Sectional Rogers-Shephard Type Inequalities for Product Measures

Michael Roysdon

Kent State University

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What is being presented are bits of papers surrounding the Rogers-Shephard inequality. Some of the results are part of a joint work with David Alonso-Gutiérrez, María A. Hernández Cifre, Jesús Yepes Nicolás, and Artem Zvavitch.

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The collection $\mathcal{K}^n$ is equipped with a natural addition, called **Minkowski addition**. That is, given $K, L \in \mathcal{K}^n$, one has

$$K + L = \{x + y \colon x \in K, y \in L\} = \{x \in \mathbb{R}^n \colon K \cap (x - L) \neq \emptyset\}.$$

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with equality if, and only if, $L = \lambda K + t$, with $\lambda \in \mathbb{R}$ and $t \in \mathbb{R}^n$.

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- Here the body $K - K$ is known as the **difference body** of the convex body K.
- In the 1950s Rogers and Shephard proved a that one can bound the volume of $K - K$ from above in terms of the volume of K.

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The answer to the above question turns out to be false. One can construction examples where the left-hand side of the above inequality may be a fixed positive constant, whereas, the μ measure of the right-hand side may be made arbitrarily small. For example, let $d\mu(x) = e^{-|x|^2/2}dx$ and K be the closed Euclidean unit ball with center far from the origin.

We notice that an inequality like $\mu(K-K) \leq \binom{2n}{n}\mu(K)$ lacks the following characteristics in general.

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Definition

For any $K \in \mathcal{K}^n$, we define the **translated-average** of the measure μ by

$$\overline{\mu}(K) = \frac{1}{\mathrm{Vol}_n(K)} \int_K \mu(-y+K)dy.$$

Theorem (Alonso-Cifre-R.-Yepes-Zvavitch, IMRN 2019)

Let μ *be a measure on* $\mathbb{R}^n$ *given by* $d\mu(x) = \phi(x)dx$, *where* $\phi\colon \mathbb{R}^n \to [0,\infty)$ *is radially decreasing, i.e, for every* $x \in \mathbb{R}^n$ *and every* $t \in [0,1]$, *one has* $\phi(tx) \geq \phi(x)$. *Then, for any* $K \in \mathcal{K}^n$, *one has*

$$\mu(K-K) \leq \binom{2n}{n} \min\{\overline{\mu}(K), \overline{\mu}(-K)\},$$

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Moreover, if ϕ is continuous at the origin, then equality holds above if, and only if, μ is a positive multiple of the Lebesgue measure on $K-K$ and K is a simplex.

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- The assumption of radially decreasing made on the density of the measure μ may not be removed in general. We illustrate this with the following example.

Corollary (Alonso-Cifre-R.-Yepes-Zvavitch, IMRN 2019)

Let μ be a measure on $\mathbb{R}^n$ given by $d\mu(x) = \phi(x)dx$, where $\phi\colon \mathbb{R}^n \to [0,\infty)$ is radially decreasing, i.e, for every $x \in \mathbb{R}^n$ and every $t \in [0,1]$, one has $\phi(tx) \geq \phi(x)$. Then, for any $K \in \mathcal{K}^n$, one has

$$\mu(K-K) \leq \binom{2n}{n} \min\left\{\sup_{z\in K} \mu(-z+K), \sup_{z\in K} \mu(z-K)\right\}.$$

An example showing that the radially decreasing assumption on the density of μ may not be removed

Example

We will use the "packing" argument that the circle of radius 2 cannot be covered with 5 or less discs of radius 1; let $B_2 = \{x \in \mathbb{R}^n \colon |x| \leq 1\}$.

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We will use the "packing" argument that the circle of radius 2 cannot be covered with 5 or less discs of radius 1; let $B_2 = \{x \in \mathbb{R}^n \colon |x| \leq 1\}$. For fixed $\delta > \varepsilon > 0$, we consider the measure μ whose density function $\phi : \mathbb{R}^2 \longrightarrow [0, \infty)$ is given by $\phi(x) = 1$, if either $x \in \delta B_2$ or $x \in 2B_2 \setminus (2 - \varepsilon) B_2$, and $\phi(x) = 0$ otherwise (see Figure).

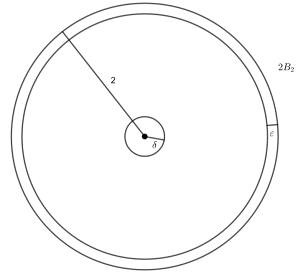

Figure: Constructing a measure for which the general Rogers-Shephard does not hold.

Example (continued)

Thus

$$\mu(B_2 - B_2) = \mu(2B_2) = \pi\delta^2 + \left(4 - (2-\varepsilon)^2\right)\pi = 4\varepsilon\pi + \pi(\delta^2 - \varepsilon^2).$$

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We note that we need at least 6 copies of the unit disk in order to cover $\mathrm{bd}(2B_2)$, which can be seen by considering a regular hexagon inscribed in $2B_2$ (see Figure).

