Non-local minimal surfaces

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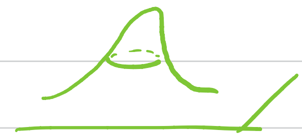
Motivation :

① <u>Motion of sets</u> Bence - Merriam - Osher (BMO scheme)

$E \subset \mathbb{R}^n$ smooth bdd set



$$\chi_E = \begin{cases} 1, & x \in E \\ 0, & x \notin E \end{cases} \qquad \varphi(x) = g(|x|), \quad \int \varphi(x)\,dx = 1$$

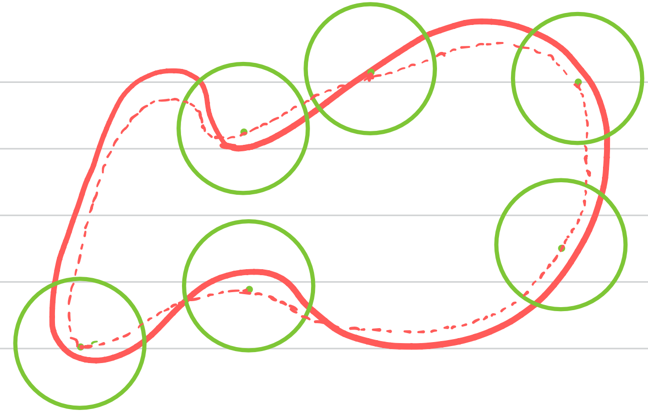
$E_0 = E$, E_k obtained from E_{k-1} as

$$E_k = \{ u > \tfrac{1}{2} \}$$

$$u := \chi_{E_{k-1}} * \varphi_\varepsilon, \qquad \varphi_\varepsilon(x) = \frac{1}{\varepsilon^n}\varphi\left(\frac{x}{\varepsilon}\right)$$

Ex: $\varphi(x) = \dfrac{\chi_{B_1}}{|B_1|}$

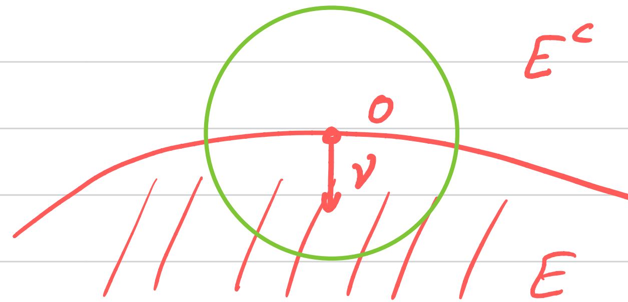
Continuous evolution of sets as $\varepsilon \to 0$.



Notation: $E^c = \mathbb{R}^n - E$

$$e(r) = \frac{\int_{\partial B_r} \chi_{E^c} - \chi_E \, d\sigma}{r^{n-1}}$$

excess function



$e(r) = c_n H r + o(r^2)$ as $r \to 0$,

$|e(r)| \leq C$, H mean curvature at 0.

$$u(0) - \frac{1}{2} = -\frac{1}{2} \int_0^{\infty} e(r)\, r^{n-1} \frac{1}{\varepsilon^n} g\left(\frac{r}{\varepsilon}\right) dr$$

$$= -\frac{1}{2} \int_0^{\infty} e(\varepsilon r)\, g(r)\, r^{n-1}\, dr$$

If $\int \varphi(x) |x|\, dx < \infty \iff \int_0^{\infty} g(r)\, r^n\, dr < \infty$.

then

$$u(0) - \frac{1}{2} = -\varepsilon \left(C' H + o(1) \right)$$

$o(1) \to 0$ as $\varepsilon \to 0$.

If $\quad g(r) = r^{-n-s}$, $\quad s \in (0,1)$ $\quad$ for large r,

$$u(0) - \frac{1}{2} = -\varepsilon^{s}\Big(\underbrace{\int_0^\infty \frac{e(r)}{r^{1+s}}\,dr}_{H_s(0)} + o(1)\Big)$$

