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Regularity of elliptic transmission problems and a new family of integro-differential operators related to the Monge-Ampère equation

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Regularity of elliptic transmission problems and a new family of integro-differential operators related to the Monge-Ampère equation

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This dissertation is divided into two main topics. First, we study transmission problems for elliptic equations, both linear and nonlinear, and prove existence, uniqueness, and optimal regularity of solutions. In our first work, we consider a problem for harmonic functions and use geometric techniques. Our second work considers viscosity solutions to fully nonlinear transmission problems. Given the nonlinear nature of these equations, our arguments are based on perturbation methods and comparison principles.

The second topic is related to nonlocal Monge-Ampère equations. We define a new family of integro-differential equations arising from geometric considerations and study some of their properties. Furthermore, we consider a Poisson problem in the full space and prove existence, uniqueness, and $C^{1,1}$ regularity of solutions. For this problem, we use tools from convex analysis and symmetrization.

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Chapter 1

Introduction

This thesis is divided into two main parts. The first part is about elliptic transmission problems and includes Chapters 2-4, where each chapter considers a different problem. The second part, in Chapter 5, deals with the construction and analysis of a new family of integro-differential operators that are motivated by a nonlocal Monge-Ampère equation.

Next, we briefly introduce transmission problems and explain the main features. Then we present the Monge-Ampère equation and discuss two different nonlocal versions. Finally, we provide an outline describing the several problems we studied.

1.1 Transmission problems

Transmission problems describe phenomena in which a physical quantity changes behavior across some surface, known as *the interface.* Historically, the study of these types of problems started in the 1950s, with the pioneering work of M. Picone in elasticity theory [49]. Further significant contributions to Picone's problem were made by J. L. Lions [39], G. Stampacchia [61] and S. Campanato [18]. In 1960, M. Schechter generalized the problem of transmission for elliptic equations with smooth coefficients and interfaces [54]. Other variations, such as the so-called *diffraction problems* in the theory of discontinuous coefficients, were given by O. A. Ladyzhenskaya and N. N. Ural'tseva [33], O. A. Oleinik [47], and M. V. Borsuk [6], among others. See [5] for a detailed exposition on classical transmission problems. Since then, many mathematicians have been interested in studying these problems due to their wide range of applications in different areas of science. For instance, they appear in electromagnetic processes, composite materials in solid mechanics, vibrating folded membranes, etc. For more recent developments, see [20, 31, 35, 36, 43] and the references therein.

Mathematically, a simple model may be described as follows: let Ω be a bounded domain in $\mathbb{R}^n$ divided into two subdomains, Ω_1 and Ω_2, by some surface Γ (the interface). On each subdomain, we prescribe some PDE and call u_1 and u_2 the solutions, respectively. For the problem to be well-posed, we need to determine some compatibility conditions on Γ. These are known as the *transmission conditions*, and they describe how both u_1 and u_2 interact with each other from each side of the interface. For example, typical transmission conditions include, $(u_1)_\nu - (u_2)_\nu = 1$ or $(u_1)_\nu = 2(u_2)_\nu$, where $(u_i)_\nu$ denotes the normal derivative of u_i. Both equations prescribe a jump between the normal derivatives of u_1 and u_2, so we expect solutions to be singular on Γ. Therefore, the primary interest is to study the regularity of solutions near the interface.

Transmission problems can be understood as two (or more) boundary value problems that have been attached via the transmission conditions. In general, it is not possible to decouple the problems, and thus, their study becomes more challenging. In contrast to free boundary problems, classical transmission problems deal with a *fixed* interface (i.e., Γ is known a priori). It is worth mentioning that several works in the literature consider the so-called *free* transmission problems, where Γ is a free boundary, in the sense that it depends on the solution itself. For instance, see [1, 25, 29, 50, 51].

In this dissertation, we consider three transmission problems with different flavors, including linear and nonlinear equations, and flat and nonflat interfaces. One of our principal features is the minimal regularity of the interface. We will explain them in more detail in Section 1.3.

1.2 Nonlocal Monge-Ampère equations

The Monge-Ampère equation arises in several problems in analysis, such as the Monge–Kantorovich optimal mass transportation problem, and in geometry, such as the prescribed Gaussian curvature problem. The classical equation prescribes the determinant of the Hessian of some convex function u in a given domain Ω. Namely,

$$\det(D^2u) = f \qquad \text{in } \Omega.$$

This is a nonlinear second order elliptic equation that degenerates whenever the Hessian of u equals 0. There are many works that study the theory of this equation and its variations. The books by Gutiérrez [28] and Figalli [24] give a detailed description of the main results and techniques.

On the other hand, in the recent years, there has been significant interest in studying nonlocal diffusion equations. These equations arise in the study of diffusion processes with long-range interactions such as Lévy processes, in probability, or particles jumping through random media, in fluid dynamics. A classical nonlocal operator is the fractional Laplacian, given by

$$\Delta^s u(x) = c_{n,s} \ \mathrm{PV} \int_{\mathbb{R}^n} \frac{u(x+y) - u(x)}{|y|^{n+2s}} \, dy.$$

The term *nonlocal* is motivated by the fact that the value of $\Delta^s u$ at the point x depends on all the values of u in $\mathbb{R}^n$, and the term *fractional* has to do with the order of the operator, $2s$, with $0 < s < 1$. For instance, this can be seen using the Fourier transform or a scaling argument. Furthermore, $c_{n,s} \approx 1 - s$ and $\Delta^s \to \Delta$, as $s \to 1$. Hence, Δ^s is a fractional analog of Δ. The integral is understood in the principal value sense to be able to cancel out the singularity from the kernel, $|y|^{-n-2s}$. Note that this requires some differentiability for the function u. Hence, the fractional Laplacian is a particular case of an *integro-differential* operator, where the latter considers more general kernels.

In the literature, there are different nonlocal versions of the Monge-Ampère operator that N. Guillen and R. W. Schwab [27], L. Caffarelli and F. Charro [8], and L. Caffarelli and L. Silvestre [13] have considered. See also [42] for a nonlocal linearized Monge-Ampère equation given by D. Maldonado and P. R. Stinga. These definitions are motivated by the following linear algebra extremal property: if B is a positive definite symmetric matrix, then

$$n \det(B)^{1/n} = \inf_{A \in \mathcal{A}} \mathrm{tr}(A^T B A),$$

where $\mathcal{A} = \{A \in M_n : A > 0,\ \det(A) = 1\}$ and M_n is the set of $n \times n$ matrices. If a convex function u is C^2 at a point x_0, then by the previous identity with $B = D^2u(x_0)$, we may write the Monge-Ampère operator as a concave envelope of linear operators. It follows that

$$n \det(D^2u(x_0))^{1/n} = \inf_{A\in\mathcal{A}} \Delta[u \circ A](A^{-1}x_0).$$

