Non-local minimal surfaces

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3 The fractional s-perimeter and nonlocal minimal sets.

<u>Definition</u>: $s \in (0,1)$, $E \subset \mathbb{R}^n$ measurable

The s-perimeter of E in Ω is

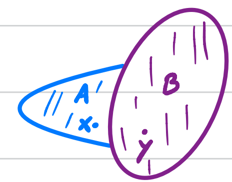
$$P_{s,\Omega}(E) = [\chi_E]_{H^{s/2}(\Omega)} = \frac{1}{2}\int_{\mathbb{R}^n \times \mathbb{R}^n \setminus (\Omega^c \times \Omega^c)} \frac{|\chi_E(x) - \chi_E(y)|^2}{|x-y|^{n+s}}\,dx\,dy$$

i.e.

$$P_{s,\Omega}(E) = \int_{\mathbb{R}^n \times \mathbb{R}^n \setminus (\Omega^c \times \Omega^c)} \frac{\chi_E(x)\,\chi_{E^c}(y)}{|x-y|^{n+s}}\,dx\,dy$$

Notation :

$$L_s(A,B) = \int_{A\times B} \frac{1}{|x-y|^{n+s}}\,dx\,dy$$

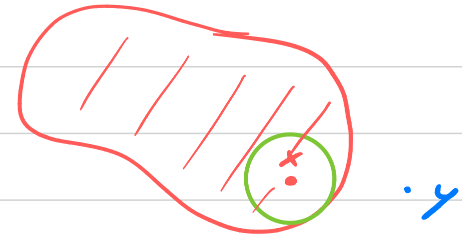


a) $L(A,B) = L(B,A)$

b) $L(A_1 \cup A_2, B) = L(A_1,B) + L(A_2,B)$ if $A_1 \cap A_2 = \emptyset$

c) $L(\lambda A, \lambda B) = \lambda^{n-s} L(A,B)$

d) E smooth bounded set.

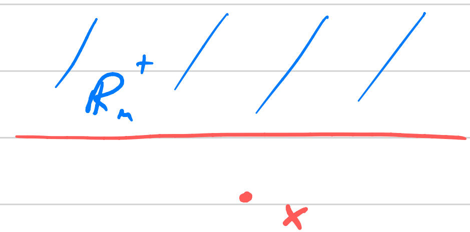


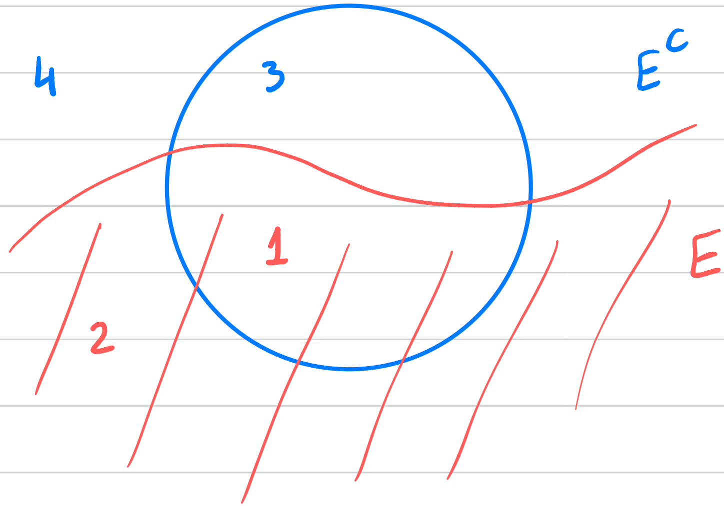
$$L(E, E^c) < \infty$$

$$\int_{B_r^c(x)} \frac{1}{|x-y|^{n+s}} \, dy \le C \, r^{-s}$$

$$\lim_{s \to 1^-} (1-s) \, L_s(E, E^c) = c_n \, P_{\mathbb{R}^n}(E)$$

$$\int_{\mathbb{R}^+_n} \frac{1}{|x-y|^{n+s}} \, dy \to c_n \, |x_n|^{-s}$$





$$P_{s,\Omega}(E) = L(\underbrace{E \cap \Omega}_{1}, \underbrace{E^c}_{3 \cup 4}) + L(\underbrace{E \cap \Omega^c}_{2}, \underbrace{E^c \cap \Omega}_{3})$$