Borderline case $\quad s = 1$

$$u(0) - \frac{1}{2} = -\varepsilon\,|\log \varepsilon|\,\big(c\,H + o(1)\big)$$

$$u_\gamma(0) = \chi_E * \partial_\gamma \varphi_\varepsilon(0) = \varepsilon^{-1}(c'' + o(1)), \quad c'' > 0.$$

$$\|D^2 u\| \leq C\,\varepsilon^{-2}$$

$0 \in \partial E$ moves in the ν direction

$c_0\,\varepsilon^2 (H + o(1))$ if $\int \varphi(x)\,|x|\,dx < \infty$

$c_0\,\varepsilon^{1+s}(H_s + o(1))$ if $\varphi(x) = |x|^{-n-s}$ for $|x|$ large

$c_0\,\varepsilon \log \varepsilon\,(H + o(1))$ if $\varphi(x) = |x|^{-n-1}$ for $|x|$ large

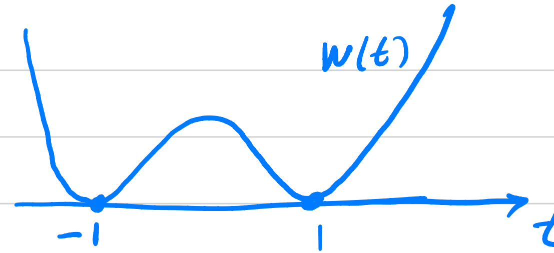
$$H_s(0) = \int_0^\infty \frac{e(r)}{r^{1+s}} \, dr$$

$$= \text{p.v.} \int_{\mathbb{R}^n} \frac{\chi_{E^c} - \chi_E}{|x|^{n+s}} \, dx$$

s - non-local mean curvature

$$H_s(0) = \Delta^{s/2} \left(\chi_{E^c} - \chi_E \right)(0).$$

② Phase - transitions



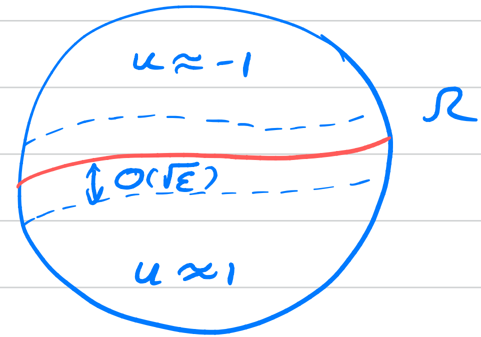
$u : \Omega \to \mathbb{R}$ density

$W : \mathbb{R} \to \mathbb{R}^+$ double-well potential

<u>Allen - Cahn energy model</u>

$$J(u, \Omega) = \int_\Omega \varepsilon |\nabla u|^2 + W(u) \, dx$$

$\varepsilon |\nabla u|^2$ - penalizes changes at small scales.



Theorem (Modica - Mortola)

As $\varepsilon \to 0$, minimizers u_ε converge on subsequences

$$u_\varepsilon \longrightarrow \chi_E - \chi_{E^c} \quad \text{in } L^1_{loc}(\Omega)$$

with E a set of minimal perimeter in Ω.

(∂E is a minimal surface)

Theorem (S. - Valdinoci)

Minimizers of

$$\varepsilon \int \frac{|u(x)-u(y)|^2}{|x-y|^{n+s}}\,dx\,dy + \int W(u)\,dx$$

converge on subsequences as $\varepsilon \to 0$ to

$$u_\varepsilon \longrightarrow \chi_E - \chi_{E^c} \quad \text{in } L^1_{loc}(\Omega)$$

with E a set of minimal s-perimeter.

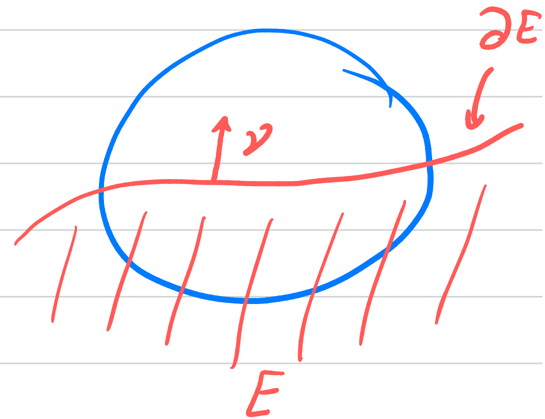
(minimizes $[\chi_E - \chi_{E^c}]_{H^{s/2}(\Omega)}$).