L. Caffarelli and F. Charro study a fractional version of $\det(D^2u)^{1/n}$, replacing the Laplacian by the fractional Laplacian in the previous identity (see also [27]). More precisely,

$$\mathcal{D}^s u(x_0) = \inf_{A\in\mathcal{A}} \Delta^s[u \circ A](A^{-1}x_0),$$

or equivalently, using the integral representation,

$$\mathcal{D}^s u(x_0) = c_{n,s} \inf_{A\in\mathcal{A}} \ \mathrm{PV} \int_{\mathbb{R}^n} \frac{u(x_0 + x) - u(x_0)}{|A^{-1}x|^{n+2s}}\, dx.$$

A different approach based on geometric considerations was given by L. Caffarelli and L. Silvestre. In fact, the authors consider kernels whose level sets are volume preserving transformations of the fractional Laplacian kernel. Namely,

$$\mathrm{MA}^s\, u(x_0) = c_{n,s} \inf_{K\in\mathcal{K}_n^s} \int_{\mathbb{R}^n} (u(x_0 + x) - u(x_0) - x \cdot \nabla u(x_0))K(x)\, dx,$$

where the infimum is taken over the family

$$\mathcal{K}_n^s = \Big\{K : \mathbb{R}^n \to \mathbb{R}_+ : |\{x \in \mathbb{R}^n : K(x) > r^{-n-2s}\}| = |B_r|,\ \forall\, r > 0\Big\}.$$

Notice that $|A^{-1}x|^{-n-2s} \in \mathcal{K}_n^s$, for any $A \in \mathcal{A}$. Therefore,

$$\mathrm{MA}^s\, u(x_0) \le \mathcal{D}^s u(x_0) \le \Delta^s u(x_0).$$

We point out that, at the nonlocal level, the operators $\mathcal{D}^s$ and MA^s are not equivalent. However, when we pass to the limit as $s \to 1$, both $\mathrm{MA}^s\, u$ and $\mathcal{D}^s u$ converge to $\det(D^2u)^{1/n}$, up to some constant.

In the last chapter of this dissertation, we will discuss a nonlocal problem related to the Monge-Ampère equation. The problem is motivated by the previous construction and the fact that there is a gap between the nonlocal Monge-Ampère operator and the fractional Laplacian. Our goal is to find a reasonable family of intermediate operators that will somehow link them.

1.3 Outline

This manuscript is organized as follows. In Chapter 2, we study a transmission problem for harmonic functions, which is motivated by the pioneering work of Schechter for smooth domains [54]. One of our main novelties is that the transmission interface has only $C^{1,\alpha}$ regularity. This minimal regularity assumption makes the problem nontrivial and challenging. For instance, to prove regularity of solutions up to the interface, the classical Schauder approach of flattening the boundary is not available. Integrating by parts, we reduce the transmission problem to a distributional Poisson equation, where the right-hand side is a measure supported on the interface. We prove existence and uniqueness of continuous solutions using techniques from potential theory. Then we prove optimal regularity up to the interface via a perturbation method. For this, we build up a new fine geometric argument based on the mean value property and the maximum principle. This is joint work with L. Caffarelli and P. R. Stinga, published in *Arch. Ration. Mech. Anal.* (2021), see [17].

Chapters 3 and 4 consider transmission problems for second order fully nonlinear equations with flat and nonflat interfaces, respectively. The theory of fully nonlinear equations started around the 1980s, and nowadays it is a *hot* research topic with many open problems. The notion of solution for these equations is understood in the *viscosity* sense. This concept was introduced by M. G. Crandall and P. L. Lions for Hamilton-Jacobi equations [22], and was generalized later on to second order fully nonlinear operators [21]. In this work, in contrast to the problems introduced in Chapter 2, we allow the operators from each side of the interface to be different, as well as having nontrivial right-hand sides. These features, especially the nonlinear character of the equations, give rise to new difficulties. For example, we cannot use variational techniques or tools such as Green's functions and representation formulas. This is joint work with P. R. Stinga that will soon be submitted for publication [60].

First, we study the flat interface case, that is, the case where the interface is a hyperplane. These problems are in the same spirit as the ones introduced by D. De Silva, F. Ferrari, and S. Salsa in [58]. Problems with flat boundaries are relatively easier to understand, and one can extend many ideas to more general domains. Furthermore, they play a fundamental role in the regularity theory of nonflat problems that we will consider in Chapter 4. One of our main results is the existence and uniqueness of viscosity solutions to flat interface transmission problems with prescribed boundary values. We point out that this problem was left open in [58]. To prove it, we follow the usual *greatest subsolution approach*, also known as Perron's method. The most challenging step is to show the comparison principle. This is possible thanks to a new maximum principle for these problems, also called the Alexandroff-Bakelman-Pucci estimate (ABP estimate). Our ideas are inspired by the remarkable book of L. Caffarelli and X. Cabré [15].

Second, we consider $C^{1,\alpha}$ interfaces. Our strategy builds on similar ideas as the ones given in Chapter 2 for the Laplace equation. In the linear case, one important ingredient is the Hölder continuity of solutions across the interface, which we obtain thanks to classical estimates for the Green's function for the Laplacian. For the fully nonlinear problem, we can also get a similar result using nonlinear techniques such as the construction of appropriate barriers and comparison principles. To obtain optimal regularity results at the interface, we require an additional closeness assumption between the operators since they may be different on each side of the interface. This condition is analogous to asking that the coefficients are sufficiently close, in the case of linear operators. Then following a perturbation argument and using flat interface problems, we are able to prove that viscosity solutions are $C^{1,\alpha}$ up to the interface.

In Chapter 5, we introduce a new family of intermediate operators between the fractional Laplacian and the nonlocal Monge-Ampère operator, studied by L. Caffarelli and L. Silvestre in [13]. Our operators are also given by infimums of integro-differential operators over a family of kernels satisfying specific geometric properties. One of the main challenges in their study is that they are not rotationally invariant, due to our construction of the kernels. This is in contrast to the nonlocal Monge-Ampère operator, where the level sets of the kernels are volume preserving transformations of balls in $\mathbb{R}^n$ (see Section 1.2). Using symmetrization techniques, we obtain representation formulas and give a connection to optimal transport. Furthermore, we consider a global Poisson problem, prescribing data at infinity, and prove existence, uniqueness, and $C^{1,1}$ regularity of solutions in the full space. This is joint work with L. Caffarelli that has been submitted for publication [16].