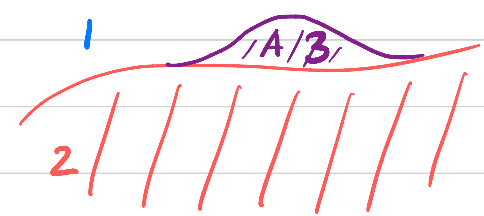
Definition: E is a <u>s-nonlocal minimal set</u> in Ω if

$$P_{s,\Omega}(E) \leq P_{s,\Omega}(F) \quad \text{if} \quad E \cap \Omega^c = F \cap \Omega^c$$

(or ∂E is a s-nonlocal min. surface)

1) variational supersolution property: $\quad A \subset E^c \cap \Omega$

$$L(A, E) - L(A, E^c - A) \leq 0$$

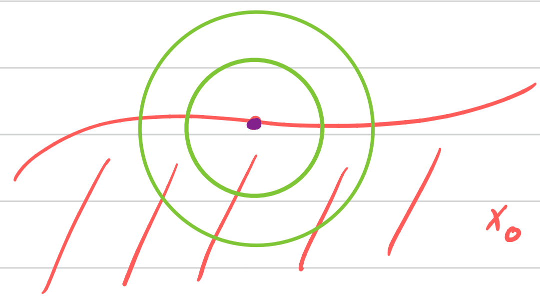


Exercise: E s-minimal in Ω $\iff$ satisfies sub/supersolution property

Heuristically, if $A = \delta_{x_0}$, $x_0 \in \partial E$ then

$$H_s(x_0) = p.v. \int_{\mathbb{R}^n} \frac{\chi_E - \chi_{E^c}}{|x-x_0|^{n+s}}\, dx = 0$$

(later)



① Proposition (Lower semicontinuity)

$$E_k \to E \text{ in } L^1_{loc}(\mathbb{R}^n) \implies \liminf P_{s,\Omega}(E_k) \geq P_{s,\Omega}(E)$$

Proof :

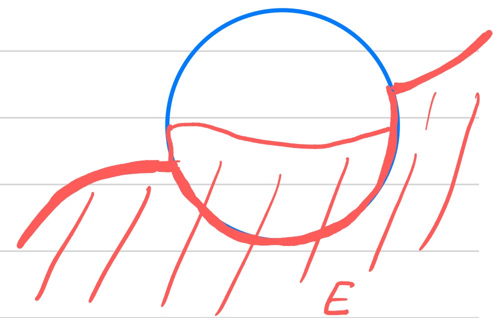
$$A_k \to A \; , \; B_k \to B \text{ in } L^1_{loc}$$

$$\chi_{A_k}(x)\, \chi_{B_k}(y) \to \chi_A(x)\, \chi_B(y) \quad \text{for a.e } (x,y)$$

$$\liminf L(A_k, B_k) \geq L(A, B) \quad \text{(Fatou's lemma)}$$

② Proposition (Existence)

Given $E_0 \subset \Omega^c$, $\exists$ a minimizer E to



$$\min_{E \cap \Omega^c = E_0} P_{s,\Omega}(E).$$

Proof $P_{s,\Omega}(E_0) < \infty$.

Let E_k be a sequence with $P_{s,\Omega}(E_k) \to \inf P_{s,\Omega}$.

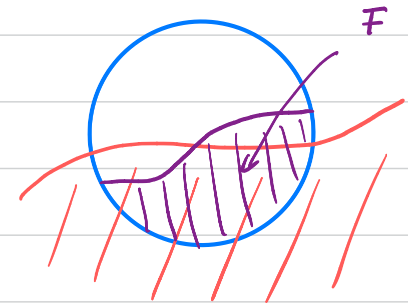
$\chi_{E_k} \in H^{s/2}_{loc}(\Omega) \implies E_k \to E$ in $L^2(\Omega)$

$\implies E_k \to E$ in $L^1(\Omega) \implies E$ is a minimizer.

