)$, $\mathbf{a} \in L^2(\mathbb{R}^n)$. *Then,*

$$(b, \mathcal{D}_{\boldsymbol{\alpha}}[\mathbf{a}]) = -(\mathcal{G}_{\boldsymbol{\alpha}}[b], \mathbf{a}). \tag{9}$$

Proof First note that, by Proposition 4, we see that, since $\mathbf{a} \in L^2(\mathbb{R}^n)$, we have $\mathcal{D}_{\boldsymbol{\alpha}}[\mathbf{a}] \in L^2(\mathbb{R}^n)$. Similarly, $\mathcal{G}_{\boldsymbol{\alpha}}[b] \in L^2(\mathbb{R}^n)$. Note that, since $\boldsymbol{\alpha}$ is antisymmetric, we have that $\boldsymbol{\alpha}^- = -\boldsymbol{\alpha}$. Now we apply (3) and obtain

$$\begin{aligned}(b, \mathcal{D}_{\boldsymbol{\alpha}}[\mathbf{a}]) &= (b, -\boldsymbol{\alpha} *. \mathbf{a}) = -(b, \alpha_1 * a_1) - \cdots - (b, \alpha_n * a_n)\\ &= (\alpha_1 * b, a_1) + \cdots + (\alpha_n * b, a_n)\\ &= -(\mathcal{G}_{\boldsymbol{\alpha}}[b], \mathbf{a}).\end{aligned}$$

□

Remark 7 This result validates the choice of the minus sign in the definition of the nonlocal gradient, so for $\boldsymbol{\alpha}$ antisymmetric, the integration by parts has a similar formulation as in the differential framework.

4 Nonlocal Helmholtz decomposition

We are now ready to prove our nonlocal Helmholtz decomposition.

Theorem 6 (Nonlocal Helmholtz decomposition) *Take* $\boldsymbol{\alpha} \in \mathcal{S}'(\mathbb{R}^n)$ *so that* $1/|\widehat{\boldsymbol{\alpha}}|^2$ *is integrable. Suppose* $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ *is a vector field decaying faster than* $1/|\widehat{\boldsymbol{\alpha}}|$ *at infinity. Then there exist potential functions* Φ *and* $\mathbf{A}$ *such that*

$$\mathbf{F} = -\mathcal{G}_{\boldsymbol{\alpha}}(\Phi) + \mathcal{C}_{\boldsymbol{\alpha}}(\mathbf{A}). \tag{10}$$

Proof Let $\mathbf{G}$ be the Fourier transform of $\mathbf{F}$. Then define the functions

$$G_{\Phi_\alpha}(\boldsymbol{\xi}) := \frac{\widehat{\boldsymbol{\alpha}}(\boldsymbol{\xi}) \cdot \mathbf{G}(\boldsymbol{\xi})}{|\widehat{\boldsymbol{\alpha}}(\boldsymbol{\xi})|^2} \quad \text{and} \quad \mathbf{G}_{\mathbf{A}_\alpha}(\boldsymbol{\xi}) := \frac{\widehat{\boldsymbol{\alpha}}(\boldsymbol{\xi}) \times \mathbf{G}(\boldsymbol{\xi})}{|\widehat{\boldsymbol{\alpha}}(\boldsymbol{\xi})|^2}.$$

Based on these, we further define

$$\Phi := \mathcal{F}^{-1}[G_{\Phi_\alpha}] \quad \text{and} \quad \mathbf{A} := \mathcal{F}^{-1}[\mathbf{G}_{\mathbf{A}_\alpha}],$$

which are well-defined by the decay condition on $\mathbf{F}$ and the integrability $1/|\boldsymbol{\alpha}|^2$. Applying (8) we see that

$$\mathbf{G}(\boldsymbol{\xi}) = \widehat{\boldsymbol{\alpha}}(\boldsymbol{\xi}) G_{\Phi_\alpha}(\boldsymbol{\xi}) - \widehat{\boldsymbol{\alpha}}(\boldsymbol{\xi}) \times \mathbf{G}_{\mathbf{A}_\alpha}(\boldsymbol{\xi}). \tag{11}$$

Indeed, using (8) on the second term, we obtain

$$\begin{aligned} \widehat{\boldsymbol{\alpha}}(\boldsymbol{\xi}) \times \mathbf{G}_{\mathbf{A}_\alpha}(\boldsymbol{\xi}) &= \widehat{\boldsymbol{\alpha}}(\boldsymbol{\xi}) \times \frac{\widehat{\boldsymbol{\alpha}}(\boldsymbol{\xi}) \times \mathbf{G}(\boldsymbol{\xi})}{|\widehat{\boldsymbol{\alpha}}(\boldsymbol{\xi})|^2} \\ &= \widehat{\boldsymbol{\alpha}}(\boldsymbol{\xi}) G_{\Phi_\alpha}(\boldsymbol{\xi}) - \frac{|\boldsymbol{\alpha}(\boldsymbol{\xi})|^2}{|\boldsymbol{\alpha}(\boldsymbol{\xi})|^2} \mathbf{G}(\boldsymbol{\xi}). \end{aligned}$$

Hence, (11) holds. Applying the inverse Fourier transform to (11) we get

$$\mathbf{F} = \mathcal{F}^{-1}[\widehat{\boldsymbol{\alpha}} G_{\Phi_\alpha}] - \mathcal{F}^{-1}[\widehat{\boldsymbol{\alpha}} \times \mathbf{G}_{\mathbf{A}_\alpha}].$$

Using (1) to rewrite the former term yields

$$\mathcal{F}^{-1}[\widehat{\boldsymbol{\alpha}} G_{\Phi_\alpha}] = \boldsymbol{\alpha} * \mathcal{F}^{-1}[G_{\Phi_\alpha}] = -\mathcal{G}_{\boldsymbol{\alpha}}\left(\mathcal{F}^{-1}[G_{\Phi_\alpha}]\right).$$

Then (6) applied to the latter term yields

$$\mathcal{F}^{-1}[\widehat{\boldsymbol{\alpha}} \times \mathbf{G}_{\mathbf{A}_\alpha}] = \boldsymbol{\alpha} *_\times \mathcal{F}^{-1}[\mathbf{G}_{\mathbf{A}_\alpha}] = -\mathcal{C}_{\boldsymbol{\alpha}}\left(\mathcal{F}^{-1}[\mathbf{G}_{\mathbf{A}_\alpha}]\right).$$

Finally, with our definitions of Φ and $\mathbf{A}$,

$$\mathbf{F} = -\mathcal{G}_{\boldsymbol{\alpha}}(\Phi) + \mathcal{C}_{\boldsymbol{\alpha}}(\mathbf{A}).$$

as desired. □

To achieve uniqueness, we first prove a lemma about the nonlocal Laplacian.