Chapter 2

Transmission problems for harmonic functions

2.1 Introduction and main results

Let Ω be a smooth, bounded domain of $\mathbb{R}^n$, $n \geq 2$. Let Ω_1 be a subdomain of Ω such that $\Omega_1 \subset\subset \Omega$ and set $\Omega_2 = \Omega \setminus \overline{\Omega}_1$. Suppose that the interface Γ between Ω_1 and Ω_2, namely, $\Gamma = \partial\Omega_1$, is a $C^{1,\alpha}$ manifold, for some $0 < \alpha < 1$. Then $\Omega = \Omega_1 \cup \Omega_2 \cup \Gamma$. For a function $u : \overline{\Omega} \to \mathbb{R}$ we denote

$$u_1 = u\big|_{\overline{\Omega}_1} \qquad \text{and} \qquad u_2 = u\big|_{\overline{\Omega}_2}.$$

We consider the problem of finding a continuous function $u : \overline{\Omega} \to \mathbb{R}$ such that

$$\begin{cases} \Delta u_1 = 0 & \text{in } \Omega_1 \\ \Delta u_2 = 0 & \text{in } \Omega_2 \\ u_2 = 0 & \text{on } \partial\Omega \\ u_1 = u_2 & \text{on } \Gamma \\ (u_1)_\nu - (u_2)_\nu = g & \text{on } \Gamma. \end{cases} \tag{2.1.1}$$

Here $g \in C^{0,\alpha}(\Gamma)$ and ν is the unit normal vector on Γ that is interior to Ω_1, see Figure 2.1. This is a transmission problem in the spirit of Schechter in [54], where Γ is the transmission interface. In contrast to our problem, [54] only deals with $\Gamma \in C^\infty$. The last two equations on (2.1.1) are called the *transmission conditions*.

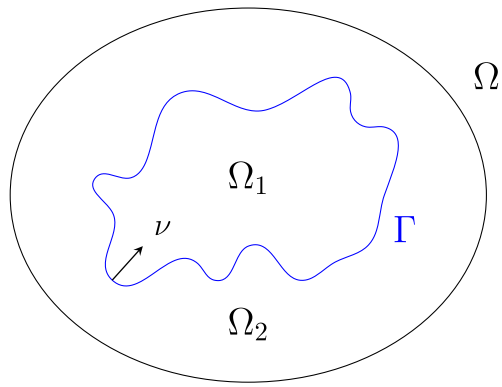

Figure 2.1: Geometry for the transmission problem (2.1.1).

If in (2.1.1) we set $g \equiv 0$ then u is a harmonic function in Ω. Therefore, in order to have a meaningful elliptic transmission condition, we assume that

$$g(x) \geq 0 \qquad \text{for all } x \in \Gamma.$$

Hence, u will not be differentiable at those points on Γ where $g > 0$. In turn, we prove that u is $C^{1,\alpha}$ from each side up to Γ. In (2.1.1) we have also imposed homogeneous Dirichlet boundary condition on $\partial\Omega$. This is not a restriction since we can always add to u a harmonic function v in Ω such that $v = \phi$ on $\partial\Omega$, to make $u_2 = \phi$ on $\partial\Omega$. The one dimensional case is excluded because one can easily find explicit solutions.

Our main result is the following.

Theorem 2.1.1. *There exists a unique classical solution u to the transmission problem* (2.1.1)*. Moreover, $u_1 \in C^{1,\alpha}(\overline{\Omega}_1)$, $u_2 \in C^{1,\alpha}(\overline{\Omega}_2)$, and there exists $C = C(n, \alpha, \Gamma) > 0$ such that*

$$\|u_1\|_{C^{1,\alpha}(\overline{\Omega}_1)} + \|u_2\|_{C^{1,\alpha}(\overline{\Omega}_2)} \leq C\|g\|_{C^{0,\alpha}(\Gamma)}.$$

The appropriate notion of solution to (2.1.1) comes from computing Δu in the sense of distributions. Indeed, if u and Γ were sufficiently smooth and $\varphi \in C_c^\infty(\Omega)$, then by Green's identities, recalling that ν is the interior normal to Ω_1, and using that $u_1 = u_2$ on Γ, we get

$$\begin{aligned}
\Delta u(\varphi) &= \int_\Omega u\Delta\varphi\,dx = \int_{\Omega_1} u_1\Delta\varphi\,dx + \int_{\Omega_2} u_2\Delta\varphi\,dx \\
&= \int_{\Omega_1} \varphi\Delta u_1\,dx - \int_{\partial\Omega_1} \big(u_1\varphi_\nu - \varphi(u_1)_\nu\big)\,dH^{n-1} \\
&\quad + \int_{\Omega_2} \varphi\Delta u_2\,dx + \int_{\partial\Omega_2} \big(u_2\varphi_\nu - \varphi(u_2)_\nu\big)\,dH^{n-1} \\
&= -\int_\Gamma \big(u_1\varphi_\nu - \varphi(u_1)_\nu\big)\,dH^{n-1} + \int_\Gamma \big(u_2\varphi_\nu - \varphi(u_2)_\nu\big)\,dH^{n-1} \\
&= \int_\Gamma \big((u_1)_\nu - (u_2)_\nu\big)\varphi\,dH^{n-1} = \int_\Gamma g\varphi\,dH^{n-1}.
\end{aligned}$$

Hence, Δu is a singular measure concentrated on Γ with density g. In Section 2.2 we show that there exists a unique distributional solution $u \in C_0(\overline{\Omega})$ to (2.1.1), where $C_0(\overline{\Omega})$ denotes the space of continuous functions on $\overline{\Omega}$ that vanish on $\partial\Omega$. Moreover, we prove that u is Log-Lipschitz on $\overline{\Omega}$, see Theorem 2.2.2. The main issue is the optimal regularity of u up to Γ. Theorem 2.1.1 will be a consequence of our next result.

Theorem 2.1.2 (Pointwise $C^{1,\alpha}$ boundary regularity)**.** *Let* $\Gamma = \{(y', \psi(y')) : y' \in B_1'\}$*, where* ψ *is a* $C^{1,\alpha}$ *function, for some* $0 < \alpha < 1$*. Assume that* $0 \in \Gamma$*. Let* $u \in C(\overline{B_1})$ *be a distributional solution to the transmission problem*

$$\Delta u = g\,dH^{n-1}\big|_\Gamma,$$

where $g \in L^\infty(\Gamma)$*,* $g \geq 0$*, and* $g \in C^{0,\alpha}(0)$*. Then there are linear polynomials*

$P(x) = A \cdot x + B$, and $Q(x) = C \cdot x + B$ such that

$$|u_1(x) - P(x)| \leq D|x|^{1+\alpha} \qquad \textit{for all } x \in \Omega_1 \cap B_{1/2},$$

$$|u_2(x) - Q(x)| \leq D|x|^{1+\alpha} \qquad \textit{for all } x \in \Omega_2 \cap B_{1/2},$$

with

$$|A| + |B| + |C| + D \leq C_0 \|\psi\|_{C^{1,\alpha}(B_1')} \big([g]_{C^\alpha(0)} + \|g\|_{L^\infty(\Gamma)} \big),$$

and $C_0 = C_0(n, \alpha) > 0$.