③ Proposition (Compactness of minimizers)

E_k nonlocal min. sets in Ω, $E_k \to E$ in $L^1_{loc}(\mathbb{R}^n)$

Then E is a nonlocal min. set in Ω.



Proof

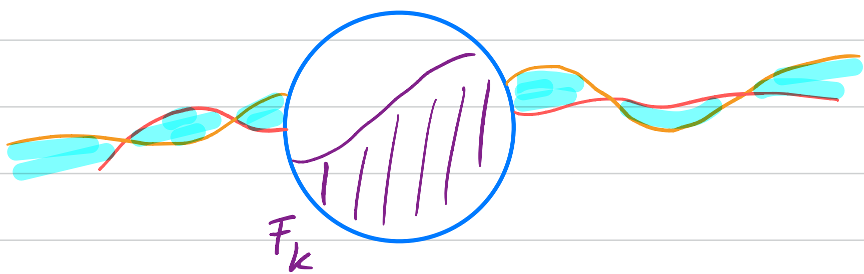
$$F_k = \begin{cases} F & \text{in } \Omega \\ E_k & \text{in } \Omega^c \end{cases}$$

$$\begin{array}{ccc} P_s(F_k) & \geq & P_s(E_k) \\ \Big\downarrow \text{ as } k\to\infty & & \text{\rotatebox{90}{$\geq$} l.s.c.} \\ P_s(F) & \geq & P_s(E) \end{array}$$

$$\left| P_s(F_k) - P_s(F) \right| \leq L(\Omega, (E_k \Delta E) \cap \Omega^c)$$

$E_k \Delta E \to 0$ in L^1_{loc} $\Longrightarrow$ $\downarrow$ as $k \to \infty$

$L(\Omega, \Omega^c) < \infty$ $\qquad 0$



④ **Proposition** (Density estimates)

$$\left.\begin{array}{l} E \quad s\text{-minimal in } \Omega \\ 0 \in \partial E \end{array}\right\} \Rightarrow \quad |E \cap B_r| \geq c\,|B_r|$$

with $c(s,n) > 0$.

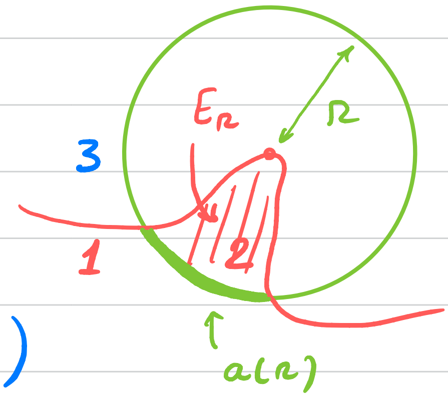
Remark: $x_0 \in \partial E$ means $|B_r(x_0) \cap E| > 0$, $|B_r(x_0) \cap E^c| > 0$ $\forall\, r > 0$.

Proof Assume $|E \cap B_1| \leq c$ small.

W.T.S $|E \cap B_{1/2}| = 0$.

Let $E_r = E \cap B_r$, $v(r) = |E_r|$

$$a(r) = \mathcal{H}^{n-1}(E \cap \partial B_r)$$



$$L(E_r, E^c) \le L(E_r, E - E_r)$$

$$L(E_r, E_r^c) \le 2L(E_r, E - E_r) \le 2L(E_r, B_r^c)$$

Sobolev ineq. $\quad L(E_r, E_r^c) \ge c|E_r|^{1-\frac{s}{n}}$

$$v(r)^{1-\frac{s}{n}} \le C \int_0^r a(\rho)(r-\rho)^{-s}\, d\rho$$

$$v(r)^{1-\frac{s}{m}} \leq C \int_0^r a(\rho)\,(r-\rho)^{-s}\,d\rho$$

Integrate $r \in [0,t]$:

$$\int_0^t v(r)^{1-\frac{s}{m}}\,dr \leq C \int_0^t a(\rho)\,(t-\rho)^{1-s}\,d\rho$$