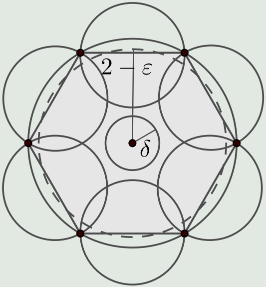

Figure: Packing the annulus of width ϵ with translated copies of B_2.

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Moreover, if we would cover $\mathrm{bd}(2B_2)$ with exactly 6 translated copies of B_2, then the covering discs would stay away from the origin.

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$$\sup_{x\in\mathbb{R}^2} \mathrm{Vol}_2\Big((x+B_2)\cap\big(2B_2\setminus(2-\varepsilon)B_2\big)\Big) = \frac{1}{6}4\pi\varepsilon + o(\varepsilon).$$

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Taking, e.g., $\delta = \sqrt{\varepsilon}/100$ we get, for ε small enough, that $\delta > \varepsilon$, and also that $4\pi\varepsilon/6 > \pi\delta^2$ and $o(\varepsilon) < \delta^2$.

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$$\begin{aligned} 6\sup_{x\in\mathbb{R}^2} \mu(x+B_2) &= 6\sup_{x\in\mathbb{R}^2} \mathrm{Vol}_2\Big((x+B_2)\cap\big(2B_2\setminus(2-\varepsilon)B_2\big)\Big) = 4\pi\varepsilon + o(\varepsilon) \\ &< 4\pi\varepsilon + \pi(\delta^2-\varepsilon^2), \end{aligned}$$

which contradicts the above theorem. This example shows that the radially decreasing assumption made on the density ϕ of μ is needed in the above theorem.

An important concept in the theory of convex bodies is the Brunn concavity principle: given a convex body K and an m-dimensional subspace H of $\mathbb{R}^n$, the function $f: H^{\perp} \to \mathbb{R}_+$ defined by

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In particular, if K is origin-symmetric, then the function f is even, and so its maximum occurs at 0; that is, Brunn's concavity principle guarantees that, for any origin-symmetric convex body K, its central section with respect to an m-dimensional subspace is of maximal m-dimensional volume.

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What occurs in the case when K is not necessarily origin-symmetric?

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Theorem (M. Rudelson, 1997)

Let $K \in \mathcal{K}^n$ and H be an m-dimensional linear subspace of $\mathbb{R}^n$. Then

$$Vol_m((K-K)\cap H) \leq [C\psi(n,m)]^m \sup_{y\in\mathbb{R}^n} Vol_m(K\cap(H+y)),$$

where $C>0$ is some absolute constant and

$$\psi(n,m) = \min\left\{\frac{n}{m}, \sqrt{m}\right\}.$$

R. Schneider introduced the following generalization of the difference body of a convex body: fix $p \in \mathbb{N}$. Given $K \in \mathcal{K}^n$, consider np-dimensional convex body, $D_p(K)$, defined by

$$D_p(K) := \{(x_1, \ldots, x_p) \in (\mathbb{R}^n)^p \colon K \cap (x_1 + K) \cap \cdots \cap (x_p + K) \neq \emptyset\}.$$

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Definition (G. Livshyts)

Let $Z \colon \mathbb{R}_+ \to \mathbb{R}_+$ be a strictly increasing differentiable function. We say that a Borel measure μ, defined on $\mathbb{R}^n$, is $Z(t)$-**concave** with respect to a class of Borel sets if, for all pairs of Borel sets A, B in $\mathbb{R}^n$ belonging to this class and every $\lambda \in [0,1]$, one has

$$Z(\mu((1-\lambda)A + \lambda B)) \geq (1-\lambda)Z(\mu(A)) + \lambda Z(\mu(B)).$$

Theorem (R., 2019+)

Let $Z\colon \mathbb{R}_+ \to \mathbb{R}_+$ *be a strictly increasing differentiable function. Suppose that* ψ *is a Borel measure on* $\mathbb{R}^n$ *that is* $Z(t)$*-concave with respect to a class of convex bodies that is invariant under translations. Fix* $p \in \mathbb{N}$ *and let* $\mu = \mu_1 \times \cdots \times \mu_p$ *be a product measure on* $(\mathbb{R}^n)^p$*, where, for each* $i = 1, \dots, p$*,* μ_i *is a measure on* $\mathbb{R}^n$ *having a radially decreasing density. For each* $i = 1, \dots, p$ *let* H_i *be an* m_i*-dimensional subspace of the i-th copy of* $\mathbb{R}^n$ *and consider the subspace* $H = H_1 \times \cdots \times H_p$ *of* $(\mathbb{R}^n)^p$*.*

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$$\tilde{C} \cdot \mu(D_p(K) \cap H) \leq \frac{1}{Z(\psi(K - g_K))} \int_{K - g_K} \prod_{i=1}^{p} \mu_i(((y + g_K) - K) \cap H_i) d\psi(y),$$

where $m = m_1 + \cdots + m_p$, g_K is the centroid of K with respect to the measure ψ, and

$$\tilde{C} := \int_0^1 (Z^{-1})'(tZ(\mu(K - g_k))(1-t)^m dt.$$

Theorem (R., 2019+)