Lemma 1 *Suppose $\widehat{\boldsymbol{\alpha}} \neq 0$ and the function f solves $-\mathcal{L}_{\boldsymbol{\alpha}}[f] = 0$. Then $f \equiv 0$.*

Proof By assumption,

$$\mathcal{L}_{\boldsymbol{\alpha}}[f] = \alpha_1 * \alpha_1 * f + \cdots + \alpha_n * \alpha_n * f = 0.$$

Taking the Fourier tranform and using (1) yields

$$(\widehat{\alpha}_1^2 + \widehat{\alpha}_2^2 + \widehat{\alpha}_3^2)\widehat{f} = 0.$$

Since $\widehat{\boldsymbol{\alpha}} \neq 0$ we obtain $\widehat{f} = 0$, so $f = 0$. □

Theorem 7 (Uniqueness for the nonlocal Helmholtz decomposition) *Suppose $\widehat{\boldsymbol{\alpha}} \neq 0$. Then the gradient potential function Φ in Theorem 6 is unique. The curl potential function $\mathbf{A}$ is unique assuming the nonlocal incompressibility condition $\mathcal{D}_{\boldsymbol{\alpha}}[\mathbf{A}] = 0$.*

Proof Suppose that there are two decompositions for $\mathbf{F}$, more precisely, there exist functions $\Phi_1, \Phi_2, \mathbf{A}_1, \mathbf{A}_2$ with

$$-\mathcal{G}_{\boldsymbol{\alpha}}[\Phi_1] + \mathcal{C}_{\boldsymbol{\alpha}}[\mathbf{A}_1] = \mathbf{F} = -\mathcal{G}_{\boldsymbol{\alpha}}[\Phi_2] + \mathcal{C}_{\boldsymbol{\alpha}}[\mathbf{A}_2]. \tag{12}$$

Taking the nonlocal divergence and using Proposition 3 we are left with

$$-\mathcal{L}_{\boldsymbol{\alpha}}[\Phi_1] = \mathcal{D}_{\boldsymbol{\alpha}}[\mathbf{F}] = -\mathcal{L}_{\boldsymbol{\alpha}}[\Phi_2],$$

which yields a nonlocal Laplace equation:

$$-\mathcal{L}_{\boldsymbol{\alpha}}[\Phi_1 - \Phi_2] = 0.$$

Apply Lemma 1 to see $\Phi_1 \equiv \Phi_2$.

Now assume that $\mathbf{A}_1$ and $\mathbf{A}_2$ have zero nonlocal divergence. Then we take the curl of (12) and apply Propositions 2 and 3 to obtain

$$\mathcal{G}_{\boldsymbol{\alpha}}[\mathcal{D}_{\boldsymbol{\alpha}}[\mathbf{A}_1]] - \mathcal{L}_{\boldsymbol{\alpha}}[\mathbf{A}_1] = \mathcal{G}_{\boldsymbol{\alpha}}[\mathcal{D}_{\boldsymbol{\alpha}}[\mathbf{A}_2]] - \mathcal{L}_{\boldsymbol{\alpha}}[\mathbf{A}_2].$$

Applying Lemma 1 to each component, we achieve the result: $\mathbf{A}_1 \equiv \mathbf{A}_2$. □

5 Nonlocal calculus

We begin by identifying the algebraic kernel (i.e., the elements for which the operator is null) of each nonlocal operator and showing parallel results to the local case. The first result is a counterpart of the classical result that shows that the only L^p function with zero gradient is the trivial one.

Proposition 8 (Kernel of the gradient) *Suppose* $\boldsymbol{\alpha} \in \mathcal{S}$ *is such that* $\widehat{\boldsymbol{\alpha}} \neq 0$. *Then*

$$\ker(\mathcal{G}_{\boldsymbol{\alpha}}) \cap L^p = \{0\}$$

for $1 \leqslant p < \infty$. *In fact,* $\ker(\mathcal{G}_{\boldsymbol{\alpha}}) \cap \mathcal{S}' = \{0\}$.

Proof It is obvious that $\mathcal{G}_{\boldsymbol{\alpha}}[0] = 0$, so we are left with the other inclusion. Note that if $f \in L^p$ and $\mathcal{G}_{\boldsymbol{\alpha}}[f] = 0$, then taking the Fourier transform and applying (1), we obtain

$$\widehat{\alpha}_1 \widehat{f} = \widehat{\alpha}_2 \widehat{f} = \widehat{\alpha}_3 \widehat{f} = 0,$$

so $\widehat{f} = 0$, which gives the result. Similarly, as long as the above operations are well-defined in $\mathcal{S}'$ the result extends from L^p to $\mathcal{S}'$. □

Proposition 9 (Kernel of the divergence) *Suppose* $\boldsymbol{\alpha} \in \mathcal{S}$ *is such that* $\widehat{\boldsymbol{\alpha}} \neq 0$. *Then the kernel of the nonlocal divergence can be identified as the space of functions that can be expressed as nonlocal curls, i.e.*

$$\ker(\mathcal{D}_{\boldsymbol{\alpha}}) = \left\{ \mathbf{w} : \mathbb{R}^3 \to \mathbb{R}^3 \,|\, \exists \mathbf{A} \textit{ such that } \mathbf{w} = \mathcal{C}_{\boldsymbol{\alpha}}[\mathbf{A}] \right\}. \tag{13}$$