The key tool to prove Theorem 2.1.2 is a *stability* result, obtained via the novel geometric approach we develop, which is based on the mean value property and the maximum principle, see Theorem 2.4.2. In fact, our idea is to explicitly construct classical solutions to problems with flat interfaces that are close to u. With this, we can transfer the regularity from classical solutions to u. Indeed, as shown in Section 2.3, solutions to flat problems have the expected optimal regularity up to the interface. More precisely, we show that if the flatness and oscillation of the interface Γ are controlled, then we can construct a solution for a flat interface problem, where the flat interface does not intersect Γ. We also quantify how close solutions must be, depending only on the geometric properties of Γ and the basic regularity of u. These ingredients are crucial for the first step in the proof of Theorem 2.1.2, see Lemma 2.5.1. To close the argument, one needs to use these approximations at each scale. Through this techniques, and similar to the case of elliptic equations [14], we are able to find that solutions to flat interface problems are asymptotically close to solutions to nonflat interface problems.

Our geometric techniques developed in Section 2.4 are constructive and quantitative, and provide a precise understanding of the underlying geometry of the transmission problem. Furthermore, this work is essentially self-contained. We believe that the tools presented here could be used in free boundary problems, an idea we will explore in the future. Finally, notice that our results are also useful in terms of numerical analysis, as our constructions give explicit rates of approximation.

The chapter is organized as follows. In Section 2.2, we prove existence, uniqueness, and global Log-Lipschitz regularity of the solution u to (2.1.1). Section 2.3 deals with the case when the transmission interface is flat. Our geometric stability result based on the mean value property is proved in Section 4. The proof of Theorems 2.1.2 and 2.1.1 are given in Sections 2.5 and 2.6, respectively. The appendix contains some basic geometric considerations about integration on Lipschitz domains.

Notation. For a point $x \in \mathbb{R}^n$ we write $x = (x', x_n)$, where $x' \in \mathbb{R}^{n-1}$, $x_n \in \mathbb{R}$. The gradient in the variables x' is denoted by ∇', dH^{n-1} is the $(n-1)$-dimensional Hausdorff measure in $\mathbb{R}^n$ and $B'_r(x')$ denotes the ball in $\mathbb{R}^{n-1}$ of radius $r > 0$ centered at x'. When the ball is centered at the origin $x' = 0'$ or $x = 0 = (0', 0)$, we will just write B'_r or B_r.

2.2 Existence, uniqueness and global Log-Lipschitz regularity

As we mentioned in the Introduction, the notion of solution to (2.1.1) comes from computing Δu in the sense of distributions.

Definition 2.2.1 (Distributional solution)**.** We say that $u \in C_0(\overline{\Omega})$ is a distributional solution to (2.1.1) if for any $\varphi \in C_c^\infty(\Omega)$ we have

$$\int_\Omega u\Delta\varphi\, dx = \int_\Gamma g\varphi\, dH^{n-1}.$$

In this case, we write

$$\Delta u = g\, dH^{n-1}\big|_\Gamma.$$

Even though the definition of distributional solution makes sense for $u \in L^1_{\mathrm{loc}}(\Omega)$, we ask u to be continuous up to the boundary so that the boundary condition $u = 0$ is well-defined.

Recall that a bounded function $u : \overline{\Omega} \to \mathbb{R}$ is in the space $\mathrm{LogLip}(\overline{\Omega})$ if

$$[u]_{\mathrm{LogLip}(\overline{\Omega})} = \sup_{\substack{x,y\in\Omega\\ x\neq y}} \frac{|u(x)-u(y)|}{|x-y||\log|x-y||} < \infty.$$

Theorem 2.2.2 (Existence, uniqueness, and Log-Lipschitz global regularity)**.** *Let Γ be a Lipschitz interface, and $g \in L^\infty(\Gamma)$. Then the unique distributional solution $u \in C_0(\overline{\Omega})$ to* (2.1.1) *is given by*

$$u(x) = \int_\Gamma G(x,y)g(y)\, dH^{n-1} \qquad \textit{for } x \in \Omega, \tag{2.2.1}$$

where $G(x,y)$ is the Green's function for the Laplacian in Ω. Furthermore, $u \in \mathrm{LogLip}(\overline{\Omega})$ and there exists $C = C(n,\Gamma,\Omega) > 0$ such that

$$\|u\|_{L^\infty(\Omega)} + [u]_{\mathrm{LogLip}(\overline{\Omega})} \le C\|g\|_{L^\infty(\Gamma)}.$$

Proof. Let u be as in (2.2.1). By using a partition of unity on Γ, it is enough to assume that $\Gamma = \psi(\mathbb{R}^{n-1})$ where $\psi : \mathbb{R}^{n-1} \to \mathbb{R}$ is a Lipschitz function and that $g(y', \psi(y'))$ has compact support in B_1' (see Appendix 2.7). Then, for any $x \in \Omega$, we have that

$$\begin{aligned}
|u(x)| &\le \int_\Gamma |G(x,y)| g(y)\, dH_y^{n-1} \\
&= \int_{B_1'} |G(x,(y',\psi(y')))| g(y',\psi(y')) \sqrt{1+|\nabla'\psi(y')|^2}\, dy' \\
&\le C(n,\Gamma)\|g\|_{L^\infty(\Gamma)} \int_{B_1'} \frac{1}{|(x'-y', x_n - \psi(y'))|^{n-2}}\, dy' \\
&\le C(n,\Gamma)\|g\|_{L^\infty(\Gamma)} \int_{B_1'} \frac{1}{|x'-y'|^{n-2}}\, dy' \\
&\le C(n,\Gamma)\|g\|_{L^\infty(\Gamma)}.
\end{aligned}$$

Thus the integral defining u in (2.2.1) is absolutely convergent and u is bounded.

Next, for any $\varphi \in C_c^\infty(\Omega)$, by Fubini's Theorem and the symmetry $G(x,y) = G(y,x)$, we get

$$\begin{aligned}
\int_\Omega u(x)\Delta\varphi(x)\, dx &= \int_\Omega \left[\int_\Gamma G(x,y) g(y)\, dH^{n-1} \right] \Delta\varphi(x)\, dx \\
&= \int_\Gamma g(y) \int_\Omega G(y,x)\Delta_x\varphi(x)\, dx\, dH^{n-1} \\
&= \int_\Gamma g(y)\varphi(y)\, dH^{n-1}.
\end{aligned}$$

Moreover, since $G(\bar{x}, y) = 0$ for $\bar{x} \in \partial\Omega$ and $y \in \Omega$, by dominated convergence we see that $u(x)$ converges to 0 as $x \in \Omega$ converges to $\bar{x}$.