② Classical minimal surfaces (see Giusti [G])

<u>Definition</u>: Perimeter of E in Ω:

$$P_\Omega(E) = [\chi_E]_{BV(\Omega)} = \int_\Omega |\nabla \chi_E|$$

$$= \sup_{\substack{|g|\le 1 \\ g \in C_0^\infty(\Omega)}} \int_\Omega \chi_E \operatorname{div} g \, dx$$



If $\partial E \in C^1$ then

$$\sup \int_{\partial E} g \cdot \nu \, d\sigma = \mathcal{H}^{n-1}(\partial E \cap \Omega)$$

key steps in the theory:

① <u>Lower semicontinuity</u>

$$E_k \to E \text{ in } L^1_{loc}(\Omega) \Rightarrow \liminf P_\Omega(E_k) \geq P_\Omega(E)$$

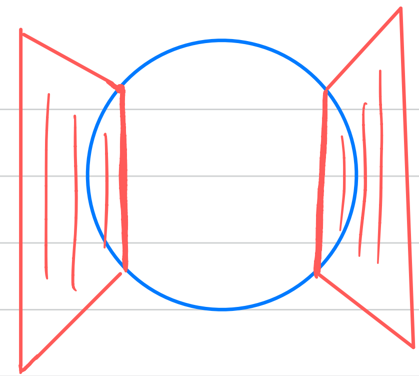
② <u>Compactness</u>

$$P_\Omega(E_k) \leq M \Rightarrow E_{k_m} \to E \text{ in } L^1_{loc}(\Omega).$$

③ <u>Existence</u>

There exists E that minimizes $P_{\mathbb{R}^n}(E)$ among all sets that are fixed outside Ω.

No uniqness !

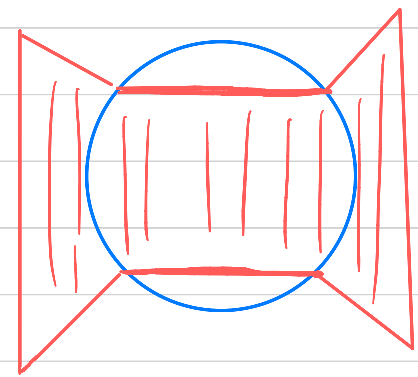


Definition : E minimizes perimeter in Ω if

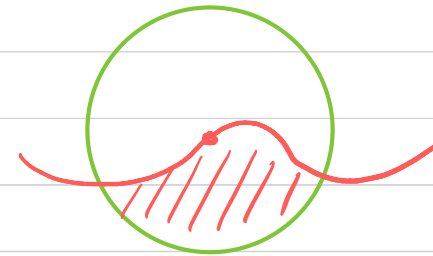
$$P_\Omega(E) \leq P_\Omega(F)$$

if $E \Delta F \subset\subset \Omega$.

(OR ∂E minimal surface in Ω)



4 Density estimates



∂E minimal in Ω, $0 \in \partial E$, then

$$1 - c_0 \geq \frac{|E \cap B_r|}{|B_r|} \geq c_0 \quad , \quad \forall\, B_r \subset \Omega$$

$c_0 > 0$ depends only on n.

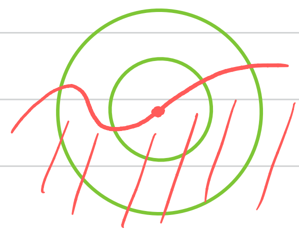
5 Compactness of minimizers

∂E_k minimal in $\Omega \implies E_{k_m} \to E$ in $L^1_{loc}(\Omega)$

∂E is minimal in Ω.

⑥ Monotonicity formula

∂E minimal, $0 \in \partial E \Rightarrow \dfrac{P_{B_r}(E)}{r^{n-1}} \nearrow$ in r

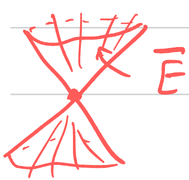


⑦ Blow-ups

∂E minimal, $0 \in \partial E \Rightarrow \lambda_k E \to \overline{E}$ in $L^1_{loc}(\mathbb{R}^n)$

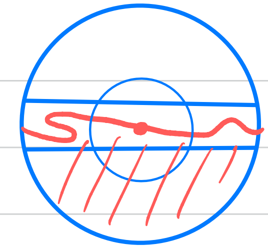
for some $\lambda_k \to \infty$.