Proof The first inclusion is Proposition 3. For the other inclusion, for a given function $\mathbf{w}$ such that $\mathcal{D}_{\boldsymbol{\alpha}}[\mathbf{w}] = 0$, we have from Theorem 6 that there exist potentials, Φ and $\mathbf{A}$ such that

$$\mathbf{w} = -\mathcal{G}_{\boldsymbol{\alpha}}[\Phi] + \mathcal{C}_{\boldsymbol{\alpha}}[\mathbf{A}].$$

By applying the nonlocal divergence operator and using again Proposition 3 we obtain that

$$\mathcal{D}_{\boldsymbol{\alpha}}[\mathbf{w}] = -\mathcal{D}_{\boldsymbol{\alpha}}[\mathcal{G}_{\boldsymbol{\alpha}}[\Phi]] + \mathcal{D}_{\boldsymbol{\alpha}}[\mathcal{C}_{\boldsymbol{\alpha}}[\mathbf{A}]] = -\mathcal{L}_{\boldsymbol{\alpha}}[\Phi].$$

Consequently, $\mathcal{L}_{\boldsymbol{\alpha}}[\Phi] = 0$. By Lemma 1 we see that $\Phi = 0$. From the nonlocal Helmholtz decomposition of Theorem 6, we have that

$$\mathbf{w} = -\mathcal{G}_{\boldsymbol{\alpha}}[\Phi] + \mathcal{C}_{\boldsymbol{\alpha}}[\mathbf{A}] = \mathcal{C}_{\boldsymbol{\alpha}}[\mathbf{A}],$$

as desired. □

Proposition 10 (Kernel of the curl) *Suppose* $\boldsymbol{\alpha} \in \mathcal{S}$ *is such that* $\widehat{\alpha}_j \neq 0$ *for* $1 \leqslant j \leqslant 3$. *Then*

$$\ker(\mathcal{C}_{\boldsymbol{\alpha}}) = \{\mathbf{v} : \mathbb{R}^3 \to \mathbb{R}^3 \,|\, \exists \Phi \text{ such that } \mathbf{v} = \mathcal{G}_{\boldsymbol{\alpha}}[\Phi]\}. \tag{14}$$

Proof The first inclusion is Proposition 3.

For the second inclusion, suppose that $\mathcal{C}_{\boldsymbol{\alpha}}[\mathbf{v}] = 0$. Then, taking the Fourier transform we see

$$\begin{pmatrix} \widehat{\alpha}_2\widehat{v}_3 - \widehat{\alpha}_3\widehat{v}_2 \\ \widehat{\alpha}_3\widehat{v}_1 - \widehat{\alpha}_1\widehat{v}_3 \\ \widehat{\alpha}_1\widehat{v}_2 - \widehat{\alpha}_2\widehat{v}_1 \end{pmatrix} = 0,$$

which gives the following system of equations:

$$\begin{cases} \widehat{\alpha}_2\widehat{v}_3 = \widehat{\alpha}_3\widehat{v}_2 \\ \widehat{\alpha}_3\widehat{v}_1 = \widehat{\alpha}_1\widehat{v}_3 \\ \widehat{\alpha}_1\widehat{v}_2 = \widehat{\alpha}_2\widehat{v}_1. \end{cases}$$

Upon rearrangement and using that $\widehat{\alpha}_j \neq 0$, we see

$$\frac{\widehat{v}_1}{\widehat{\alpha}_1} = \frac{\widehat{v}_2}{\widehat{\alpha}_2} = \frac{\widehat{v}_3}{\widehat{\alpha}_3}. \tag{15}$$

To conclude the proof, we define $\Phi := -\mathcal{F}^{-1}(\widehat{v}_1/\widehat{\alpha}_1)$. Note that by (1) and (15)

$$\mathcal{F}(\mathcal{G}_{\boldsymbol{\alpha}}[\Phi]) = \mathcal{F}(\boldsymbol{\alpha})\frac{\widehat{v}_1}{\widehat{\alpha}_1} = \begin{pmatrix} \widehat{\alpha}_1(\widehat{v}_1/\widehat{\alpha}_1) \\ \widehat{\alpha}_2(\widehat{v}_2/\widehat{\alpha}_2) \\ \widehat{\alpha}_3(\widehat{v}_3/\widehat{\alpha}_3) \end{pmatrix} = \widehat{\mathbf{v}}.$$

Hence, $\mathcal{G}_{\boldsymbol{\alpha}}[\Phi] = \mathbf{v}$. □

Remark 8 These identities are important in the study of the local Helmholtz decomposition, because they allow us to create Poisson problems. In particular, using Propositions 2, 9, and 10, take the curl of the Helmholtz decomposition

$$\mathcal{C}_{\boldsymbol{\alpha}}[\mathbf{F}] = -\mathcal{C}_{\boldsymbol{\alpha}}[\mathcal{G}_{\boldsymbol{\alpha}}[\Phi]] + \mathcal{C}_{\boldsymbol{\alpha}}[\mathcal{C}_{\boldsymbol{\alpha}}[\mathbf{A}]],$$

to yield a nonlocal Poisson equation:

$$-\mathcal{L}_{\boldsymbol{\alpha}}[\mathbf{A}] = \mathcal{C}_{\boldsymbol{\alpha}}[\mathbf{F}] - \mathcal{G}_{\boldsymbol{\alpha}}[\mathcal{D}_{\boldsymbol{\alpha}}[\mathbf{A}]].$$

We can also take the divergence of the Helmholtz decomposition to see

$$\mathcal{D}_{\boldsymbol{\alpha}}[\mathbf{F}] = -\mathcal{D}_{\boldsymbol{\alpha}}[\mathcal{G}_{\boldsymbol{\alpha}}[\Phi]] + \mathcal{D}_{\boldsymbol{\alpha}}[\mathcal{C}_{\boldsymbol{\alpha}}[\mathbf{A}]],$$

which further gives

$$-\mathcal{L}_{\boldsymbol{\alpha}}[\Phi] = \mathcal{D}_{\boldsymbol{\alpha}}[\mathbf{F}].$$

The well-posedness of the Poisson problem will be discussed in detail in Sect. 7 of this paper.

In the next proposition we will identify the adjoints of the nonlocal operators introduced in Sect. 3.1 above.

