Non-local minimal surfaces

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The fractional s-perimeter

Definition: $s \in (0,1)$, $E \subset \mathbb{R}^n$ measurable

$$P_{s,\Omega}(E) = \int_{\mathbb{R}^n \times \mathbb{R}^n \setminus (\Omega^c \times \Omega^c)} \frac{\chi_E(x)\,\chi_{E^c}(y)}{|x-y|^{n+s}}\,dx\,dy$$

Definition: E is a s-nonlocal minimal set in Ω if

$$P_{s,\Omega}(E) \le P_{s,\Omega}(F) \quad \text{if} \quad E \cap \Omega^c = F \cap \Omega^c$$

(or ∂E is a s-nonlocal min. surface)

1. Lower semicontinuity
2. Existence
3. Compactness of minimizers
4. Density estimates
5. Euler-Lagrange equation $\quad H_{s,E} = 0$ if $\partial E \in C^2$.
6. Extension problem $\quad u = \chi_E - \chi_{E^c} \rightsquigarrow U(X)$.

$$[u]_{H^{s/2}(\mathbb{R})} \longrightarrow \int_{\mathbb{R}^{n+1}_+} |\nabla U|^2 z^{1-s} \, dX$$

Theorem (Monotonicity formula)

E nonlocal min. set in Ω, U the extension of $u = \chi_E - \chi_{E^c}$.

Then

$$\Phi_U(r) = r^{s-n} \int_{B_r^+} |\nabla U|^2 z^{1-s} \, dx$$

is monotone increasing in r as long as $B_r \subset \Omega$.

$\Phi_U(r)$ const. $\iff$ U 0-homogeneous

Proof.

$$\frac{d}{dr}\Phi_U(r)\Big|_{r=1} = \int_{\partial B_1^+} |\nabla U|^2 z^{1-s}\, d\sigma - (n-s)\int_{B_1^+} |\nabla U|^2 z^{1-s}\, dx$$

Let $\tilde{U}$ be the 0-homog extension of U from ∂B_1^+ to B_1^+.

$$\int_{\partial B_1^+} |\nabla U|^2 z^{1-s}\, d\sigma \geq \int_{\partial B_1^+} |\nabla \tilde{U}|^2 z^{1-s}\, d\sigma$$

$$\int_{B_1^+} |\nabla U|^2 z^{1-s}\, dx \leq \int_{B_1^+} |\nabla \tilde{U}|^2 z^{1-s}\, dx$$

$$\frac{d}{dr}\Phi_U(r) \geq \int_{\partial B_1^+} |\nabla \tilde{U}|^2 z^{1-s}\, d\sigma - (n-s)\int_{B_1^+} |\nabla \tilde{U}|^2 z^{1-s}\, dX$$

$$\parallel$$

$$\frac{d}{dr}\Phi_{\tilde{U}}(r)\Big|_{r=1} = 0$$

If equality holds then $|\nabla U| = |\nabla \tilde{U}|$

$\Rightarrow \; |\partial_r U| = 0 \;\Rightarrow\;$ U is constant in the radial direction.

Corollary (Blow-ups)

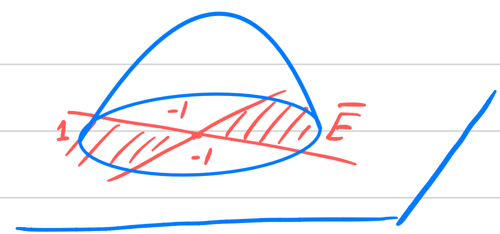
E nonlocal minimal set near $0 \in \partial E$. Then $\exists\, \lambda_k \to 0$

$$\lambda_k^{-1} E \to \bar{E} \quad \text{in } L^1_{loc}(\mathbb{R}^n)$$

$$U(\lambda_k x) \to \bar{U}(x) \quad \text{in } L^1_{loc}(\mathbb{R}^{n+1}_+)$$

with $\bar{E}$, $\bar{U}$ 0-homog. and $\bar{U}$ is the extension of $\bar{E}$.

$\bar{E}$ - blow-up cone of E at 0

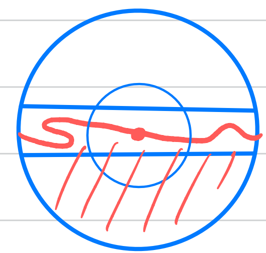


⑤ Improvement of flatness

Theorem Let E be a nonlocal minimal set in B_1.

If $\{x_n \le -\varepsilon_0\} \subset E \subset \{x_n \le \varepsilon_0\}$ in B_1

with $\varepsilon_0(s,n)$ small, then $\partial E \cap B_{1/2}$ is a $C^{1,\alpha}$ graph.