Moreover, $\overline{E}$ is a minimal cone (homog. of deg 0)



$\overline{E}$ – blow-up cone at 0

8 Flatness implies regularity



∂E minimal in B_1 and

$$\{x_n \le -\varepsilon_0\} \subset E \subset \{x_n \le \varepsilon_0\}, \quad \varepsilon_0 \text{ small.}$$

Then ∂E is smooth in $B_{1/2}$.

E is a half-space $\Rightarrow$ E is smooth near 0.

9 Rigidity of cones

Simons theorem: Minimal cones are half-spaces if $n \le 7$.

Bombieri - De Giorgi - Giusti: $\{x_1^2 + \dots + x_4^2 < x_5^2 + \dots + x_8^2\}$ is minimal in $\mathbb{R}^8$.

(10) Dimension reduction

∂E minimal $\Longrightarrow$ ∂E is smooth except on a closed singular set of dim. $n-8$.

(11) Minimal graphs

Let Ω be mean convex, $\varphi : \partial\Omega \to \mathbb{R}$ continuous.

$\exists!$ E minimizes perimeter in $\Omega \times \mathbb{R}$ with bry data given by subgraph of φ.

$\partial E = \{ (x, u(x)), x \in \Omega \}$ $\quad u \in C^\infty(\Omega) \cap C^0(\bar{\Omega})$.

③ The fractional s-perimeter and nonlocal minimal sets.

<u>Definition</u> : $s \in (0,1)$, $E \subset \mathbb{R}^n$ measurable

The s-perimeter of E in Ω is

$$P_{s,\Omega}(E) = [\chi_E]_{H^{s/2}(\Omega)} = \frac{1}{2} \int_{\mathbb{R}^n \times \mathbb{R}^n \setminus (\Omega^c \times \Omega^c)} \frac{|\chi_E(x) - \chi_E(y)|^2}{|x-y|^{n+s}} \, dx\, dy$$

i.e.

$$P_{s,\Omega}(E) = \int_{\mathbb{R}^n \times \mathbb{R}^n \setminus (\Omega^c \times \Omega^c)} \frac{\chi_E(x)\, \chi_{E^c}(y)}{|x-y|^{n+s}} \, dx\, dy$$

Notation :

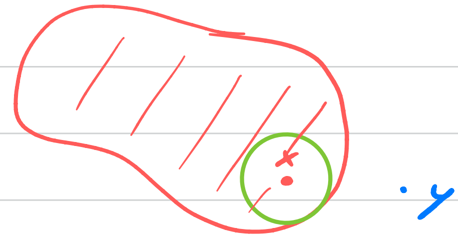
$$L_s(A,B) = \int_{A\times B} \frac{1}{|x-y|^{n+s}}\,dx\,dy$$

a) $L(A,B) = L(B,A)$

b) $L(A_1 \cup A_2, B) = L(A_1,B) + L(A_2,B)$ if $A_1 \cap A_2 = \emptyset$

c) $L(\lambda A, \lambda B) = \lambda^{n-s} L(A,B)$

d) E smooth bounded set.