Proposition 11 *Let $\boldsymbol{\alpha} \in \mathcal{S}$ be antisymmetric. Then, in L^2 we have*

$$\mathcal{G}_{\boldsymbol{\alpha}}^* = -\mathcal{D}_{\boldsymbol{\alpha}} \quad \text{and} \quad \mathcal{D}_{\boldsymbol{\alpha}}^* = -\mathcal{G}_{\boldsymbol{\alpha}}.$$

Furthermore, $-\mathcal{L}_{\boldsymbol{\alpha}}$ and $\mathcal{C}_{\boldsymbol{\alpha}}$ are self-adjoint in L^2.

Proof The adjoints of $\mathcal{D}_{\boldsymbol{\alpha}}$ and $\mathcal{G}_{\boldsymbol{\alpha}}$ follow immediately from Proposition 5. To show that $-\mathcal{L}_{\boldsymbol{\alpha}}$ is self-adjoint, suppose $a, b \in L^2(\mathbb{R}^n)$. Using Proposition 5, we obtain

$$(-\mathcal{L}_{\boldsymbol{\alpha}}[a], b) = (\mathcal{G}_{\boldsymbol{\alpha}}[a], \mathcal{G}_{\boldsymbol{\alpha}}[b]) = -(a, \mathcal{D}_{\boldsymbol{\alpha}}[\mathcal{G}_{\boldsymbol{\alpha}}[b]]) = (a, -\mathcal{L}_{\boldsymbol{\alpha}}[b]),$$

so $-\mathcal{L}_{\boldsymbol{\alpha}}$ is self-adjoint in L^2.

Now, to show that $\mathcal{C}_{\boldsymbol{\alpha}}$ is self-adjoint in L^2, we take $\mathbf{a}, \mathbf{b} \in L^2(\mathbb{R}^3)$. Using (3) and the fact that $\boldsymbol{\alpha}$ is antisymmetric, we see

$$\begin{aligned}
(\mathcal{C}_{\boldsymbol{\alpha}}[\mathbf{a}], \mathbf{b}) &= \int_{\mathbb{R}^3} \mathcal{C}_{\boldsymbol{\alpha}}[\mathbf{a}] \cdot \mathbf{b} \, dx \\
&= -\int_{\mathbb{R}^3} [b_1(\alpha_2 * a_3) - b_1(a_2 * \alpha_3)] + [b_2(a_1 * \alpha_3) - b_2(\alpha_1 * a_3)] \\
&\quad + [b_3(\alpha_1 * a_2) - b_3(a_1 * \alpha_2)] \, dx \\
&= -\int_{\mathbb{R}^3} a_1[\alpha_2 * b_3 - b_2 * \alpha_3] + a_2[b_1 * \alpha_3 - \alpha_1 * b_3] \\
&\quad + a_3[\alpha_1 * b_2 - b_1 * \alpha_2] \, dx \\
&= \int_{\mathbb{R}^3} \mathcal{C}_{\boldsymbol{\alpha}}[\mathbf{b}] \cdot \mathbf{a} \, dx \\
&= (\mathbf{a}, \mathcal{C}_{\boldsymbol{\alpha}}[\mathbf{b}]),
\end{aligned}$$

as was to be shown. □

6 Connections to the classical framework

6.1 Nonlocal decomposition generalizes the local framework

A natural question to ask is whether a proof of the local Helmholtz decomposition can be recovered from the above proof by a suitable choice of $\boldsymbol{\alpha}$. Indeed, if we consider the case where $\boldsymbol{\alpha} = -\nabla\delta_0$, where δ_0 is the Dirac mass centered at the origin, then we have a proof of the local Helmholtz decomposition.

To be explicit, take $\boldsymbol{\alpha} = -\nabla\delta_0$. Begin by noting that, by (2)

$$\mathcal{F}[-\nabla\delta_0] = -i\boldsymbol{\xi}.$$

Now we apply the same method as in the proof of Theorem 6. Define

$$G_{\Phi_\alpha}(\boldsymbol{\xi}) = \frac{-i\boldsymbol{\xi} \cdot \mathbf{G}(\boldsymbol{\xi})}{|\boldsymbol{\xi}|^2} \quad \text{and} \quad \mathbf{G}_{\mathbf{A}_\alpha}(\boldsymbol{\xi}) = \frac{-i\boldsymbol{\xi} \times \mathbf{G}(\boldsymbol{\xi})}{|\boldsymbol{\xi}|^2}.$$

Then

$$\mathbf{G}(\boldsymbol{\xi}) = -i\boldsymbol{\xi} G_{\Phi_\alpha}(\boldsymbol{\xi}) + i\boldsymbol{\xi} \times \mathbf{G}_{\mathbf{A}_\alpha}(\boldsymbol{\xi}).$$

Applying the inverse Fourier Transform gives

$$\mathbf{F}(\mathbf{x}) = -\frac{1}{(2\pi)^3}\int_{\mathbb{R}^3} i\boldsymbol{\xi} G_{\Phi_{\boldsymbol{\alpha}}}(\boldsymbol{\xi})e^{i\boldsymbol{\xi}\cdot\mathbf{x}}\,d\boldsymbol{\xi} + \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3} i\boldsymbol{\xi}\times\mathbf{G}_{\mathbf{A}_{\boldsymbol{\alpha}}}(\boldsymbol{\xi})e^{i\boldsymbol{\xi}\cdot\mathbf{x}}\,d\boldsymbol{\xi}.$$

Defining

$$\Phi = \mathcal{F}^{-1}[G_{\Phi_{\boldsymbol{\alpha}}}] \quad \text{and} \quad \mathbf{A} = \mathcal{F}^{-1}[\mathbf{G}_{\mathbf{A}_{\boldsymbol{\alpha}}}],$$

we see that

$$\mathbf{F}(\mathbf{x}) = -\nabla\Phi + \nabla\times\mathbf{A},$$

which is the local Helmholtz decomposition. With regard to the decay condition in Theorem 6, here that translates to $\mathbf{F}$ decaying faster than $1/|\mathbf{x}|$ at infinity, which is a common assumption for obtaining a Helmholtz decomposition in the full space.