Corollary: If $\bar{E}$ is a half-space then

∂E is a $C^{1,\alpha}$ graph near 0.

Proposition

Fix $\alpha \in (0,s)$ and ∂E a viscosity solution of the E-L equation, $0 \in \partial E$. If

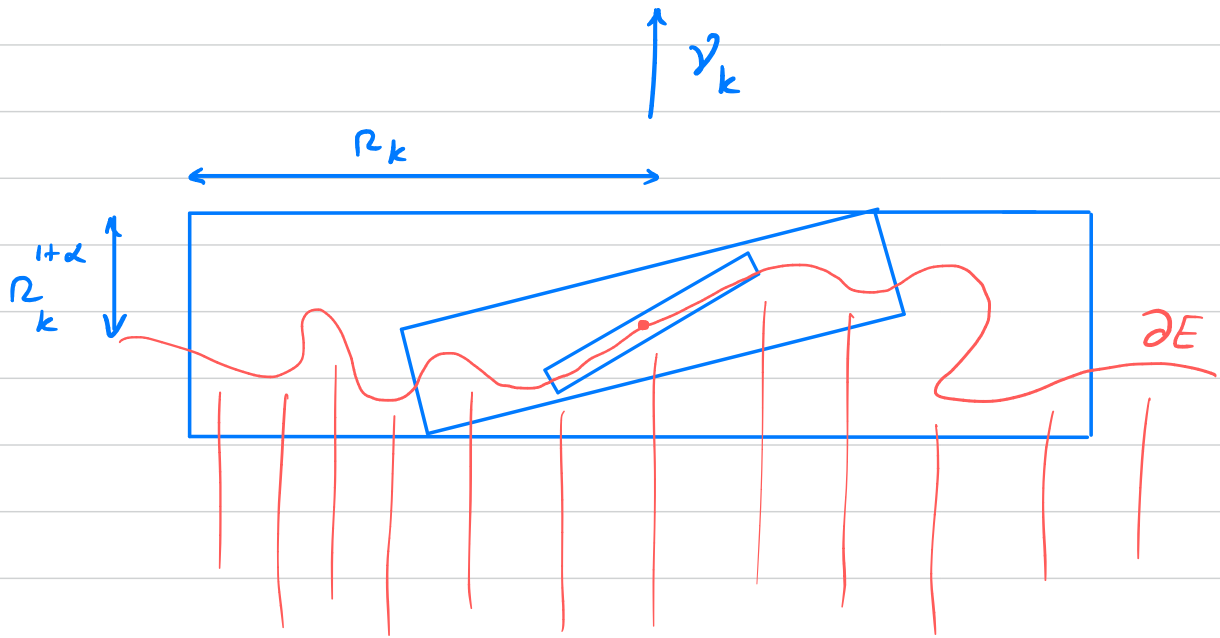
$$\{x \cdot \nu \le -r^{1+\alpha}\} \subset E \subset \{x \cdot \nu \le r^{1+\alpha}\} \quad \text{in } B_r$$

holds for $r = r_k = 2^{-k}$, $\nu = \nu_k$

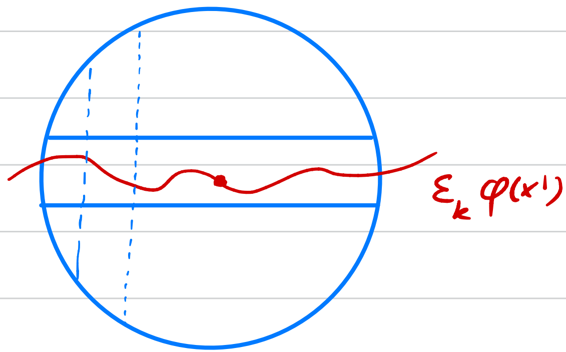
$k = 0, 1, 2, \ldots, k_0$ $\qquad k_0(s, \alpha, n)$ large

then it holds for all k.

$k = 0, 1, 2, \dots, k_0$



$$|\nu_k - \nu_{k+1}| \le C_m r_k^{\alpha} \implies \nu_k \to \bar{\nu}$$



$$|x - x_0| = |x' - x_0'| + O(\varepsilon_k^2)$$

$$H_s(x_0) = 2\varepsilon_k \int_{B_M'(x_k')} \frac{\varphi(x') - \varphi(x_0')}{|x' - x_0'|^{n+s}}\, dx' + O(\varepsilon_k M^{\alpha - s})$$

Higher regularity $C^{1,\alpha} \to C^{\infty}$ obtained by

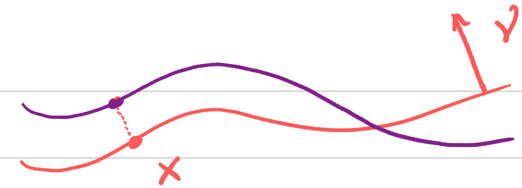
Barrios - Figalli - Valdinoci

$$\int_{\mathbb{R}^{n-1}} \mathcal{F}\left(\frac{\varphi(x') - \varphi(x'_0)}{|x' - x'_0|}\right) |x' - x'_0|^{-(n-1+s)} \, dx' = 0 .$$

<u>Open problem</u>: Is ∂E analytic?

