On Calderon-Zygmund type estimates for nonlocal PDE

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NSFE 2022

based on joint works with M.M. Fall, W. Lewkeeratiyutkul, S. Khomrutai, T. Mengesha, S. Yeepo

Funding: Simons Foundation, NSF Career DMS-2044898

Classical (=local) version

Let $u \in W^{1,2}(\Omega, \mathbb{R})$ be a solution to

$$\operatorname{div}(K D u) \equiv \sum_{\ell=1}^{n} \partial_\ell (K \partial_\ell u) = f \quad \text{in } \Omega \subset \mathbb{R}^n$$

where $K(x) : \Omega \to [0, \infty)$ is bounded & measurable

$$\inf_{x \in \Omega} K(x) > 0 \qquad \text{(elliptic)}$$

Calderon-Zygmund theory:

$$f \in W^{s,p} \overset{?}{\Rightarrow} u \in W^{s+2,p} \qquad \text{(CZ)}$$

- Here $s < 0$ is ok: $f \in W^{-1,p} \equiv (W^{1,p'})^* \overset{?}{\Rightarrow} u \in W^{1,p}$

Answer: Depends on regularity of K.

Classical (=local) version – approach by commutator

Let $u \in W^{1,2}(\mathbb{R}^n, \mathbb{R})$ be a solution to

$$\sum_{\ell=1}^{n} \partial_\ell (K \partial_\ell u) = f \quad \text{in } \mathbb{R}^n$$

Apply $(-\Delta)^{-\frac{1}{2}}$, set $w := (-\Delta)^{\frac{1}{2}} u$, $g := (-\Delta)^{-\frac{1}{2}} f$.

$$\sum_{\ell=1}^{n} \mathcal{R}_\ell (K \mathcal{R}_\ell w) = g \quad \text{in } \mathbb{R}^n$$

Where $\mathcal{R}_\ell = (-\Delta)^{-\frac{1}{2}} \partial_\ell$ is the ℓ-th Riesz transform.

$$(-\Delta)^{-\frac{1}{2}} h(x) = c \int_{\mathbb{R}^n} |x - z|^{1-n} h(z)\, dz$$

$$\mathcal{R}_\ell h(x) = c \int_{\mathbb{R}^n} \frac{(x-z)_\ell}{|x-z|^{n+1}} h(z)\, dz$$

Classical (=local) version – approach by commutator

Let $w \in L^2(\mathbb{R}^n, \mathbb{R})$ be a solution to

$$\sum_{\ell=1}^{n} \mathcal{R}_\ell(K \mathcal{R}_\ell w) = g \quad \text{in } \mathbb{R}^n$$

We have $\sum_{\ell=1}^{n} \mathcal{R}_\ell \mathcal{R}_\ell = -I$, so

$$-K\, w + \sum_{\ell=1}^{n} [\mathcal{R}_\ell, K](\mathcal{R}_\ell w) = g$$

where $[T, a](b)$ is the commutator $T(ab) - aT(b)$.
So

$$w = -\frac{1}{K} g + \frac{1}{K} \sum_{\ell=1}^{n} [\mathcal{R}_\ell, K](\mathcal{R}_\ell w)$$

Classical (=local) version – approach by commutator

Let $w \in L^2(\mathbb{R}^n, \mathbb{R})$ be a solution to

$$w = -\frac{1}{K}g + \frac{1}{K}\sum_{\ell=1}^{n}[\mathcal{R}_\ell, K](\mathcal{R}_\ell w)$$

Main observation: If K is C^γ-Hölder continuous then

$$\begin{aligned}
|[\mathcal{R}_\ell, K][\tilde{w}](x)| =& \left|\int_{\mathbb{R}^n} \frac{(x-y)_\ell\,(K(x)-K(z))}{|x-z|^{n+1}} w(z)dz\right| \\
\leq& [K]_{C^\gamma} \left|\int_{\mathbb{R}^n} |x-z|^{\gamma-n}\,|w(z)|dz\right| \\
\approx& [K]_{C^\gamma}(-\Delta)^{-\frac{\gamma}{2}}|w|(x)
\end{aligned}$$

So

$$w = -\frac{1}{K}g + \text{compact distortion}$$

and even for any $0 \leq \beta < \gamma$

$$|D|^\beta w = -|D|^\beta\left(\frac{1}{K}g\right) + \text{compact distortion}$$