Observe that, also with this choice of $\boldsymbol{\alpha}$, our nonlocal operators become their local counterparts. This aspect ties in with aspects of convergence, since the Dirac mass (and its derivatives) correspond to an interaction horizon that shrinks to zero (with appropriate scaling), thus "localizing" the nonlocal operators. We turn now to a more detailed discussion of these convergence aspects.

6.2 Convergence of nonlocal operators to classical

Lemma 2 (Convergence of nonlocal gradients in 1D) *Let $f : \mathbb{R} \to \mathbb{R}$ with $f \in C^3(\mathbb{R})$ and suppose that α_δ is a family of antisymmetric kernels such that $\alpha \geqslant 0$ for all $x \geqslant 0$ and additionally supp$(\alpha_\delta) \subseteq B_\delta(0)$ for $\delta > 0$. (For simplicity of notation we will drop the subscript δ on the kernel α in the sequel). Additionally, we impose the normalization condition*

$$\int_{B_\delta(0)} y\alpha(y)\,dy = 1. \tag{16}$$

Then we have the following approximation estimate between the nonlocal and local (one-dimensional) gradients:

$$|\mathcal{G}_\alpha[f](x) - f'(x)| = |-(\alpha * f)(x) - f'(x)| \leqslant \frac{\delta^2}{6}\sup_{c\in B_\delta(x)}|f'''(c)|. \tag{17}$$

Moreover, if f''' is bounded on $\mathbb{R}$, then

$$\|-\alpha * f - f'\|_\infty \leqslant \frac{\delta^2}{6}\|f'''\|_\infty. \tag{18}$$

Proof Using the fact that supp$(\alpha) \subseteq B_\delta(0)$ we get

$$(\alpha * f)(x) = \int_{\mathbb{R}} f(x-y)\alpha(y)\,dy = \int_{B_\delta(0)} f(x-y)\alpha(y)\,dy.$$

By writing a Taylor expansion with Lagrange remainder for f near x we obtain

$$f(x-y) = f(x) - yf'(x) + \frac{y^2}{2}f''(x) - \frac{y^3}{6}f'''(c),$$

for some $c \in (x-\delta, x+\delta)$. Hence

$$\int_{B_\delta(0)} f(x-y)\alpha(y)\,dy$$

$$= \int_{B_\delta(0)} \left[f(x) - yf'(x) + \frac{y^2}{2} f''(x) - \frac{y^3}{6} f'''(c) \right] \alpha(y)\, dy.$$

Note that, by the antisymmetry of α, we have

$$\int_{B_\delta(0)} \alpha(y)\, dy = 0 = \int_{B_\delta(0)} y^2 \alpha(y)\, dy.$$

Therefore,

$$-(\alpha * f)(x) = f'(x) \int_{B_\delta(0)} y\alpha(y)\, dy + \frac{f'''(c)}{6} \int_{B_\delta(0)} y^3 \alpha(y)\, dy.$$

Using (16), we obtain

$$-(\alpha * f)(x) - f'(x) = \frac{f'''(c)}{6} \int_{B_\delta(0)} y^3 \alpha(y)\, dy.$$

Notice that

$$\left| \int_{B_\delta(0)} y^3 \alpha(y)\, dy \right| \leqslant \int_{B_\delta(0)} y^2 \left| y\alpha(y) \right| \, dy \leqslant \delta^2 \int_{B_\delta(0)} \left| y\alpha(y) \right| \, dy,$$

since $y \in B_\delta(0)$. Note that, because α is antisymmetric and $\alpha \geqslant 0$ for $x \geqslant 0$, we know that $\alpha \leqslant 0$ for $x \leqslant 0$, hence, $y\alpha = |y\alpha|$. This gives, from our normalization condition,

$$\left| \int_{B_\delta(0)} y^3 \alpha(y)\, dy \right| \leqslant \delta^2 \int_{B_\delta(0)} \left| y\alpha(y) \right| \, dy = \delta^2.$$

With the constant C chosen as

$$C := \frac{1}{6} \sup_{c \in B_\delta(x)} |f'''(c)|,$$

we have

$$\left| -(\alpha * f)(x) - f'(x) \right| \leqslant C\delta^2.$$

To establish the uniform convergence we take the supremum on all of $\mathbb{R}$. □

Remark 9 Note that in estimate (17) we may take $\sup_{c \in B_\delta(x)} |f'''(c)| = \max_{c \in B_\delta(x)} |f'''(c)|$ since $f \in C^3(\mathbb{R})$. Additionally, this bound may be chosen independently of δ, by taking the supremum over $B_1(x)$.

In order to prove convergence to the classical counterparts we will need to impose an additional assumption on the structure $\boldsymbol{\alpha}$.

Assumption 1 Suppose the kernel $\boldsymbol{\alpha}$ has the form

$$\boldsymbol{\alpha}(x_1, \ldots, x_n) = (\alpha_1(x_1), \ldots, \alpha_n(x_n)), \tag{19}$$

for $\alpha_i : \mathbb{R} \to \mathbb{R}$ for all $1 \le i \le n$.

Remark 10 While (19) restricts the form of the kernels, the condition is natural as it is reminiscent of the structure of the classical operator

$$\nabla = \left(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n} \right).$$

As we investigate convergence of the nonlocal gradient to its classical counterpart, it is sensible to enforce that each component is an operator acting on only its respective variable.

As we will see in the proof below, each component converges to the derivative in its variable, which proves that we may never take $\boldsymbol{\alpha}$ in full generality. To construct a simple counterexample, we permute the derivatives; for example, take $\boldsymbol{\alpha}(x_1, \dots, x_n) = (\alpha_1(x_2), \alpha_2(x_1), \alpha_3(x_3), \dots, \alpha_n(x_n))$. This will result in the nonlocal gradient converging to the local gradient with the first and second components swapped.

Remark 11 If we use Assumption 1 on $\boldsymbol{\alpha}$, then we have for some function $f : \mathbb{R}^n \to \mathbb{R}$ and $1 \leqslant i \leqslant n$ that

$$\begin{aligned}(\alpha_i * f)(\mathbf{x}) &= \int_{\mathbb{R}^n} \alpha_i(y_1 - x_1, \dots, y_i - x_i, \dots, y_n - x_n) f(\mathbf{y})\, d\mathbf{y} \\ &= \int_{\mathbb{R}^n} \alpha_i(y_1, y_2, \dots, y_i - x_i, \dots, y_n) f(\mathbf{y})\, d\mathbf{y},\end{aligned}$$

because α_i is only a function of y_i. We will employ the more compact notation

$$\alpha_i * f = \alpha_i *_i f,$$

where $*_i$ is used to denote a convolution in only the variable x_i.