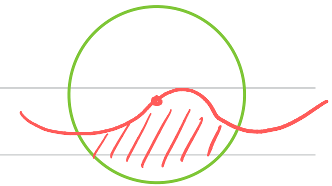
$$\leq C\, t^{1-s}\, v(t)$$

Set $t_k = \frac{1}{2} + 2^{-k}$, $v_k = v(t_k)$

$$(t_k - t_{k+1})\; v_{k+1}^{1-\frac{s}{m}} \leq C\, v_k$$

$v_k \to 0$ if v_0 sufficiently small.

④ Density estimates



$$\left.\begin{array}{l} E \quad s\text{-minimal in } \Omega \\ 0 \in \partial E \end{array}\right\} \Rightarrow |E \cap B_r| \geq c\,|B_r|$$

with $c(s,n) > 0$.

Consequences: $x_0 \in \partial E$,

a) $L\left(E \cap B_r(x_0),\, E^c \cap B_r(x_0)\right) \geq c\, r^{n-s}$.

b) $\mathcal{H}^{n-s}(\partial E) = 0$.

c) $E \cap B_r(x_0)$, $E^c \cap B_r(x_0)$ contain balls of radius cr.

d) $E_k \to E$ in L^1_{loc} $\Longrightarrow$ $\partial E_k \to \partial E$ uniformly

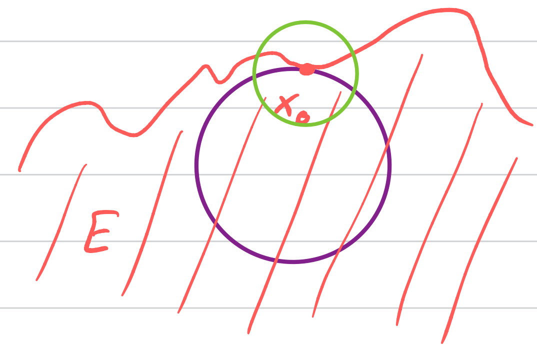
⑤ Proposition (E-L equation)

E variational supersolution
E interior tangent ball at $x_0 \in \partial E$ $\Longrightarrow$ $H_s(x_0) = \int_{\mathbb{R}^n} \frac{\chi_E - \chi_{E^c}}{|x-x_0|^{n+s}} \, dx \leq 0$

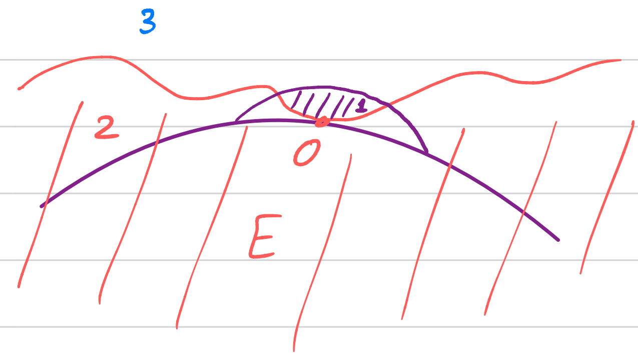
Remark: $H_s(x_0) = \int_0^\infty \frac{e(r)}{r^{1+s}} \, dr$

$$e(r) = \frac{\int_{\partial B_r(x_0)} \chi_E - \chi_{E^c} \, d\sigma}{r^{n-1}}$$

