Shape optimization of the first Steklov eigenvalue on 2-Ddomains with a hole

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Outline

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1. Introduction
2. Shape calculus

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3. Steklov case

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Steklov eigenvalue problem

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$$\begin{aligned} -\Delta u &= 0 && \text{in } \Omega \\ \partial_n u &= \sigma u && \text{on } \partial\Omega \end{aligned} \tag{1}$$

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is known as **Steklov eigenvalue problem**. This problem describes the vibration of a free membrane with its whole mass distributed on the boundary.

Variational characterization of eigenvalues

The Steklov eigenvalues have the following variational characterization

$$\sigma_n(\Omega) = \inf_{E_n} \sup_{0 \neq u \in E_n} \frac{\int_\Omega |\nabla u|^2 dx}{\int_{\partial\Omega} u^2 ds}, \quad n = 1, 2, \ldots \tag{2}$$

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where the infimum is taken over all $n-$dimensional subspaces E_n of the Sobolev space $H^1(\Omega)$ that are orthogonal to constants on $\partial\Omega$ and satisfy

$$\sigma_0 = 0 < \sigma_1 \leq \sigma_2 \ldots \rightarrow +\infty \tag{3}$$

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$$\sigma_1(\Omega) = \min \left\{ \frac{\int_\Omega |\nabla u|^2 dx}{\int_{\partial\Omega} u^2 ds} : u \in H^1(\Omega). \int_{\partial\Omega} u ds = 0 \right\} \tag{4}$$

Steklov eigenvalues of an annulus

We look at the *Steklov* eigenvalue problem, given by

$$\begin{aligned} -\Delta u &= 0, \quad in \quad \Lambda_\epsilon \\ \partial_n u &= \sigma u, \quad on \quad \partial\Lambda_\epsilon, \end{aligned} \tag{5}$$

where $\partial_n u$ is the outer normal derivative, $u \not\equiv 0$, and $\Lambda_\epsilon = \{x : \epsilon \leq |x| \leq 1\}$.

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The Steklov eigenvalues on this domain are given by

$$\sigma_p(\Lambda_\epsilon) = \frac{p}{2}\left(\frac{1+\epsilon}{\epsilon}\right)\left(\frac{1+\epsilon^{2p}}{1-\epsilon^{2p}}\right) \pm \frac{p}{2}\sqrt{\left(\frac{1+\epsilon}{\epsilon}\right)^2\left(\frac{1+\epsilon^{2p}}{1-\epsilon^{2p}}\right)^2 - \frac{4}{\epsilon}}. \tag{6}$$

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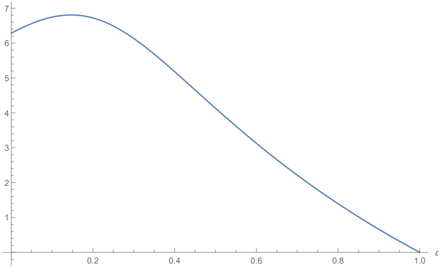
Question

How large can σ_1 be on a **non-simply** *connected bounded planar domain with two boundary components and given perimeter?*

Is there such an Ω that maximizes $\sigma_1(\Omega)|\Omega|$?

Conjecture (Girouard-Polterovich, [3])

When restricting to bounded connected planar domains with two boundary components, the expectation is that the best planar annulus is the one that realizes the max on the curve of the function $\sigma_1(\Lambda_\epsilon)|\partial\Lambda_\epsilon|$, where $\sigma_1(\Lambda_\epsilon)$ is given by equation (6)



Normalized eigenvalue $\sigma_1(\Omega_\epsilon)|\partial\Omega_\epsilon|$. The max is attained at the solution ϵ_0 of $\frac{d(\sigma_1(\Omega_\epsilon)|\partial\Omega_\epsilon|)}{d\epsilon} = 0$. Numerically we get $\epsilon_0 \approx 0.146721$

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Definition 1

Given a nonempty subset D of $\mathbb{R}^N$ and consider its power set $\mathcal{P}(D)$. A *shape functional* is a map

$$J : \mathcal{A} \to \mathbb{R}$$

from some *admissible family* $\mathcal{A}$ of sets in $\mathcal{P}(D)$ into $\mathbb{R}$. The set D will be referred to as the underlying *holdall*.