Theorem 12 (Convergence of nonlocal gradient) *Suppose the vector kernel $\boldsymbol{\alpha}$ has the structure given in (19). Furthermore for $1 \leqslant i \leqslant n$, we impose the normalization condition:*

$$\int_{B_\delta(0)} y_i \alpha_i(y_i)\, dy_i = 1.$$

Then for $f \in C^3(\mathbb{R}^n)$ there exists a constant $C = C(x, f_{x_1x_1x_1}, \dots, f_{x_nx_nx_n})$ such that

$$|\mathcal{G}_{\boldsymbol{\alpha}}[f](\mathbf{x}) - \nabla f(\mathbf{x})| \leqslant C\delta^2.$$

Uniform convergence is obtained if all homogeneous third order derivatives of f are bounded.

Proof First we know that, for $1 \leqslant j \leqslant n$

$$|-\alpha_j *_j f(\mathbf{x}) - f_{x_j}(\mathbf{x})| \leqslant C_j\delta^2.$$

by Remark 11 and then Lemma 2. Now,

$$\begin{aligned}|\mathcal{G}_{\boldsymbol{\alpha}}[f](\mathbf{x}) - \nabla f(\mathbf{x})| &= \left|\langle -\alpha_1 * f(\mathbf{x}) - f_{x_1}(\mathbf{x}), \dots, -\alpha_n * f(\mathbf{x}) - f_{x_n}(\mathbf{x})\rangle\right| \\ &= \left|\langle -\alpha_1 *_1 f(\mathbf{x}) - f_{x_1}(\mathbf{x}), \dots, -\alpha_n *_n f(\mathbf{x}) - f_{x_n}(\mathbf{x})\rangle\right| \\ &\leqslant \delta^2\sqrt{C_1^2 + \cdots + C_n^2} \\ &= C\delta^2.\end{aligned}$$

Assuming that all homogeneous third order derivatives of f are bounded, the convergence becomes uniform. □

Similarly, we can prove the convergence of the nonlocal divergence and curl as given by:

Theorem 13 (Convergence of nonlocal divergence and curl) *Suppose $\boldsymbol{\alpha}$ has the structure (19). Furthermore suppose that for $1 \leqslant i \leqslant n$ we have*

$$\int_{B_\delta(0)} y_i \alpha_i(y_i)\, dy_i = 1.$$

Then for $\mathbf{f} \in C^3(\mathbb{R}^n)$ *and some constant* $C = C(x, \mathbf{f}_{x_1x_1x_1}, \ldots, \mathbf{f}_{x_nx_nx_n})$*, we have*

$$|\mathcal{D}_{\boldsymbol{\alpha}}[\mathbf{f}](\mathbf{x}) - \nabla \cdot \mathbf{f}(\mathbf{x})| \leqslant C\delta^2$$

and similarly when $n = 3$*,*

$$|\mathcal{C}_{\boldsymbol{\alpha}}[\mathbf{f}](\mathbf{x}) - \nabla \times \mathbf{f}(\mathbf{x})| \leqslant C\delta^2.$$

If all homogeneous third order derivatives of $\mathbf{f}$ *are bounded, then the convergence is uniform.*

Finally, we present the convergence of the nonlocal Laplacian, which is only linear with respect to δ.

Theorem 14 *Let the nonlocal kernel be given by*

$$\boldsymbol{\alpha}(\mathbf{x}) = \left(\frac{2-\beta_1}{2}\delta^{\beta_1-2}x_1^{-\beta_1}, \ldots, \frac{2-\beta_n}{2}\delta^{\beta_n-2}x_n^{-\beta_n}\right).$$

Then the corresponding nonlocal Laplacian converges uniformly to the classical Laplacian, assuming boundedness of the homogeneous third and fourth derivatives for f*. More precisely, there exists* $C > 0$ *such that*

$$\|\mathcal{L}_{\boldsymbol{\alpha}}[f] - \Delta f\|_\infty \leqslant C\delta.$$

Proof We first show this in the 1D case; take $\alpha(x) = \omega(\delta)/x^\beta$ for some appropriate constant β and function $\omega(\delta)$. First, to find ω, we seek to fulfill the normalization condition

$$\int_{B_\delta(0)} \frac{y\omega(\delta)}{y^\beta}\,dy = 1,$$

which yields

$$\omega(\delta) = \frac{2-\beta}{2\,\delta^{2-\beta}}.$$

From here, we use the triangle inequality to see

$$\|\alpha * \alpha * f - f''\|_\infty \leqslant \|\alpha * \alpha * f + \alpha * f'\|_\infty + \|-\alpha * f' - f''\|_\infty.$$

The latter term converges at a rate of δ^2 by Lemma 2 assuming $f^{(4)}$ is bounded. The first term, after applying Young's inequality for convolutions (4), becomes

$$\|\alpha * \alpha * f + \alpha * f'\|_\infty \leqslant \|\alpha\|_1 \|-\alpha * f - f'\|_\infty.$$

By the boundedness of f''' we see again from Lemma 2 that $\|-\alpha * f - f'\|_\infty$ converges at a rate of δ^2. However,

$$\|\alpha\|_1 = \frac{2-\beta}{(1-\beta)\delta}.$$

Combining these results we find for this choice of α,[1] that we achieve linear convergence for the second derivative in 1D assuming boundedness of the third and fourth derivatives of f, as given by the estimate

$$\|\alpha * \alpha * f - f''\|_\infty \leqslant C\delta^2\left(1 + \frac{2-\beta}{(1-\beta)\delta}\right) \leqslant C\delta.$$

From this 1D case we can use the triangle inequality to get the result. □

1 This type of integrable kernel is common in the literature associated with the theory of peridynamics, see for example [1, 18].