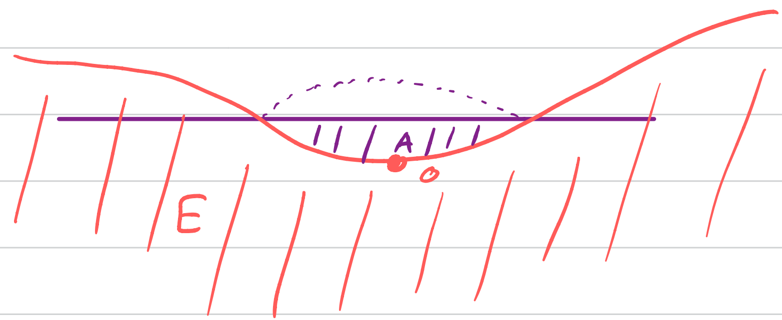
$|e(r)| \leq C, \qquad e(r) \geq -Cr$



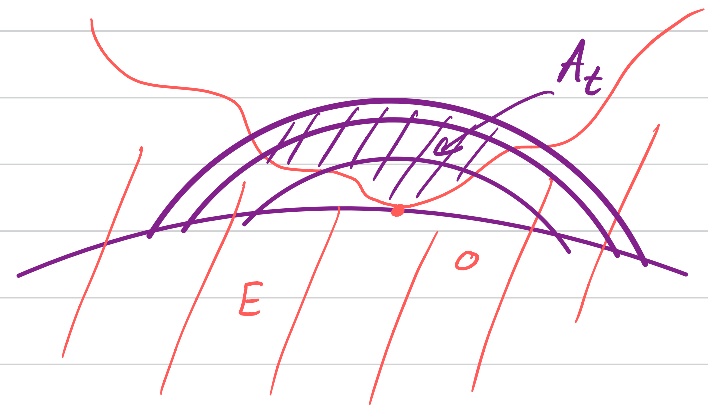
$$L(\underbrace{A}_{1}, \underbrace{E}_{2}) - L(\underbrace{A}_{1}, \underbrace{E^c - A}_{3}) \leq 0$$



Case $\mathbb{R}^n_- \subset E$



Proof (Calibration argument)



Assume $B_1(-e_n) \subset E$.

$$A_t = E^c \cap B_{\frac{1}{2}+t}(-\tfrac{1}{2}e_n)$$

$$A_t' = E^c \cap \partial B_{\frac{1}{2}+t}(-\tfrac{1}{2}e_n)$$

Claim:

$$L(A_t, E) - L(A_t, E^c - A_t) = \int_{A_t} H_s(x)\, dx$$

with $H_s(x)$ the s-curvature of $A_\rho \cup E$, $x \in A_\rho'$.

If $H_s(0) > 0$ then $H_s(x) > 0$ for x near 0.

We prove the claim for truncated kernels $K_\varepsilon = \frac{1}{|x|^{n+s}} \chi_{B_\varepsilon^c}$.

$$L_k(A_t, E) - L_k(A_t, E^c - A_t) = \int_{A_t} H_{s,k}(x)\, dx$$

and get the result by letting $\varepsilon \to 0$.

Differentiate in t:

$$\int_{\mathbb{R}^n} \left(\int_{A_t'} L_k(x-y)\, d\sigma_x \right) \left(\chi_E(y) - \chi_{E^c - A_t}(y) \right) dy$$

$$+ \int_{\mathbb{R}^n} \left(\int_{A_t'} L_k(x-y)\, d\sigma_y \right) dx \quad =$$

$$= \int_{A_t'} \int_{\mathbb{R}^n} L_k(x-y) \left(\chi_{A_t \cup E}(y) - \chi_{E^c - A_t}(y) \right) dy \, d\sigma_x$$

$$= \int_{A_t'} H_{s,k}(x) \, d\sigma_x$$

$$= \frac{d}{dt} \int_{A_t} H_{s,k} \, dx$$

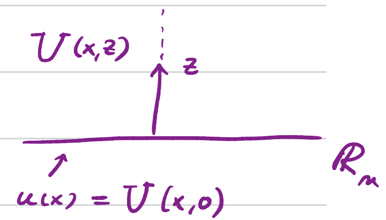
4 The extension problem and monotonicity formula

Caffarelli - Silvestre extension:

$x \in \mathbb{R}^n$ $\qquad$ $X = (x,z) \in \mathbb{R}^{n+1}_+ \,, \quad z \geq 0.$

$$u \in H^{s/2}(\mathbb{R}^n) \rightsquigarrow U \in H^1(\mathbb{R}^{n+1}_+ \,, z^{1-s}\,dx)$$

$$\begin{cases} \operatorname{div}(z^{1-s} \nabla U) = 0 \\ U(x,0) = u(x) \end{cases}$$



$u \in C_0^\infty(\mathbb{R}^n)$

$$\Delta^{s/2} u(x) = c_{n,s} \lim_{z \to 0} z^{1-s} U_z$$

$$[u]_{H^{s/2}} = C_{n,s} \int_{\mathbb{R}^{n+1}_+} |\nabla U|^2 \, z^{1-s} \, dx$$

$u \in C_0^\infty$

$$[u]_{H^{s/2}} = \frac{1}{2} \int_{\mathbb{R}^n \times \mathbb{R}^n} \frac{(u(x) - u(y))^2}{|x-y|^{n+s}} \, dx \, dy$$

$$= - \int_{\mathbb{R}^n} u(x) \int_{\mathbb{R}^n} \frac{u(y) - u(x)}{|y-x|^{n+s}} dy \quad dx$$

$$= - \int_{\mathbb{R}^n} u(x) \quad \Delta^{s/2} u(x) \quad dx$$

$$= - C_{n,s} \lim_{z \to 0} \int_{\mathbb{R}^n} U(x,z) \; z^{1-s} \, U_z(x,z) \, dx$$

$$= C_{n,s} \int_{\mathbb{R}^{n+1}_+} |\nabla U|^2 \, z^{1-s} \, dX$$

The extension problem makes sense for

$$u \in H^{s/2}_{loc}(\mathbb{R}^n) \quad \text{and}$$

$$\int_{\mathbb{R}^n} \frac{|u(x)|}{(1+|x|)^{n+s}} \, dx < \infty$$

① <u>Proposition</u> (Extension problem)

E nonlocal min. set in Ω, and U the extension of

$$u = \chi_E - \chi_{E^c}.$$

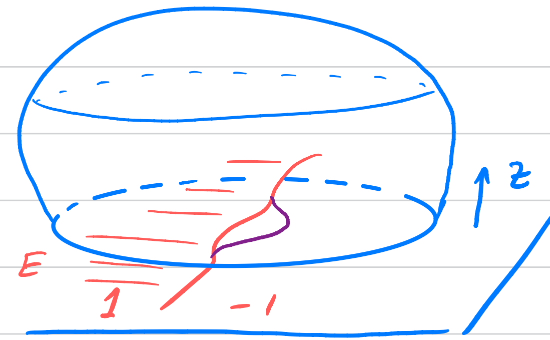
Then U minimizes locally the energy

$$\int_{\mathbb{R}^{n+1}_+} |\nabla U|^2 \, z^{1-s} \, dX$$

among all compact perturbations V with trace

$$v = \chi_F - \chi_{F^c} \qquad \text{and} \qquad F \Delta E \subset \Omega.$$

Proof



Obvious if E has compact support and then $\chi_E \in H^{s/2}$.

Let W have trace $v-u$ on $z=0$ and support in B_M.

Let U_R, V_R be the extensions of $\varphi_R u$, $\varphi_R v$,
φ_R cut off function.

$$J(U_R + W) \geq J(V_R)$$

$$\mathcal{J}(u_R + w) - \mathcal{J}(u_R) \geq \mathcal{J}(v_R) - \mathcal{J}(u_R)$$

$$\|$$

$$\mathcal{J}(u_R + w, B_{\Omega}) - \mathcal{J}(u_R, B_{\Omega}) = c\left([\varphi_R v]_{H^{s/2}} - [\varphi_R u]_{H^{s/2}}\right)$$

$$= c\left([\varphi_R v]_{H^{s/2}(\Omega)} - [\varphi_R u]_{H^{s/2}(\Omega)}\right)$$

We let $R \to \infty$

$$\mathcal{J}(u + w, B_{\Omega}) - \mathcal{J}(u, B_{\Omega}) \geq 0.$$

Properties of the extension: $\quad 0 \in \partial E$, E minimal in B_1

1) $|U| \le 1$, $\quad U$ unif. Lipschitz on compact sets of $\mathbb{R}^{n+1}_+$.