The constant ε_0 can be taken independent of s as $s \to 1$. In this case minimality needs to be used in the Harnack inequality.

The linearized operator

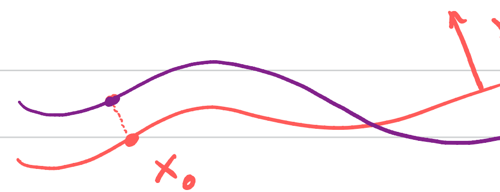


Deform $\partial E \in C^2$ in the normal direction:

$$E \longrightarrow E_t \qquad\qquad x \longrightarrow x + t\,\eta(x)\,\nu$$

$$H_{\partial E_t}(x) = H_{\partial E}(x) + t\,\underbrace{\left(\Delta \eta + |A|^2 \eta\right)}_{\text{Jacobi operator}} + o(t^2)$$

$$x + t\eta\nu - t\eta(x_0)\nu(x_0)$$



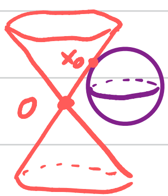
$$\frac{d}{dt} H_{s,E}(x_0)\Big|_{t=0} = \int_{\partial E} \frac{\eta(x) - \eta(x_0)\nu(x_0)\cdot\nu(x)}{|x-x_0|^{n+s}}\,dx$$

$$= \underbrace{\int_{\partial E} \frac{\eta(x)-\eta(x_0)}{|x-x_0|^{n+s}}\,dx}_{\text{"}\Delta_{\partial E}^{\frac{1+s}{2}}\eta\text{"}} + \eta(x_0)\underbrace{\int_{\partial E} \frac{1-\nu(x_0)\cdot\nu(x)}{|x-x_0|^{n+s}}\,dx}_{\text{"}|A|_s^2\text{"}}$$

Consequences of the Flatness Theorem:

1) The half-space is the cone of least energy

$$\Phi(\bar{E}) \geq \Phi(\mathbb{R}^n_+) + \delta$$ if $\bar{E}$ is a nontrivial cone.



x_0 is a regular point.

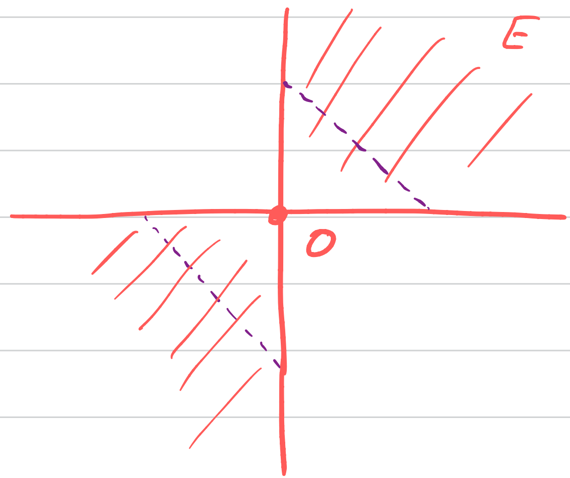
2) Dimension reduction (Federer):

There exists a first dimension $n_0 \in [2, \infty]$ for which a non-trivial cone with smooth cross-section exists.

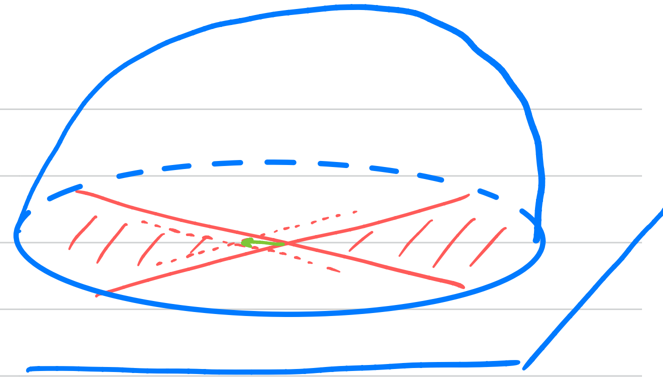
E min. set $\Longrightarrow$ ∂E is smooth outside a closed set of dimension $n - n_0$.

Regularity of cones

Theorem : The half-space is the only s-minimal cone in 2D.