Classical (=local) version – approach by commutator

Let $w \in L^2(\mathbb{R}^n, \mathbb{R})$ be a solution to

$$|D|^\beta w = -|D|^\beta \left(\frac{1}{K} g\right) + \text{compact distortion}$$

Thus we get (CZ):

- If $f \in W^{s,p}$ then $g = (-\Delta)^{-\frac{1}{2}} f \in W^{1+s,p}$ then
- if $\beta < 1 + s$: $|D|^\beta w \in L^p$, thus
- $(-\Delta)^{\frac{1}{2}} u = w \in W^{\beta,p}$
- $u \in W^{t,p}$ whenever $t < 2 + s$.

Fun fact: also works with $\beta = 0$ if $K \in VMO$, i.e. if

$$\limsup_{r\to 0} r^{-2n} \int_{B(x,r)} \int_{B(x,r)} |K(z_1) - K(z_2)| dz_1 dz_2 = 0 \quad \forall x \in \mathbb{R}^n.$$

Nonlocal version

$$\mathrm{div}(K Du) = f \quad \text{becomes} \quad \mathrm{div}_\alpha(K d_\alpha u) = f$$

where div_α and d_α are type of nonlocal divergence Gunzburger-Du et al., Mazowiecka-S. (div-curl).

$$\int_{\mathbb{R}^n} \frac{K(x,y)(u(x) - u(y))}{|x-y|^{n+2\alpha}}\, dy = f(x).$$

Many previous results for some cases of CZ: e.g.
Dong-Kim ($K(x-y)$), Brasco–Lindgren, Cozzi, and very recently: Nowak (for VMO!!)

Approach? Same!

$$\int_{\mathbb{R}^n} \frac{K(x,y)(u(x)-u(y))}{|x-y|^{n+2\alpha}}\,dy = f(x).$$

For $\alpha_1+\alpha_2=2s$ apply $(-\Delta)^{-\frac{\alpha_1}{2}}$, set $w := (-\Delta)^{\frac{\alpha_2}{2}}u$,
$g := (-\Delta)^{-\frac{\alpha_1}{2}} f$.
Then

$$\int_{\mathbb{R}^n} A_K(z_1,z_2)w(z_2)dz_2 = g(z_1)$$

where

$$A_K(z_1,z_2) =$$
$$\int\limits_{\mathbb{R}^n\times\mathbb{R}^n} \frac{K(x,y)\,\big(|x-z_1|^{\alpha_1-n}-|y-z_1|^{\alpha_1-n}\big)\,\big(|x-z_2|^{\alpha_2-n}-|y-z_2|^{\alpha_2-n}}{|x-y|^{n+2\alpha}}$$

Commutator? Similar!

$$\int_{\mathbb{R}^n} A_K(z_1, z_2) w(z_2) dz_2 = g(z_1)$$

is written as

$$K(z_1, z_1) w(z_1) + \int_{\mathbb{R}^n} \left(A_K(z_1, z_2) - K(z_1, z_2)\delta_{z_1,z_2}\right) w(z_2) dz_2 = g(z_1)$$

Main observation:

$$\left(A_K(z_1, z_2) - K(z_1, z_2)\delta_{z_1,z_2}\right) =$$
$$\int_{\mathbb{R}^n \times \mathbb{R}^n} \frac{\left(K(x,y) - K(z_1, z_2)\right) \left(|x - z_1|^{\alpha_1 - n} - |y - z_1|^{\alpha_1 - n}\right) \left(|x - z_2|^{\alpha_2 - n} - \right.}{|x - y|^{n+2\alpha}}$$

Similar as before this is a compact operator!