2) $c\, r^{n-s} \le J(U, B_r^+) \le C_1\, r^{n-s} \quad \forall\, r \le 1.$

3) $\| U(x,z) - u(x) \|^2_{L^2(B_1)} \le C_1\, z^s$

$$\left(g(z) - g(0)\right)^2 \le C\, z^s \int_0^z (g')^2\, t^{1-s}\, dt$$

Theorem (Monotonicity formula)

E nonlocal min. set in Ω, U the extension of $u = \chi_E - \chi_{E^c}$.

Then
$$\Phi_U(r) = r^{s-n} \int_{B_r^+} |\nabla U|^2 z^{1-s} \, dx$$

is monotone increasing in r as long as $B_r \subset \Omega$.

$\Phi_U(r)$ const. $\iff$ U 0-homogeneous

Proof.

$$\frac{d}{dr}\Phi_U(r)\Big|_{r=1} = \int_{\partial B_1^+} |\nabla U|^2 z^{1-s}\, d\sigma - (n-s)\int_{B_1^+} |\nabla U|^2 z^{1-s}\, dx$$

Let $\tilde{U}$ be the 0-homog extension of U from ∂B_1^+ to B_1^+.

$$\int_{\partial B_1^+} |\nabla U|^2 z^{1-s}\, d\sigma \geq \int_{\partial B_1^+} |\nabla \tilde{U}|^2 z^{1-s}\, d\sigma$$

$$\int_{B_1^+} |\nabla U|^2 z^{1-s}\, dx \leq \int_{B_1^+} |\nabla \tilde{U}|^2 z^{1-s}\, dx$$

$$\frac{d}{dr}\Phi_U(r) \geq \int_{\partial B_1^+} |\nabla \tilde{U}|^2 z^{1-s}\, d\sigma - (n-s)\int_{B_1^+} |\nabla \tilde{U}|^2 z^{1-s}\, dX$$

$$\parallel$$

$$\frac{d}{dr}\Phi_{\tilde{U}}(r)\Big|_{r=1} = 0$$

If equality holds then $|\nabla U| = |\nabla \tilde{U}|$

$\Rightarrow |\partial_r U| = 0 \Rightarrow U$ is constant in the radial direction.

Corollary (Blow-ups)

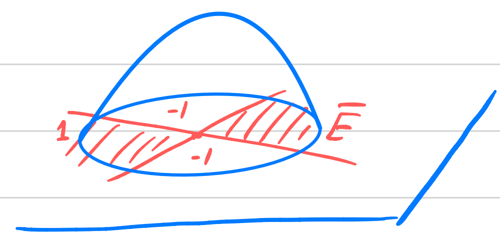
E nonlocal minimal set near $0 \in \partial E$. Then $\exists\, \lambda_k \to 0$

$$\lambda_k^{-1} E \to \bar{E} \quad \text{in } L^1_{loc}(\mathbb{R}^n)$$

$$U(\lambda_k x) \to \bar{U}(x) \quad \text{in } L^1_{loc}(\mathbb{R}^{n+1}_+)$$

with $\bar{E}$, $\bar{U}$ 0-homog. and $\bar{U}$ is the extension of $\bar{E}$.

$\bar{E}$ - blow-up cone of E at 0



Proof: $U_k = U(\lambda_k x)$ the extension of $E_k = \lambda_k^{-1} E$.

Then $U_k \to \overline{U}$ uniformly on compact sets.

Recall $\| U(x,z) - u(x) \|_{L^2(B_1)} \leq C_1 z^s$

which gives $u_k \to \overline{u}$ in $L^2_{loc}(\mathbb{R}^n)$.