Proof Let U be the extension
φ a cutoff function
e unit direction



$$U_\varepsilon(x) = U(x + \varepsilon\varphi(x)e)$$

$$U_{-\varepsilon}(x) = U(x - \varepsilon\varphi(x)e)$$

$$J(U_\varepsilon, B_1^+) + J(U_{-\varepsilon}, B_1^+) = 2J(U, B_1^+) + o(\varepsilon^2)$$

$$U_+ = \max\{U_\varepsilon, U_{-\varepsilon}\}, \quad U_- = \min\{U_\varepsilon, U_{-\varepsilon}\}$$

$$J(U_+, B_1^+) + J(U_-, B_1^+) = 2J(U, B_1^+) + o(\varepsilon^2)$$

$$\mathcal{J}(U_+, B_1^+) \leq \mathcal{J}(U, B_1^+) + O(\varepsilon^2)$$

If U_+ is not a minimizer in $\{z>0\}$ then we can modify it such that

$$\mathcal{J}(\tilde{U}_+, B_1^+) \leq \mathcal{J}(U_+, B_1^+) - \sigma \varepsilon^{2-s}, \quad \sigma > 0.$$

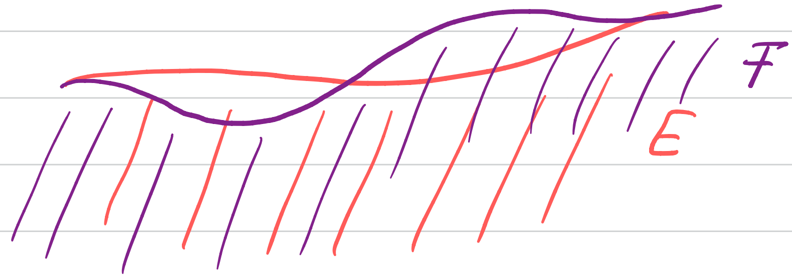
<u>Theorem</u> The half-space is the only s-minimal cone in dimension $n \leq 7$ and s close to 1.

Davila - Del Pino - Wei stability of Lawson cones

they are unstable up to dimension $n \leq 6$.

<u>Theorem</u> E s-minimal in B_1 then

$$P_{B_{1/2}}(E) \leq C(s,n).$$

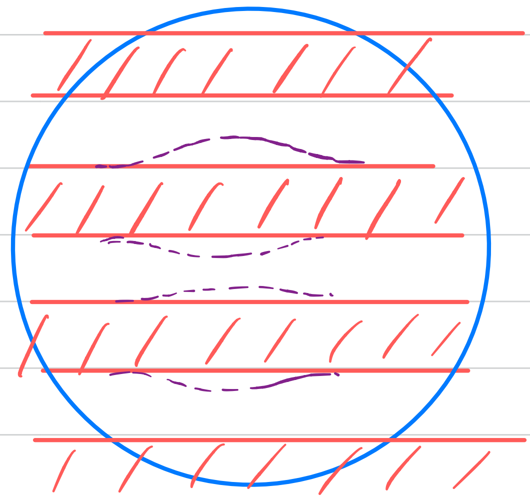


$$P_s(E) + P_s(F) = P_s(E \cup F) + P_s(E \cap F) - L(F \setminus E, E \setminus F)$$

<u>Theorem</u> E is s-stable in B_1 then

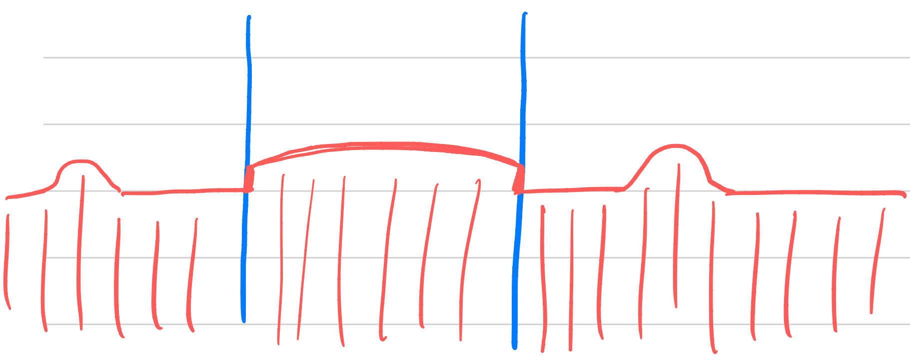
$$P_{B_{1/2}}(E) \leq C(n,s).$$

Major differences with classical minimal surfaces:



1) stability

2) boundary regularity



Open questions

1) classification of minimal cones (in dimension $n=3$)

2) study of singularities for nonlocal mean curvature flow

3) Construction of special solutions