Commutator estimate

Theorem (Mengesha-S-Yeepo)

If $K \in L^\infty \cap C^\sigma$, for all $\varepsilon \ll 1$

$$\left|\int_{\mathbb{R}^n\times\mathbb{R}^n} \left(A_K(z_1,z_2) - K(z_1,z_2)\delta_{z_1,z_2}\right) f(z_1)\, g(z_2) dz_1 dz_2\right|$$

$$\lesssim \int_{\mathbb{R}^n} (-\Delta)^{-\frac{\sigma-\varepsilon}{2}} |f|(x)\, (-\Delta)^{-\frac{\varepsilon}{2}} |g|(x)\, dx$$

$$\left(A_K(z_1,z_2) - K(z_1,z_2)\delta_{z_1,z_2}\right) =$$

$$\int_{\mathbb{R}^n\times\mathbb{R}^n} \frac{\left(K(x,y) - K(z_1,z_2)\right)\left(|x-z_1|^{\alpha_1-n} - |y-z_1|^{\alpha_1-n}\right)\left(|x-z_2|^{\alpha_2-n} -\right.}{|x-y|^{n+2\alpha}}$$

Conclusion - Mengesha-S-Yeepo

$s \leq 0$, K Hölder continuous and bounded: equation becomes

$$w(z_1) = \frac{1}{K(z_1, z_1)} g(z_1) + \text{compact perturbation}$$

- If $f \in W^{s,p}$ then $g(z_1) = (-\Delta)^{-\frac{\alpha_2}{2}} f \in W^{\alpha_2+s,p}$
- Then $w \in W^{\alpha_2+s,p}$
- Then $u = (-\Delta)^{-\frac{\alpha_1}{2}} w \in W^{\alpha_1+\alpha_2+s,p}$
- Then $u \in W^{2\alpha+s,p}$

decoupling of regularity of K and $2\alpha + s$ because α_1 and α_2 can be chosen freely.
works as long as $2\alpha + s < 2$ (if > 2 needs more reg for K)

Calderon-Zygmund kernel theorem

$$A_{K,\alpha_1,\alpha_2}(z_1,z_2) = \int\limits_{\mathbb{R}^n\times\mathbb{R}^n} \frac{K(x,y)\left(|x-z_1|^{\alpha_1-n}-|y-z_1|^{\alpha_1-n}\right)\left(|x-z_2|^{\alpha_2-n}-|y-z_2|^{\alpha_2-n}\right)}{|x-y|^{n+(\alpha_1+\alpha_2)}}$$

Theorem (Lewkeeratiyutkul- Khomrutai -S-Yeepo)

Let $K \in L^\infty(\mathbb{R}^n\times\mathbb{R}^n)$, $(\alpha_1+\alpha_2)\in(0,2)$

$$T := f(\cdot) \mapsto \int_{\mathbb{R}^N} A_{K,\alpha_1,\alpha_2}(\cdot,z_2)f(z_2)dz_2$$

is a Calderon-Zygmund operator, and

$$\|Tf\|_{L^p} \lesssim \|K\|_{L^\infty}\,\|f\|_{L^p(\mathbb{R}^n)}$$

$$\|Tf\|_{BMO} \lesssim \|K\|_{L^\infty}\,\|f\|_{L^\infty(\mathbb{R}^n)}$$

$$\|Tf\|_{L(1,\infty)} \lesssim \|K\|_{L^\infty}\,\|f\|_{L^1(\mathbb{R}^n)}$$

Simon Nowak's result

- Simon Nowak obtained essentially same results assuming $K \in VMO$ (different argument)
- So should we hope that if $K \in VMO$ then T is compact, where

$$\langle Tf, g\rangle = \int_{\mathbb{R}^n \times \mathbb{R}^n} (A_K(z_1, z_2) - K(z_1, z_2)\delta_{z_1,z_2})\, f(z_1)\, g(z_2) dz_1 dz_2$$

Is there a Coifman-Rochberg-Weiss-type off-diagonal-commutator theorem??

Different Kernel: Fall, Mengesha, S., Yeepo

Think about

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{(u(x)-u(y))\,(\varphi(x)-\varphi(y))}{|\Phi(x)-\Phi(y)|^{n+2\alpha}} |\det(D\Phi(x)||\det(D\Phi(y)| = f[\varphi]$$

Equation for $u \circ \Phi^{-1}$, e.g. where $\Phi : \mathbb{R}^n \to \mathcal{M}$ is diffeo.
Leads to the kernel

$$\frac{|\Phi(x)-\Phi(y)|}{|x-y|} \approx \left| D\phi(x)\frac{x-y}{|x-y|}\right|.$$

So we should maybe consider instead kernels (cf. Fall: Schauder theory) of the form

$$K\left(x, \frac{x-y}{|x-y|}, |x-y|\right)$$

Different Kernel: Fall, Mengesha, S., Yeepo

CZ-theory for:

$$\int_{\mathbb{R}^n} K\left(x, \frac{x-y}{|x-y|}, |x-y|\right) \frac{(u(x)-u(y))}{|x-y|^{n+2\alpha}} dy = f(x)$$

- Dong-Kim: If $K = K(x-y)$ then full CZ-theory

$$f \in W^{s,p} \Rightarrow u \in W^{s+2\alpha,p}$$

(Dong-Kim prove $s = 0$ case $f \in L^p$ – since convolution operator commutes with differentiation: follows for any s by interpolation.)