Now we show that $u \in \mathrm{LogLip}(\overline{\Omega})$. Since u is harmonic in $\Omega \setminus \Gamma$, we only need to prove the regularity of u near Γ. Suppose that $x_1, x_2 \in K$, where

$K \subset \Omega$ is a compact set containing Γ. Let $0 < d << 1$. If $|x_1 - x_2| \geq d$ then

$$|u(x_1) - u(x_2)| \leq \frac{2\|u\|_{L^\infty(\Omega)}}{d} d \leq C|x_1 - x_2|.$$

Assume next that $|x_1 - x_2| = \delta < d$. If $n \geq 3$ then, since $B_{2\delta}(x_1) \subset B_{4\delta}(x_2)$, by classical estimates for the Green's function,

$$\begin{aligned}
|u(x_1) - u(x_2)| &\leq \int_\Gamma |G(x_1, y) - G(x_2, y)||g(y)|\, dH^{n-1} \\
&\leq C_{n,K}\|g\|_{L^\infty(\Gamma)} \bigg[\int_{B_{2\delta}(x_1)\cap\Gamma} \frac{1}{|x_1 - y|^{n-2}}\, dH^{n-1} \\
&\quad + \int_{B_{4\delta}(x_2)\cap\Gamma} \frac{1}{|x_2 - y|^{n-2}}\, dH^{n-1} + \int_{\Gamma\setminus(B_{2\delta}(x_1)\cap\Gamma)} \frac{|x_1 - x_2|}{|x_1 - y|^{n-1}}\, dH^{n-1} \bigg] \\
&\leq C_{n,K,\Gamma}\|g\|_{L^\infty(\Gamma)} \bigg[\int_{B'_{2\delta}(x'_1)} \frac{1}{|x'_1 - y'|^{n-2}}\, dy' + \int_{B'_{4\delta}(x'_2)} \frac{1}{|x'_2 - y'|^{n-2}}\, dy' \\
&\quad + |x_1 - x_2| \int_{B'_1\setminus B'_{2\delta}(x'_1)} \frac{1}{|x'_1 - y'|^{n-1}}\, dy' \bigg] \\
&\leq C_{n,K,\Gamma}\|g\|_{L^\infty(\Gamma)} \big(|x_1 - x_2| + |x_1 - x_2||\log|x_1 - x_2|| \big).
\end{aligned}$$

The estimate in dimension $n = 2$ follows the same lines.

For uniqueness, if $u, v \in C_0(\overline{\Omega})$ are distributional solutions then

$$\int_\Omega (u - v)\Delta\varphi\, dx = 0 \quad \text{for every } \varphi \in C_c^\infty(\Omega).$$

Hence, $u - v \in C_0(\overline{\Omega})$ is harmonic in Ω and, as a consequence, $u \equiv v$. $\square$

Remark 2.2.3. Note that if $u \in \mathrm{LogLip}(\overline{\Omega})$ then $u \in C^{0,\gamma}(\overline{\Omega})$ for every $0 < \gamma < 1$ and there exists $C = C(\Omega, \gamma) > 0$ such that

$$[u]_{C^{0,\gamma}(\overline{\Omega})} \leq C[u]_{\mathrm{LogLip}(\overline{\Omega})}.$$

2.3 Flat interface problems

For the next results, we fix the following notation. For $a \in \mathbb{R}$ we denote

$$
\begin{aligned}
B_{r,a} &= B_r(0', a) \\
B_{r,a}^+ &= B_r(0', a) \cap \{x_n > a\} \\
B_{r,a}^- &= B_r(0', a) \cap \{x_n < a\} \\
T_{r,a} &= \{x \in B_r(0', a) : x_n = a\} \\
T_a &= B_1 \cap \{x_n = a\} \\
T_a^+ &= \{x_n \geq a\} \\
T_a^- &= \{x_n \leq a\}.
\end{aligned}
$$

When $a = 0$, we use the simplified notation $T = T_0$ and $B_r^\pm = B_{r,0}^\pm$.

Theorem 2.3.1 (Flat problem)**.** *Let $r > 0$ and $a \in \mathbb{R}$. Given $0 < \alpha, \gamma < 1$, let $g \in C^{0,\alpha}(T_{r,a})$ and $f \in C^{0,\gamma}(\partial B_{r,a})$. Then there exists a unique solution $v \in C^\infty(B_{r,a} \setminus T_{r,a}) \cap C^{0,\gamma}(\overline{B_{r,a}})$ to the flat transmission problem*

$$
\begin{cases}
\Delta v = g \, dH^{n-1}\big|_{T_{r,a}} & \text{in } B_{r,a} \\
v = f & \text{on } \partial B_{r,a}
\end{cases}
$$

that satisfies the global estimate

$$
\|v\|_{C^{0,\gamma}(\overline{B_{r,a}})} \leq C\big(\|g\|_{C^{0,\alpha}(T_{r,a})} + \|f\|_{C^{0,\gamma}(\partial B_{r,a})}\big),
$$

where $C = C(n, \alpha, \gamma, r) > 0$. Moreover, if we let $v^\pm = v\chi_{\overline{B_{r,a}^\pm}}$, then $v^\pm \in C^{1,\alpha}(\overline{B_{r/2,a}^\pm})$ and

$$
\|v^\pm\|_{C^{1,\alpha}(\overline{B_{r/2,a}^\pm})} \leq C\big(\|g\|_{C^{0,\alpha}(T_{r,a})} + \|f\|_{L^\infty(\partial B_{r,a})}\big),
$$

where $C = C(n, \alpha, r) > 0$. *If* $g \in C^{k-1,\alpha}(T_{r,a})$, $k \geq 1$, *then* $v \in C^{k,\alpha}(\overline{B^{\pm}_{r/2,a}})$ *and*

$$\|v^{\pm}\|_{C^{k,\alpha}(\overline{B^{\pm}_{r/2,a}})} \leq C\big(\|g\|_{C^{k-1,\alpha}(T_{r,a})} + \|f\|_{L^{\infty}(\partial B_{r,a})}\big),$$

where $C = C(n, \alpha, r, k) > 0$.

Proof. By subtracting from v the harmonic function h in $B_{r,a}$ that coincides with f on $\partial B_{r,a}$, it is enough to assume that $f = 0$ on $\partial B_{r,a}$. We consider only the case $k = 1$, that is, $g \in C^{0,\alpha}(T_{r,a})$. When $k \geq 1$ the proof is completely analogous. Moreover, it is sufficient to prove the result for $a = 0$ and $r = 1$. Indeed suppose that g is as in the statement, and let $\tilde{g}$ be defined on T, so that

$$g(x', x_n) = r^{-1}\tilde{g}\big(r^{-1}x', r^{-1}(x_n - a)\big),$$

whenever $x \in T_{r,a}$. If $\tilde{v}$ is the corresponding solution in B_1, then

$$v(x', x_n) = \tilde{v}\big(r^{-1}x', r^{-1}(x_n - a)\big) \qquad \text{for } x \in \overline{B_{r,a}}$$

is the unique solution to $\Delta v = g\, dH^{n-1}\big|_{T_{r,a}}$ such that $v = 0$ on $\partial B_{r,a}$. Moreover, we have the following control of the norms:

$$\begin{aligned}
\|v^{\pm}\|_{C^{1,\alpha}(\overline{B^{\pm}_{r/2,a}})} &= \|\tilde{v}^{\pm}\|_{L^{\infty}(\overline{B^{\pm}_{1/2}})} + r^{-1}\|\nabla\tilde{v}^{\pm}\|_{L^{\infty}(\overline{B^{\pm}_{1/2}})} + r^{-(1+\alpha)}[\nabla\tilde{v}^{\pm}]_{C^{0,\alpha}(\overline{B^{\pm}_{1/2}})} \\
&\leq \max\{1, r^{-1}, r^{-(1+\alpha)}\}\|\tilde{v}^{\pm}\|_{C^{1,\alpha}(\overline{B^{\pm}_{1/2}})} \\
&\leq C\max\{1, r^{-1}, r^{-(1+\alpha)}\}\|\tilde{g}\|_{C^{0,\alpha}(T)} \\
&\leq C\max\{1, r^{-1}, r^{-(1+\alpha)}\}\big(r\|g\|_{L^{\infty}(T_{r,a})} + r^{1+\alpha}[g]_{C^{0,\alpha}(T_{r,a})}\big) \\
&\leq C\|g\|_{C^{0,\alpha}(T_{r,a})},
\end{aligned}$$

and, similarly,

$$\|v\|_{C^{0,\gamma}(\overline{B_{r,a}})} \leq C\|g\|_{C^{0,\alpha}(T_{r,a})},$$

where $C > 0$ is as in the statement.

Let v^+ be the solution to the mixed boundary value problem

$$\begin{cases} \Delta v^+ = 0 & \text{in } B_1^+ \\ v^+ = 0 & \text{on } \partial B_1^+ \setminus T \\ v^+_{x_n} = g/2 & \text{on } T. \end{cases}$$

By classical elliptic regularity, $v^+ \in C^\infty(B_1^+) \cap C^{1,\alpha}(\overline{B_{1/2}^+})$ and

$$\|v^+\|_{C^{1,\alpha}(\overline{B_{1/2}^+})} \leq C_0\|g\|_{C^{0,\alpha}(T)},$$

for some $C_0 = C_0(n) > 0$. Furthermore, $v^+ \in C^{0,\gamma}(\overline{B_1^+})$. Indeed, consider the solution w to

$$\begin{cases} \Delta w = 0 & \text{in } B_2^+ \\ w = 0 & \text{on } \partial B_2^+ \setminus T_{2,0} \\ w_{x_n} = \tilde{g}/2 & \text{on } T_{2,0}, \end{cases}$$

where $\tilde{g} = g$ on T with $\|\tilde{g}\|_{C^{0,\alpha}(T_{2,0})} \leq \tilde{C}\|g\|_{C^{0,\alpha}(T)}$, for some constant $\tilde{C} > 0$. Then $w \in C^\infty(B_2^+) \cap C^{1,\alpha}(\overline{B_1^+})$ with

$$\|w\|_{C^{1,\alpha}(\overline{B_1^+})} \leq C_1\|\tilde{g}\|_{C^{0,\alpha}(T_{2,0})} \leq C_1\tilde{C}\|g\|_{C^{0,\alpha}(T)},$$

where $C_1 = C_1(n)$. Define $u(x) = v^+(x) - w(x)$, for $x \in \overline{B_1}$, and consider the even reflection extension of u to $\overline{B_1}$ given by $\tilde{u}(x', x_n) = u(x', |x_n|)$. It follows that $\tilde{u}$ is harmonic in B_1 and $\tilde{u} = -\tilde{w}$ on ∂B_1, where $\tilde{w}$ is the even reflection of w to $\overline{B_1}$. Since $\tilde{w} \in \mathrm{Lip}(\overline{B_1})$, by using the Poisson kernel in B_1 (see [26]), it can be checked that

$$\|\tilde{u}\|_{C^{0,\gamma}(\overline{B_1})} \leq C\|\tilde{w}\|_{C^{0,\gamma}(\partial B_1)} \leq C\|g\|_{C^{0,\alpha}(T)},$$

where $C = C(n, \alpha, \gamma) > 0$. Therefore, $v^+ \in C^{0,\gamma}(\overline{B_1^+})$, with the corresponding estimate for $\|v^+\|_{C^{0,\gamma}(\overline{B_1^+})}$ as in the statement. Next, the function $v^-(x', x_n) = v^+(x', -x_n)$ solves

$$\begin{cases} \Delta v^- = 0 & \text{in } B_1^- \\ v^- = 0 & \text{on } \partial B_1^- \setminus T \\ v^-_{x_n} = -g/2 & \text{on } T, \end{cases}$$

and $v^- \in C^\infty(B_1^-) \cap C^{1,\alpha}(\overline{B_{1/2}^-}) \cap C^{0,\gamma}(\overline{B_1^-})$. It follows that $v = v^+ \chi_{\overline{B_1^+}} + v^- \chi_{\overline{B_1^-}}$ is the unique distributional solution to $\Delta v = g \, dH^{n-1}\big|_T$ such that $v = 0$ on ∂B_1. Furthermore, $v \in C^\infty(B_1 \setminus T) \cap C^{0,\gamma}(\overline{B_1})$ and $v^\pm \in C^{1,\alpha}(\overline{B_{1/2}^\pm})$ with

$$\|v\|_{C^{0,\gamma}(\overline{B_1})} \leq C(n, \alpha, \gamma) \|g\|_{C^{0,\alpha}(T)}$$

and

$$\|v^\pm\|_{C^{1,\alpha}(\overline{B_{1/2}^\pm})} \leq C(n, \alpha) \|g\|_{C^{0,\alpha}(T)}.$$

□

Corollary 2.3.2. *Given $|a| < 1/4$, $c_0 > 0$, and $f \in C^{0,\gamma}(\partial B_1)$, with $0 < \gamma < 1$, there exists a unique solution $v \in C^\infty(B_1 \setminus T_a) \cap C^{0,\gamma}(\overline{B_1})$ to*

$$\begin{cases} \Delta v = c_0 \, dH^{n-1}|_{T_a} & \text{in } B_1 \\ v = f & \text{on } \partial B_1 \end{cases}$$

such that

$$\|v\|_{C^{0,\gamma}(\overline{B_1})} \leq C\big(c_0 + \|f\|_{C^{0,\gamma}(\partial B_1)}\big),$$

where $C = C(n, \gamma) > 0$ and, for any $k \geq 1$,

$$\|v^\pm\|_{C^{k,\alpha}(\overline{B_{1/2}} \cap T_a^\pm)} \leq C\big(c_0 + \|f\|_{L^\infty(\partial B_1)}\big),$$

where $C = C(n, \alpha, k) > 0$.