7 Wellposedness of the nonlocal Poisson problem on $\mathbb{R}^n$

Theorem 15 (Wellposedness) *The Poisson problem*

$$-\mathcal{L}_{\boldsymbol{\alpha}}[u] = f. \tag{20}$$

has a solution given by

$$u = \mathcal{F}^{-1}\left(\frac{\widehat{f}}{|\widehat{\boldsymbol{\alpha}}|^2}\right) = f * \mathcal{F}^{-1}\left(\frac{1}{|\widehat{\boldsymbol{\alpha}}|^2}\right) \tag{21}$$

whenever the inverse Fourier transforms exist. Additionally, if $\widehat{\boldsymbol{\alpha}} \neq 0$ then the solution is unique.

Proof Applying the Fourier transform to (20) yields

$$|\widehat{\boldsymbol{\alpha}}|^2 \widehat{u} = \widehat{f},$$

which yields either equality of (21). For uniqueness suppose that there are two functions u_1, u_2 that satisfy (20). Then if $w := u_1 - u_2$, we see that

$$-\mathcal{L}_{\boldsymbol{\alpha}}[w] = 0,$$

which, by Lemma 1 implies that $w \equiv 0$, which gives uniqueness. □

In the following we provide an example where wellposedness holds, by examining the interplay between the decay for f and $\boldsymbol{\alpha}$.

Example 1 Take $f(\mathbf{x}) = e^{-b|\mathbf{x}|^2}$ and $\boldsymbol{\alpha}(\mathbf{x}) = \left\langle e^{-a|\mathbf{x}|^2}, \ldots, e^{-a|\mathbf{x}|^2} \right\rangle$ in (20), where $\boldsymbol{\alpha}$ is a function in $\mathbb{R}^n$ and $a, b > 0$. Since, for $c > 0$

$$\mathcal{F}\left(e^{-c|\mathbf{x}|^2}\right) = \left(\frac{\pi}{c}\right)^{n/2} e^{-|\boldsymbol{\xi}|^2/4c},$$

we have,

$$|\widehat{\boldsymbol{\alpha}}|^2 = n\left(\frac{\pi}{a}\right)^n e^{-|\boldsymbol{\xi}|^2/2a} \qquad \text{and} \qquad \widehat{f} = \left(\frac{\pi}{b}\right)^{n/2} e^{-|\boldsymbol{\xi}|^2/4b}.$$

Then

$$\frac{\widehat{f}}{|\widehat{\boldsymbol{\alpha}}|^2} = C(n) e^{-|\boldsymbol{\xi}|^2\left(\frac{1}{4b} - \frac{1}{2a}\right)}.$$

For the above function to have an inverse Fourier transform, we need

$$2b < a.$$

The next example highlights the dependence of the admissible set of forcing terms on $\boldsymbol{\alpha}$. For simplicity, we will take $n = 1$.

Example 2 Suppose $\alpha(x) = e^{-|x|}$. Then

$$\mathcal{F}(\alpha)(\xi) = \frac{2}{1 + \xi^2}.$$

Applying (21) we get

$$u = f * \left(\frac{1}{4}\mathcal{F}^{-1}(1 + 2\xi^2 + \xi^4)\right) = \frac{1}{4} f * (\delta_0 + 2\partial^2\delta_0 + \partial^4\delta_0)$$

$$= \frac{1}{4}\left(f + 2f'' + f^{(4)}\right),$$

where we have used (2) and the fact that, in the distributional sense, $\mathcal{F}(\delta_0) = 1$. Therefore, to find a solution u we need to impose $f \in C^4(\mathbb{R})$. In this case the solution is unique, because $\widehat{\alpha} \neq 0$.

This example shows a remarkable fact, that a kernel like $e^{-|x|}$ which is a "nice" function (continuous, in $L^p(\mathbb{R})$ for all $1 \leqslant p \leqslant \infty$ and infinitely differentiable in the classical sense everywhere except at 0, with $e^{-|x|} \in W^{1,p}(\mathbb{R})$ for all $1 \leqslant p \leqslant \infty$) may yield unexpected results, namely, that in order to ensure well-posedness one must impose high regularity (here, C^4) for the forcing term.

Remark 12 Given the availability of the integration-by-parts result, we can also show uniqueness using energy methods, i.e. (20) admits at most one solution $u \in L^2$ if $\boldsymbol{\alpha}$ is antisymmetric and $\widehat{\boldsymbol{\alpha}} \neq 0$.

Indeed, assuming that two solutions $u_1, u_2 \in L^2$ exist, define $w := u_1 - u_2$. Hence, $-\mathcal{L}_{\boldsymbol{\alpha}}[w] = 0$. Taking the L^2 inner product with w yields

$$(-\mathcal{L}_{\boldsymbol{\alpha}}[w], w) = 0.$$

Applying Proposition 5 (integration by parts) gives

$$\|\mathcal{G}_{\boldsymbol{\alpha}}[w]\|_{L^2} = (\mathcal{G}_{\boldsymbol{\alpha}}[w], \mathcal{G}_{\boldsymbol{\alpha}}[w]) = 0.$$

Therefore, $\mathcal{G}_{\boldsymbol{\alpha}}[w] = 0$, which, by Proposition 8, implies that $w \equiv 0$, so solutions are unique.

8 Conclusions and new directions

One of the main contributions of the paper is the introduction of a nonlocal framework with kernel dependent operators, which enables a large degree of adjustability in physical applications. As it is the case with other integral operators, the assumptions on input functions are minimal (integrability is sufficient), while the output maintains the same regularity. This fact allows multiple applications of these operators[2] (nonlocal equivalents of higher order differentials) to be applied to functions that are even discontinuous. As a particular example, the domain of a nonlocal Laplacian could be as large as L^2, with systems well-posed over spaces of integrable functions with no prescribed regularity. While the same applies to a nonlocal biharmonic, we lose convergence to its classical counterpart, as each iteration of the nonlocal operator reduces the rate of convergence (no convergence is expected even for equivalents for third order nonlocal operators).

The framework introduced naturally extends the classical framework. In fact, this is the first nonlocal framework for which the kernels for nonlocal divergence and curl have been identified. This fact, together with the additional properties and results shown in the paper, provides a promising path for this new framework's employability in fluid dynamics, or other applications where incompressibility, for example, is a desired property.