Claim: $J(U_k, B_1^+) \to J(\overline{U}, B_1^+)$

Follows by density estimates

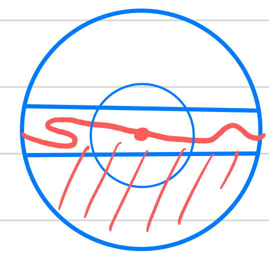
and $c r^{n-s} \leq J(\overline{U}, B_r) \leq C_1 r^{n-s}$.

⑤ Improvement of flatness

Theorem Let E be a nonlocal minimal set in B_1.

If $\{x_n \le -\varepsilon_0\} \subset E \subset \{x_n \le \varepsilon_0\}$ in B_1

with $\varepsilon_0(s,n)$ small, then $\partial E \cap B_{1/2}$ is a $C^{1,\alpha}$ graph.



Corollary: If $\bar{E}$ is a half-space then ∂E is a $C^{1,\alpha}$ graph near 0.

Proposition

Fix $\alpha \in (0,s)$ and ∂E a viscosity solution of the E-L equation, $0 \in \partial E$. If

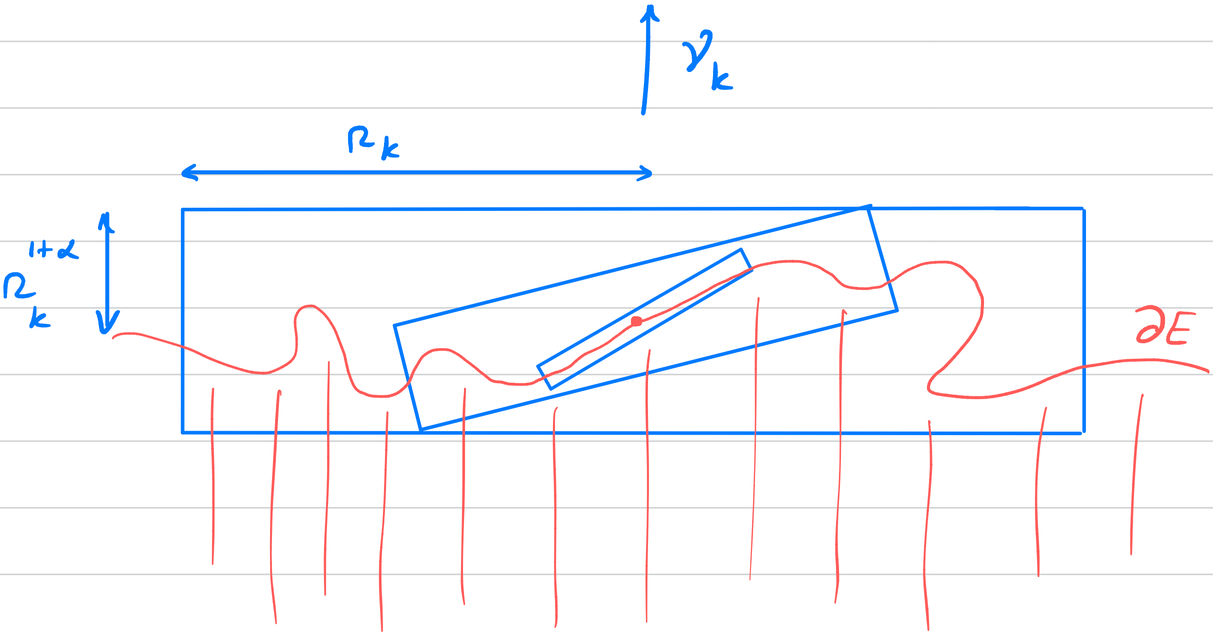
$$\{x \cdot \nu \leq -r^{1+\alpha}\} \subset E \subset \{x \cdot \nu \leq r^{1+\alpha}\} \quad \text{in } B_r$$

holds for $r = r_k = 2^{-k}$, $\nu = \nu_k$

$k = 0, 1, 2, \ldots, k_0$ $\quad k_0(s, \alpha, n)$ large

then it holds for all k.

$k = 0, 1, 2, \dots, k_0$



$$|\nu_k - \nu_{k+1}| \le C_n r_k^{\alpha} \implies \nu_k \to \bar{\nu}$$