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We consider

$$\mathcal{A} = \{\Omega \subset \mathbb{R}^2 : \Omega = \mathbb{D} \setminus \Omega_\delta, \Omega_\delta \text{ and simply connected}\}$$

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For any vector field $V \in W^{3,\infty}(\Omega, \mathbb{R}^N)$, the Eulerian derivative of the domain functional $J(\Omega)$ at Ω in the direction of a vector field V is defined as the limit

$$dJ(\Omega; V) = \lim_{t\to 0} \frac{J(\Omega_t) - J(\Omega)}{t} \tag{7}$$

where $\Omega_t = T_t(V)(\Omega) = \{T_t(x) = x + tV(x) : t \in [0, \epsilon]\}$.

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From this definition it is possible to conclude that for a function u lying in a suitable function space, the functional

$$J(\Omega_t) = \int_{\partial\Omega} u(t,x)dS \tag{8}$$

has Eulerian derivative given by

$$dJ(\Omega; V) = \int_{\partial\Omega} u'(\Omega; V)dS + \int_{\partial\Omega} \left(\frac{\partial u}{\partial n} + Hu\right) V_n dS, \tag{9}$$

where $u' = \partial_t u(t,x)|_{t=0}$ is the shape derivative, V_n is the normal component of the vector field V at $t = 0$, $H = \Delta b$ is the mean curvature on $\partial\Omega$ and b is the oriented distance to Ω.

Critical Domain for $J(\Omega)$

Remark: The vector field V that we consider here, leaves fixed the outer boundary and satisfies

$$V_n = \frac{1}{|\partial\Omega|}\int_{\partial\Omega} V_n + \left(V_n - \frac{1}{|\partial\Omega|}\int_{\partial\Omega} V_n\right). \tag{10}$$

This allows us to consider deformations that preserve the length of the inner boundary once a radial deformation has been applied at a given "instant".

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Can we find a critical shape for the shape functional $J(\Omega) = \sigma_1(\Omega)|\partial\Omega|$?. i.e can we find an Ω_c such that $dJ(\Omega_c; V) = 0$?

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The answer to the second question is work in progress.

The Shape Derivative

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Theorem 3 (Dambrine-Kateb-Lamboley-2016,[1])

If σ has multiplicity m,then there exists m analytic branches $t \mapsto \sigma_i(t)$, $i = 1, 2, \ldots, m$ with $\sigma_i(0) = \sigma$,

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Theorem 4 (Dambrine-Kateb-Lamboley,[1])

Let σ be a multiple eigenvalue of order $m \geq 2$. Then each $t \mapsto \sigma_i(t)$ for $i \in [\![1, N]\!]$ has a derivative near 0, and the values of $(\sigma_i'(0))_{i\in[\![1,N]\!]}$ are the eigenvalues of the matrix $M(V_n) = (M_{jk})_{1\leq j,k\leq m}$ defined by

$$M_{jk} = \int_{\partial\Omega} V_n \left(\nabla_\tau u_j \cdot \nabla_\tau u_k - \partial_n u_j \partial_n u_k - \sigma H u_j u_k + \beta \left(HI_d - 2D^2 b\right) \nabla_\tau u_j \cdot \nabla_\tau u_k\right) d\sigma$$

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Proposition 2.1 (Product Rule)

Let σ be a multiple eigenvalue of order $m \geq 2$. Then each $t \mapsto \sigma_i(t)|\partial\Omega_t|$ for $i = 1 \ldots N$ has a derivative near 0, and the values of $(\sigma_i(t)|\partial\Omega_t|)'_{t=0}$, $i = 1 \ldots N$ satisfy

$$(\sigma(t)|\partial\Omega_t|)'_{t=0}\vec{d} = (|\partial\Omega|M + K(V)\sigma Id)\vec{d} \tag{11}$$

where M is the matrix defined by Theorem 4, $K(V) = \left.\frac{d|\partial\Omega_t|}{dt}\right|_{t=0}$, and $\vec{d}$ is the vector of coefficients in the decomposition of $u(0,x) = \sum_{i=1}^{m} d_i u_i$ where u_i are the corresponding basis for the eigenspace. Here $'$ denotes the shape derivative in the direction of V.