Let $Z: \mathbb{R}_+ \to \mathbb{R}_+$ be a strictly increasing differentiable function. Suppose that ψ is a Borel measure on $\mathbb{R}^n$ that is $Z(t)$-concave with respect to a class of convex bodies that is invariant under translations. Fix $p \in \mathbb{N}$ and let $\mu = \mu_1 \times \cdots \times \mu_p$ be a product measure on $(\mathbb{R}^n)^p$, where, for each $i = 1, \dots, p$, μ_i is a measure on $\mathbb{R}^n$ having a radially decreasing density. For each $i = 1, \dots, p$ let H_i be an m_i-dimensional subspace of the i-th copy of $\mathbb{R}^n$ and consider the subspace $H = H_1 \times \cdots \times H_p$ of $(\mathbb{R}^n)^p$. Then, for all convex bodies K belonging to this class,

$$\tilde{C} \cdot \mu(D_p(K) \cap H) \leq \frac{1}{Z(\psi(K - g_K))} \int_{K - g_K} \prod_{i=1}^{p} \mu_i(((y + g_K) - K) \cap H_i) d\psi(y),$$

where $m = m_1 + \cdots + m_p$, g_K is the centroid of K with respect to the measure ψ, and

$$\tilde{C} := \int_0^1 (Z^{-1})'(tZ(\mu(K - g_k))(1-t)^m dt.$$

Note that choosing $Z(t) = t^{1/n}$, $p = 1$, $H = \mathbb{R}^n$, and ψ the Lebesgue measure, we recover the Rogers-Shephard type inequality due to A-C-R-Y-Z.

By choosing $Z(t) = t^{1/n}$, $p = 1$, letting H be any m-dimensional subspace of $\mathbb{R}^n$, letting ψ be the Lebesgue measure, and replacing K by $-K$ in the previous theorem, we obtain the following generalization of Rudelson's inequality when the minimum is taken to be $\frac{n}{m}$.

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Corollary (R., 2019+)

Let μ be a measure on $\mathbb{R}^n$ having a radially decreasing density and let $K \in \mathcal{K}^n$. Then, for any m-dimensional subspace H of $\mathbb{R}^n$,

$$\mu((K-K)\cap H) \leq \left[c\frac{n}{m}\right]^m \sup_{y\in\mathbb{R}^n} \mu(K\cap(H+y)),$$

where $c > 0$ is some absolute constant.

A functional Roger-Shephard type inequality

Definition

Let $p \in [-\infty, \infty]$ and $a, b, s, t, \geq 0$. Consider the averages for all $a, b \geq 0$,

$$M_p(a,b;s,t) = \begin{cases} [sa^p + tb^p]^{1/p} & \text{if } p \in (-\infty,\infty), p \neq 0, \\ a^s b^t & \text{if } p = 0, \\ \max\{a,b\} & \text{if } p = \infty, \\ \min\{a,b\} & \text{if } p = -\infty. \end{cases}$$

A function $f : \mathbb{R}^n \to [0,\infty)$ is said to be **p-concave** if, for every $x, y \in \mathbb{R}^n$ and all $\lambda \in [0,1]$, one has

$$f(\lambda x + (1-\lambda)y) \geq M_p(f(x), f(y); \lambda, 1-\lambda),$$

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Theorem (A. Colesanti, 2005)

Given an integrable p-concave function $f : \mathbb{R}^n \to [0, \infty)$, with $p \in [-\infty, 0)$, one has

$$\int_{\mathbb{R}^n} \Delta_p f(x) dx \leq \binom{2n}{n} \int_{\mathbb{R}^n} f(x) dx.$$

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Let μ be the measure on $\mathbb{R}^n$ having a radially decreasing density, and let $f : \mathbb{R}^n \to [0, \infty)$ be a μ-integrable $\frac{1}{s}$-concave function for some $s \in \mathbb{N}$.

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Let μ be the measure on $\mathbb{R}^n$ having a radially decreasing density, and let $f : \mathbb{R}^n \to [0,\infty)$ be a μ-integrable $\frac{1}{s}$-concave function for some $s \in \mathbb{N}$. Then, for any m-dimensional subspace H of $\mathbb{R}^n$

$$\int_H \Delta_{\frac{1}{s}} f(x) d\mu(x) \leq \left[c_1 \frac{n}{m}\right]^m \sup_{y \in \mathbb{R}^n} \int_{H+y} f(x) d\mu(x).$$

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If, instead, f is taken to be an integrable logarithmically-concave function assuming its maximum at the origin, and μ is taken to be the Lebesgue measure, then, for any m-dimensional subspace H of $\mathbb{R}^n$,

$$\int_H \Delta_0 f(x)dx \le \left[c_2 \min\left\{\frac{n}{m}, \sqrt{m}\right\}\right]^m \sup_{y\in\mathbb{R}^n} \left\{ \frac{\|f\|_\infty^2}{\sup_{x_0\in H+y} f(x_0)} \int_{H+y} f(x)dx \right\}.$$

Here $c_1, c_2 > 0$ are some absolute constants.

Thanks

Thank you to the organizers and everyone for listening! :)