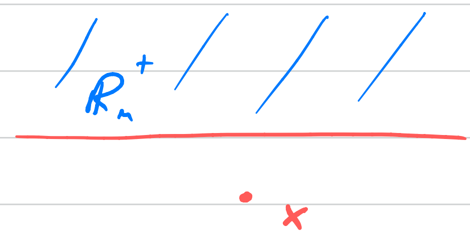


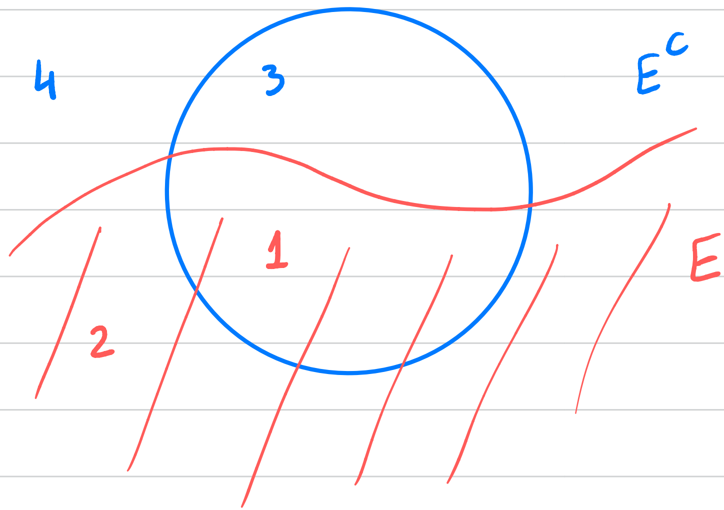
$$L(E, E^c) < \infty$$

$$\int_{B_r^c(x)} \frac{1}{|x-y|^{n+s}}\, dy \le C\, r^{-s}$$

$$\lim_{s \to 1^-} (1-s)\, L_s(E, E^c) = c_n\, P_{\mathbb{R}^n}(E)$$

$$\int_{\mathbb{R}^+_n} \frac{1}{|x-y|^{n+s}}\, dy \to c_n\, |x_n|^{-s}$$





$$P_{S,\Omega}(E) = L(\underbrace{E \cap \Omega}_{1}, \underbrace{E^c}_{3 \cup 4}) + L(\underbrace{E \cap \Omega^c}_{2}, \underbrace{E^c \cap \Omega}_{3})$$

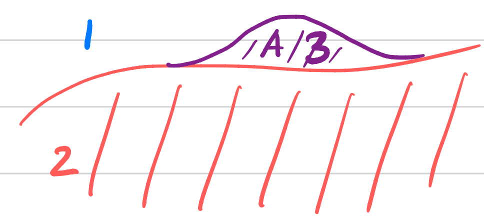
Definition: E is a <u>s-nonlocal minimal set</u> in Ω if

$$P_{s,\Omega}(E) \le P_{s,\Omega}(F) \quad \text{if} \quad E \cap \Omega^c = F \cap \Omega^c$$

(or ∂E is a s-nonlocal min. surface)

1) variational supersolution property: $\quad A \subset E^c \cap \Omega$

$$L(A, E) - L(A, E^c - A) \le 0$$



Exercise: E s-minimal in Ω $\iff$ satisfies sub/supersolution property

Heuristically, if $A = \delta_{x_0}$, $x_0 \in \partial E$ then

$$H_s(x_0) = \text{p.v.} \int_{\mathbb{R}^n} \frac{\chi_E - \chi_{E^c}}{|x-x_0|^{n+s}} \, dx = 0$$

(Lecture 2)

① Proposition (Lower semicontinuity)

$$E_k \to E \text{ in } L^1_{loc}(\mathbb{R}^n) \implies \liminf P_{s,\Omega}(E_k) \geq P_{s,\Omega}(E)$$

Proof :

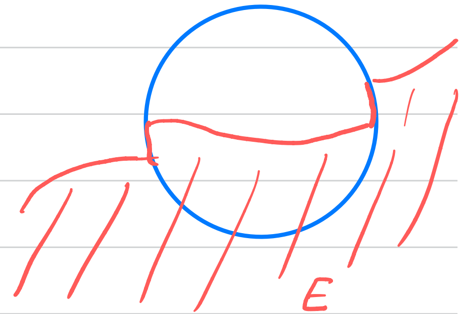
$$A_k \to A \ , \ B_k \to B \text{ in } L^1_{loc}$$

$$\chi_{A_k}(x)\, \chi_{B_k}(y) \to \chi_A(x)\, \chi_B(y) \quad \text{for a.e } (x,y)$$

$$\liminf L(A_k, B_k) \geq L(A, B) \quad \text{(Fatou's lemma)}$$

② Proposition (Existence)

Given $E_0 \subset \Omega^c$, $\exists$ a minimizer E to



$$\min_{E \cap \Omega^c = E_0} P_{s,\Omega}(E) .$$

Proof $P_{s,\Omega}(E_0) < \infty$.

Let E_k be a sequence with $P_{s,\Omega}(E_k) \to \inf P_{s,\Omega}$.

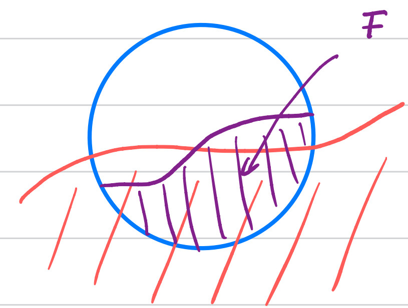
$\chi_{E_k} \in H^{s/2}_{loc}(\Omega) \implies E_k \to E$ in $L^2(\Omega)$

$\implies E_k \to E$ in $L^1(\Omega) \implies E$ is a minimizer.