That is, the derivatives of $(\sigma_i(t)|\partial\Omega_t|)'_{t=0}$ are the eigenvalues of the matrix

$$|\partial\Omega|M + K(V)\sigma Id.$$

In other words, the product rule holds in the sense described by equation (11). That is

$$\begin{aligned}(\sigma(t)|\partial\Omega_t|)'_{t=0}\vec{d} &= |\partial\Omega|M\vec{d} + K(V)\sigma\vec{d}\\ &= |\partial\Omega|\sigma'(0)\vec{d} + K(V)\sigma\vec{d},\end{aligned}$$

where we have used $\sigma'(0) = (\sigma_i'(0))$ is the set of eigenvalues of M and where M is defined in Theorem 4.

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With this proposition at hand we proved the following result

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Proposition 3.1

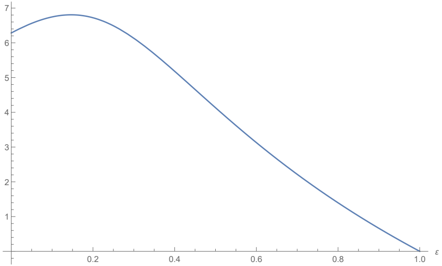
For each $\epsilon \in (0,1)$, $\Lambda_\epsilon = \mathbb{D} \setminus B(0,\epsilon)$ *provides a critical domain among the class of annular domains* $\Omega = \mathbb{D} \setminus \Omega_\delta$, *with* Ω_δ *simply connected subset of* $\mathbb{D}$ *and* $|\partial\Omega_\delta| = 2\pi\epsilon$.

Proposition 3.2

The annulus $\Lambda_{\epsilon_0} = \mathbb{D} \setminus B(0,\epsilon_0)$, *where* ϵ_0 *is the unique root in* $(0,1)$ *of the polynomial* $\Pi(\epsilon) = \epsilon^6 - 10\epsilon^5 + 23\epsilon^4 - 12\epsilon^3 + 23\epsilon^2 - 10\epsilon + 1$ *is a critical domain in* $\mathcal{A}$.

Our contribution

We proved that locally an annulus is a critical domain for $\sigma_1(\Omega)|\partial\Omega|$.This annulus is a candidate for a local maximizer.



Normalized eigenvalue $\sigma_1(\Lambda_\epsilon)|\partial\Lambda_\epsilon|$. The max is attained at the unique root in the interval $(0,1)$ of the polynomial equation

$$\epsilon^6 - 10\epsilon^5 + 23\epsilon^4 - 12\epsilon^3 + 23\epsilon^2 - 10\epsilon + 1 = 0$$

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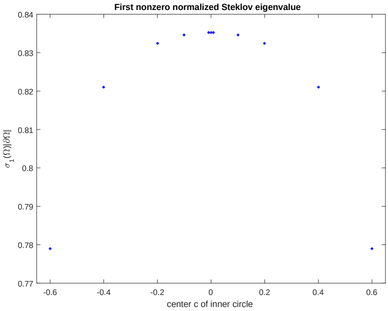
- Numerical experiments can be done using **FreeFem++** to study how the graph of an eigenvalue changes for different annuli.



Thanks!

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Shape Derivative of eigenfunctions

Shape Derivative of the eigenfunctions for the W-L eigenvalue problem.

Theorem 5 (Dambrine-Kateb-Lamboley-2016,[1])

Let Ω, V and T_t as in theorem (3), if $t \mapsto (\lambda(t), u_t)$ is one of the smooth eigenpair path $(\lambda_i(t), u_{i,t})$ of Ω_t for the Wentzell problem, then the shape derivative $u' = (\partial_t u_t)|_{=0}$ of the eigenfunction satisfies

$$\begin{aligned} \Delta u' &= 0 \quad in\ \Omega, \\ -\beta\Delta_\tau u' + \partial_n u' - \lambda u' &= \beta\Delta_\tau(V_n\partial_n u) - \beta div_\tau(V_n(2D^2 b - HId)\nabla_\tau u) \\ &\quad + div_\tau(V_n\nabla_\tau u) + \lambda'(0)u + \lambda V_n(\partial_n u + Hu) \quad on\ \partial\Omega \end{aligned} \tag{12}$$

Where b is the signed distance function and H is the mean curvature of $\partial\Omega$.