Proof. The global $C^{0,\gamma}$ estimate follows immediately from Theorem 2.3.1 with $g = c_0$. Hence, we only need to show the $C^{k,\alpha}$ estimate. Fix $k \geq 1$. By Theorem 2.3.1 with $r = 4$, there is a unique solution $w \in C^\infty(B_{4,a} \setminus T_{4,a}) \cap C^{0,\gamma}(\overline{B_{4,a}})$ to $\Delta w = c_0 \, dH^{n-1}\big|_{T_{4,a}}$ such that $w = 0$ on $\partial B_{4,a}$. Moreover, $\|w^\pm\|_{C^{k,\alpha}(\overline{B^\pm_{2,a}})} \leq Cc_0$, for some $C = C(n, \alpha, k) > 0$. Let h be the harmonic function in B_1 such that $h = w - f$ on ∂B_1. Then $h \in C^\infty(B_1) \cap C^{0,\gamma}(\overline{B_1})$, and

$$\|h\|_{C^{k,\alpha}(\overline{B_{1/2}})} \leq C\big(\|w\|_{L^\infty(\partial B_1)} + \|f\|_{L^\infty(\partial B_1)}\big) \leq C\big(c_0 + \|f\|_{L^\infty(\partial B_1)}\big),$$

where $C = C(n, \alpha, k) > 0$. Define $v = w - h$ on $\overline{B_1}$. Then v is the unique solution to $\Delta v = g \, dH^{n-1}|_{T_a}$ with $v = f$ on ∂B_1. Moreover,

$$\|v^\pm\|_{C^{k,\alpha}(\overline{B_{1/2}} \cap T^\pm_a)} \leq \|w^\pm\|_{C^{k,\alpha}(\overline{B^\pm_{2,a}})} + \|h\|_{C^{k,\alpha}(\overline{B_{1/2}})} \leq C\big(c_0 + \|f\|_{L^\infty(\partial B_1)}\big),$$

since $\overline{B_{1/2}} \cap T^\pm_a \subset \overline{B^\pm_{2,a}}$. $\square$

2.4 The stability result

In this section we prove our stability result, Theorem 2.4.2. As we mentioned at the beginning, our argument is based on the mean value property and, therefore, it is self-contained.

Fix $\varepsilon > 0$, and define the sets

$$\Omega_\varepsilon = \{x \in \Omega : d(x, \partial\Omega) > \varepsilon\} \quad \text{and} \quad \Gamma_\varepsilon = \{x \in \Omega : d(x, \Gamma) < \varepsilon\}.$$

Consider the average function:

$$u_\varepsilon(x) = \frac{1}{|B_\varepsilon|} \int_{B_\varepsilon(x)} u(y) \, dy \qquad \text{for} \quad x \in \Omega_\varepsilon.$$

Proposition 2.4.1 (Properties of averages)**.** *Let u be the distributional solution given in Theorem 2.2.2. The following properties hold:*

(i) *If $B_\varepsilon(x) \cap \Gamma = \varnothing$, then $u_\varepsilon(x) = u(x)$.*

(ii) *$u_\varepsilon \to u$ uniformly in compact subsets of Ω, as $\varepsilon \to 0$.*

(iii) *If $g \in L^\infty(\Gamma)$, then $g_\varepsilon \in C_c(\Gamma_\varepsilon)$, where*

$$g_\varepsilon(x) = \frac{1}{|B_\varepsilon|} \int_{\Gamma \cap B_\varepsilon(x)} g(y)\, dH^{n-1} \qquad \text{for } \ x \in \Gamma_\varepsilon.$$

Moreover, $\Delta u_\varepsilon(x) = g_\varepsilon(x)$ for any $x \in \Omega_\varepsilon$.

Proof. Since u is harmonic outside of Γ, (i) is immediate by the mean value property.

For (ii), recall by Remark 2.2.3 that $u \in C^{0,\gamma}(\overline{\Omega})$. Therefore,

$$|u_\varepsilon(x) - u(x)| \leq \frac{1}{|B_\varepsilon|} \int_{B_\varepsilon(x)} |u(y) - u(x)|\, dy \leq C \|g\|_{L^\infty(\Gamma)} \varepsilon^\gamma \to 0,$$

as $\varepsilon \to 0$.

We now show (iii). If $g \in L^\infty(\Gamma)$, then by dominated convergence, $g_\varepsilon \in C_c(\Gamma_\varepsilon)$. Moreover, for any $\varphi \in C_c^\infty(\Omega)$, we have

$$
\begin{aligned}
(\Delta u_\varepsilon)(\varphi) &= \int_\Omega u_\varepsilon(x)\Delta\varphi(x)\,dx \\
&= \frac{1}{|B_\varepsilon|}\int_{B_\varepsilon}\int_\Omega u(x+y)\Delta\varphi(x)\,dx\,dy \\
&= \frac{1}{|B_\varepsilon|}\int_{B_\varepsilon}\int_\Omega u(z)\Delta\varphi(z-y)\,dz\,dy \\
&= \frac{1}{|B_\varepsilon|}\int_{B_\varepsilon}\int_\Gamma g(z)\varphi(z-y)\,dH^{n-1}_z\,dy \\
&= \frac{1}{|B_\varepsilon|}\int_\Gamma\left[\int_{B_\varepsilon}\varphi(z-y)\,dy\right]g(z)\,dH^{n-1}_z \\
&= \frac{1}{|B_\varepsilon|}\int_\Gamma\left[\int_\Omega \chi_{B_\varepsilon}(z-y)\varphi(y)\,dy\right]g(z)\,dH^{n-1}_z \\
&= \frac{1}{|B_\varepsilon|}\int_\Omega\int_\Gamma \chi_{B_\varepsilon}(z-y)\,g(z)\,dH^{n-1}_z\,\varphi(y)\,dy \\
&= \int_\Omega\left[\frac{1}{|B_\varepsilon|}\int_{\Gamma\cap B_\varepsilon(y)} g(z)\,dH^{n-1}_z\right]\varphi(y)\,dy = \int_\Omega g_\varepsilon(y)\varphi(y)\,dy.
\end{aligned}
$$

□

Theorem 2.4.2 (Stability). *Let $0<\varepsilon,\theta<1/2$ and $0<\delta,\gamma<1$ be given, and let $\Gamma=\{(y',\psi(y')):y'\in B_1'\}$, where ψ is a Lipschitz function. Assume that Γ is $\theta\varepsilon$-flat in B_1, in the sense that*

$$\Gamma\subset\{x\in B_1:|x_n|<\theta\varepsilon\},$$

and that Γ is also ε-horizontal in B_1, that is,

$$1-\varepsilon\le\nu(x)\cdot(0',1)=\left(1+|\nabla'\psi(x')|^2\right)^{-1/2}\le 1,$$

for every $x\in\Gamma$, where $\nu(x)$ denotes the upward pointing normal on Γ. Then there exists $C=C(n,\gamma)>0$ such that for any $u\in C(\overline{B_1})$ and $g\in L^\infty(\Gamma)$

satisfying

$$\begin{cases} \Delta u = g\, dH^{n-1}\big|_{\Gamma} & \text{in } B_1 \\ |u| \le 1 & \text{in } B_1 \\ |g-1| \le \delta & \text{on } \Gamma, \end{cases}$$

the classical solution $v \in C^\infty(B_{3/4} \setminus T_{-\theta\varepsilon}) \cap C^{0,\gamma}(\overline{B_{3/4}})$ *to the flat problem*

$$\begin{cases} \Delta v = dH^{n-1}\big|_{T_{-\theta\varepsilon}} & \text{in } B_{3/4} \\ v = u & \text{on } \partial B_{3/4} \end{cases}$$

satisfies

$$|u - v| \le C(\theta + \delta + \varepsilon^\gamma) \qquad \text{in } B_{1/2}.$$