An interesting fact shown in Example 2 shows that relatively smooth choices of kernels can give rise to unexpected behaviors. Further studies are needed to identify kernels that yield elliptic-type properties for the nonlocal Laplacian introduced here. Additional properties of these nonlocal operators that resemble the classical chain and product rule would be useful

[2] Note that, in contrast with other existing nonlocal theories, the fact that both input and output are one-point functions, does not raise any difficulties in applying the same operator multiple times to the same function.

in developing additional tools for this theory. Of additional importance would be the study of these nonlocal operators for nonhomogeneous kernels (i.e. α depends separately on x, for example, $\alpha(x, y) = \alpha_0(x)\mu(x - y)$).

An immediate direction of research which will be pursued elsewhere is the extension of the nonlocal framework to bounded domains. This is a nontrivial exercise, as the operators rely on convolutions with kernels on the entire space. One may consider restricting the support of the kernel (or the domain for the convolution) to bounded domains, however, the formulation of boundary conditions will have to be carefully considered in order to ensure well-posedness of solutions. Additionally, the coercivity of the doubly-nonlocal operator which defines the Laplacian will have to be established separately, as some of the most general existing results [24] will not cover this setting. At the same time, the formulation of nonlocal boundary value problems is of high interest in a variety of applications where convolutions appear naturally (such as diffusion, image processing).

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Declarations

Conflict of interest The authors declare that they have no conflict of interest nor competing interests.

References

1. Silling, S.: Reformulation of elasticity theory for discontinuities and long-range forces. J. Mech. Phys. Solids **48**, 175–209 (2000)
2. Silling, S.A.: Linearized theory of peridynamic states. J. Elast. **99**(1), 85–111 (2010)
3. Andreu-Vaillo, F., Mazón, J.M., Rossi, J.D., Toledo-Melero, J.J.: Nonlocal Diffusion Problems. Volume 165 of Mathematical Surveys and Monographs. American Mathematical Society, Providence, RI (2010)
4. Carrillo, C., Fife, P.: Spatial effects in discrete generation population models. J. Math. Biol. **50**(2), 161–188 (2005)
5. Gilboa, G., Osher, S.: Nonlocal operators with applications to image processing. Multiscale Model. Simul. **7**(3), 1005–1028 (2009)
6. Mirman, R.: Maxwell's equations and Helmholtz's theorem. Am. J. Phys. **33**(6), 503–504 (1965)
7. Chorin, A.J.: A numerical method for solving incompressible viscous flow problems. J. Comput. Phys. **135**(2), 118–125 (1997)
8. Palit, B., Basu, A., Mandal, M. K.: Applications of the discrete Hodge Helmholtz decomposition to image and video processing. In: International Conference on Pattern Recognition and Machine Intelligence, pp. 497–502. Springer (2005)
9. Bhatia, H., Norgard, G., Pascucci, V., Bremer, P.-T.: The Helmholtz-Hodge decomposition-a survey. IEEE Trans. Vis. Comput. Graphics **19**(8), 1386–1404 (2013)
10. Caffarelli, L.: Non-local diffusions, drifts and games. In: Holden, H., Karlsen, K. (eds.) Nonlinear Partial Differential Equations, pp. 37–52. Springer, Berlin (2012)
11. Carpinteri, A., Cornetti, P., Sapora, A.: A fractional calculus approach to nonlocal elasticity. Eur. Phys. J. Special Topics **193**(1), 193–204 (2011)
12. Du, Q., Gunzburger, M., Lehoucq, R.B., Zhou, K.: A nonlocal vector calculus, nonlocal volume-constrained problems, and nonlocal balance laws. Math. Models Methods Appl. Sci. **23**(03), 493–540 (2013)
13. D'Elia, M., Flores, C., Li, X., Radu, P., Yu, Y.: Helmholtz-Hodge decompositions in the nonlocal framework. J. Peridyn. Nonlocal Model. **2**, 401–418 (2020)

14. Laskin, N.: Nonlocal quantum mechanics: fractional calculus approach. In: Tarasov, V.E. (ed.) Applications in Physics, p. 207. De Gruyter, Berlin (2019)
15. Lee, H., Du, Q.: Nonlocal gradient operators with a nonspherical interaction neighborhood and their applications. ESAIM: Math. Model. Numer. Anal. **54**(1), 105–128 (2020)
16. Gunzburger, M., Lehoucq, R.B.: A nonlocal vector calculus with application to nonlocal boundary value problems. Multiscale Model. Simul. **8**, 1581–1598 (2010)
17. Shankar, R.: On a nonlocal extension of differentiation. J. Math. Anal. Appl. **440**(2), 516–528 (2016)
18. Foss, M.D., Radu, P.: Bridging local and nonlocal models: Convergence and regularity. In: Voyiadjis, G.Z. (ed.) Handbook of Nonlocal Continuum Mechanics for Materials and Structures. Springer International Publishing, Cham (2019)
19. Mengesha, T., Du, Q.: The bond-based peridynamic system with Dirichlet-type volume constraint. Proc. R. Soc. Edinb. Sect. A **144**(1), 161–186 (2014)
20. Radu, P., Toundykov, D., Trageser, J.: A nonlocal biharmonic operator and its connection with the classical analogue. Arch. Ration. Mech. Anal. **223**(2), 845–880 (2017)
21. Du, Q., Gunzburger, M., Lehoucq, R.B., Zhou, K.: Analysis and approximation of nonlocal diffusion problems with volume constraints. SIAM Rev. **54**(4), 667–696 (2012)
22. Radu, P., Wells, K.: A doubly nonlocal Laplace operator and its connection to the classical Laplacian. J. Integr. Equ. Appl. **31**(3), 379–409 (2019)
23. Bobaru, F., Duangpanya, M.: The peridynamic formulation for transient heat conduction. Int. J. Heat Mass Transfer **53**(19–20), 4047–4059 (2010)
24. Foss, M. D.: Nonlocal Poincaré inequalities for integral operators with integrable nonhomogeneous kernels, arXiv preprint arXiv:1911.10292 (2019)