③ Proposition (Compactness of minimizers)

E_k nonlocal min. sets in Ω, $E_k \to E$ in $L^1_{loc}(\mathbb{R}^n)$

Then E is a nonlocal min. set in Ω.



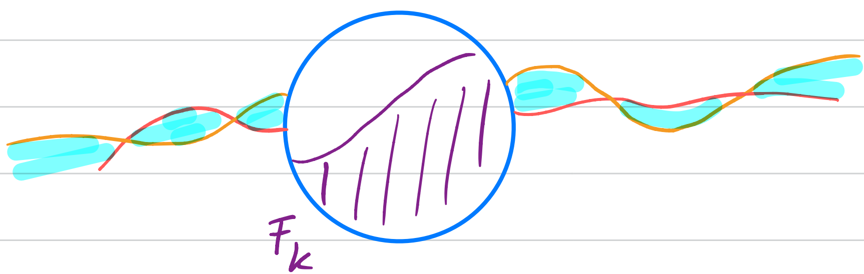
Proof $\quad F_k = \begin{cases} F & \text{in } \Omega \\ E_k & \text{in } \Omega^c \end{cases}$

$$\begin{array}{ccc} P_s(F_k) & \geq & P_s(E_k) \\ \Big\downarrow \text{as } k\to\infty & & \text{VI} \quad \text{l.s.c.} \\ P_s(F) & \geq & P_s(E) \end{array}$$

$$\left| P_s(F_k) - P_s(F) \right| \leq L\left(\Omega, (E_k \Delta E) \cap \Omega^c\right)$$

$E_k \Delta E \to 0$ in L^1_{loc} $\qquad \downarrow$ as $k \to \infty$

$L(\Omega, \Omega^c) < \infty \implies 0$



④ **Proposition** (Density estimates)

$$\left.\begin{array}{l} E \quad s\text{-minimal in } \Omega \\ 0 \in \partial E \end{array}\right\} \Rightarrow \quad |E \cap B_r| \geq c\,|B_r|$$

with $c(s,n) > 0$.

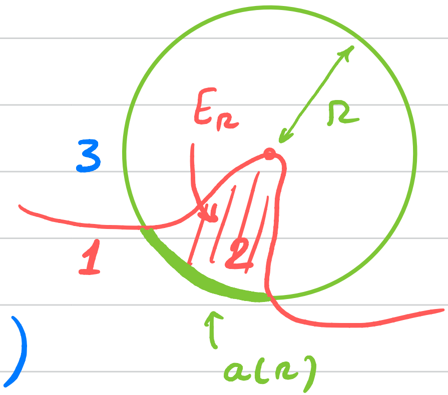
Remark: $x_0 \in \partial E$ means $|B_r(x_0) \cap E| > 0$, $|B_r(x_0) \cap E^c| > 0$ $\forall\, r > 0$.

Proof Assume $|E \cap B_1| \leq c$ small.

W.T.S $|E \cap B_{1/2}| = 0$.

Let $E_r = E \cap B_r$, $v(r) = |E_r|$

$$a(r) = \mathcal{H}^{n-1}(E \cap \partial B_r)$$



$$L(E_r, E^c) \le L(E_r, E - E_r)$$

$$L(E_r, E_r^c) \le 2L(E_r, E - E_r) \le 2L(E_r, B_r^c)$$

Sobolev ineq. $L(E_r, E_r^c) \ge c\,|E_r|^{1-\frac{s}{n}}$

$$v(r)^{1-\frac{s}{n}} \le C \int_0^r a(\rho)\,(r-\rho)^{-s}\,d\rho$$

$$v(r)^{1-\frac{s}{m}} \leq C \int_0^r a(\rho)\,(r-\rho)^{-s}\,d\rho$$

Integrate $r \in [0,t]$:

$$\int_0^t v(r)^{1-\frac{s}{m}}\,dr \leq C \int_0^t a(\rho)\,(t-\rho)^{1-s}\,d\rho$$

$$\leq C\, t^{1-s}\, v(t)$$

Set $t_k = \frac{1}{2} + 2^{-k}$, $v_k = v(t_k)$

$$(t_k - t_{k+1})\, v_{k+1}^{1-\frac{s}{m}} \leq C\, v_k$$

$v_k \to 0$ if v_0 sufficiently small.