If $K(x,h,r)$ uniformly continuous in x for small r, freezing argument

$$\int_{\mathbb{R}^n} K\left(x_0, \frac{x-y}{|x-y|}, 0\right) \frac{(u(x)-u(y))}{|x-y|^{n+2\alpha}}dy$$
$$= f(x) + \int_{\mathbb{R}^n} \left(K\left(x_0, \frac{x-y}{|x-y|}, 0\right) - K\left(x, \frac{x-y}{|x-y|}, |x-y|\right)\right) \frac{(u(x)-u(}{|x-y|^{n+}}$$

Then we can compare our solution to Dong-Kim-problem plus small variation.

Use Banach Fixed-point theorem to show there is a unique solution of the right regularity

Provides a generalization of Dong-Kim-type kernels (previous result was not)

Fractional p-Laplacian:

$$\mathrm{div}(|\nabla u|^{p-2}\nabla u) = f.$$

If $u \in C^1$ around some x_0 and $|\nabla u(x_0)| > 0$ then any regularity theory around x_0.

$$\int_{\mathbb{R}^n} \frac{|u(x)-u(y)|^{p-2}\,(u(x)-u(y))}{|x-y|^{n+\alpha p}}\,dy$$
$$\approx \int_{\mathbb{R}^n} \frac{|Du(x)\frac{x-y}{|x-y|}|^{p-2}\,(u(x)-u(y))}{|x-y|^{n+\alpha p-(p-2)}}\,dy$$
$$\approx \int_{\mathbb{R}^n} \frac{K(x,x-y)\,(u(x)-u(y))}{|x-y|^{n+2\beta}}\,dy$$

for $\beta := \frac{\alpha p-(p-2)}{2}$.
If $\beta \in (0,1)$ we can apply our theory? yes: but $K(x,h) = 0$ for some h, just not so many!

Degenerate Kernel

Theorem (Fall, Mengesha, S., Yeepo)

Dong-Kim-result holds if K is nonnegative and bounded, and degenerate only in the following sense:
For some $v \in \mathbb{S}^{n-1}$,

$$\forall \sigma \in [0,1): \ \exists \lambda = \lambda(\sigma) > 0 : \quad \lambda < \inf_{h \in S(\sigma,v)} K(h).$$

Here $S(\sigma, v)$ denotes the the part of the sphere $\mathbb{S}^{n-1}$ which is bounded away from the $v^{\perp}$,

$$S(\sigma, v) := \{h \in \mathbb{R}^n : |h| = 1, |\langle h, v\rangle| \geq \sigma\}$$

Applies in particular if for some $v \neq 0$.

$$K(h) = |\langle v, h\rangle|$$

Proof Same as Dong-Kim: a priori estimate by Fourier transform.

Consequence

Corollary

Our theorem applies also if

$$\forall \sigma \in [0,1): \ \exists \lambda = \lambda(\sigma) > 0: \quad \lambda < \inf_{x} \inf_{h \in S(\sigma, v)} K(x, h).$$

Corollary

Regularity for (some regime of) the fractional p-Laplace around x_0 if $|\nabla u(x_0)| > 0$.

Further Problems

- Boundary regularity
- *VMO*-result via commutator?
- unbounded and degenerate K (for frac p-Laplacian)
- parabolic version?

Thank you for your attention

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- Yeepo, Lewkeeratiyutkul, Khomrutai, Schikorra: On the Calderon-Zygmund property of Riesz-transform type operators arising in nonlocal equations. Commun. Pure Appl. Anal. 20 (2021), no. 9, 2915
- Fall, Mengesha, Schikorra, Yeepo, Calderon-Zygmund theory for non-convolution type nonlocal equations with continuous coefficient, arXiv:2109.04879