Remark 2.4.3. The interface for the flat problem in Theorem 2.4.2 is $T_{-\theta\varepsilon} = B_{3/4} \cap \{x_n = -\theta\varepsilon\}$, which lies below Γ in the x_n-direction. To approximate u with the solution to a flat problem where the interface lies above Γ in the x_n-direction, it is enough to consider the classical solution v to

$$\begin{cases} \Delta v = dH^{n-1}\big|_{T_{\theta\varepsilon}} & \text{in } B_{3/4} \\ v = u & \text{on } \partial B_{3/4}. \end{cases}$$

In this case, the same conclusion as in Theorem 2.4.2 holds.

Before we give the proof, we need the following geometric result.

Lemma 2.4.4. *Let* Γ *be as in Theorem 2.4.2. Define* $M = 1 + 2\theta$ *and let* $x \in B_{3/4 - M\varepsilon}$ *be such that* $\operatorname{dist}(x, \Gamma) < \varepsilon$*. Then*

$$\begin{aligned} \{y' : (y', \psi(y')) \in B_\varepsilon(x)\} &\subset B'_{((M\varepsilon)^2 - (x_n + \theta\varepsilon)^2)^{1/2}}(x') \\ &= \{y' : (y', -\theta\varepsilon) \in B_{M\varepsilon}(x)\} \end{aligned} \tag{2.4.1}$$

and

$$\{y' : (y', \psi(y')) \in B_{M\varepsilon}(x)\} \supset B'_{(\varepsilon^2-(x_n+\theta\varepsilon)^2)^{1/2}}(x')$$
$$= \{y' : (y', -\theta\varepsilon) \in B_\varepsilon(x)\}. \qquad (2.4.2)$$

We illustrate this result in Figure 2.2.

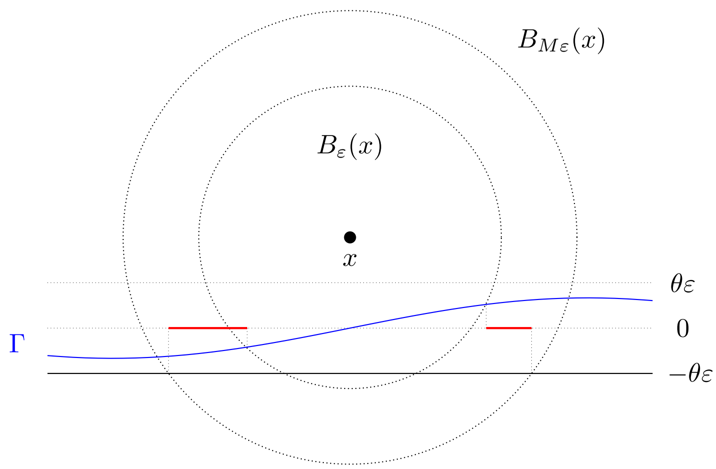

Figure 2.2: The red set is $\{y' : (y', -\theta\varepsilon) \in B_{M\varepsilon}(x)\} \setminus \{y' : (y', \psi(y')) \in B_\varepsilon(x)\}$.

Proof. If x is as in the statement then, by the flatness condition on Γ, we have $|x_n| < (1+\theta)\varepsilon$. Let us prove (2.4.1). Suppose first that $-\theta\varepsilon < x_n < \theta\varepsilon$. Then

$$\{y' : (y', \psi(y')) \in B_\varepsilon(x)\} \subset \{y' : (y', x_n) \in B_\varepsilon(x)\} = B'_\varepsilon(x').$$

Since

$$(M\varepsilon)^2 - (x_n + \theta\varepsilon)^2 = (1+2\theta)^2\varepsilon^2 - (x_n+\theta\varepsilon)^2$$
$$\geq (1 + 4\theta + 4\theta^2)\varepsilon^2 - (2\theta\varepsilon)^2 = \varepsilon^2 + 4\theta\varepsilon^2 > \varepsilon^2,$$

we see that $B'_\varepsilon(x') \subset B'_{((M\varepsilon)^2-(x_n+\theta\varepsilon)^2)^{1/2}}(x')$, and the conclusion follows. Assume now that $\theta\varepsilon \leq x_n < (1+\theta)\varepsilon$. Notice that

$$\{y' : (y', \psi(y')) \in B_\varepsilon(x)\} \subset \{y' : (y', \theta\varepsilon) \in B_\varepsilon(x)\} = B'_{(\varepsilon^2-(x_n-\theta\varepsilon)^2)^{1/2}}(x').$$

Since

$$\begin{aligned}(M\varepsilon)^2 - &(x_n+\theta\varepsilon)^2 - (\varepsilon^2 - (x_n-\theta\varepsilon)^2) \\ &= (1+2\theta)^2\varepsilon^2 - (x_n^2 + 2\theta\varepsilon x_n + (\theta\varepsilon)^2) - \varepsilon^2 + (x_n^2 - 2\theta\varepsilon x_n + (\theta\varepsilon)^2) \\ &= 4\theta\varepsilon^2 + 4\theta^2\varepsilon^2 - 4\theta\varepsilon x_n \geq 4\theta\varepsilon^2 \geq 0,\end{aligned}$$

we find that $B'_{(\varepsilon^2-(x_n-\theta\varepsilon)^2)^{1/2}}(x') \subset B'_{((M\varepsilon)^2-(x_n+\theta\varepsilon)^2)^{1/2}}(x')$, as desired. The last case is when $-(1+\theta)\varepsilon < x_n \leq -\theta\varepsilon$. Here it is clear that, since $M > 1$,

$$\begin{aligned}\{y' : (y', \psi(y')) \in B_\varepsilon(x)\} &\subset \{y' : (y', -\theta\varepsilon) \in B_\varepsilon(x)\} \\ &= B'_{(\varepsilon^2-(x_n+\theta\varepsilon)^2)^{1/2}}(x') \\ &\subset B'_{((M\varepsilon)^2-(x_n+\theta\varepsilon)^2)^{1/2}}(x').\end{aligned}$$

This concludes the proof of (2.4.1).

For (2.4.2), notice that if $x_n \geq (1-\theta)\varepsilon$, then the inclusion follows as $\{y' : (y', -\theta\varepsilon) \in B_\varepsilon(x)\} = \varnothing$. We therefore assume that $-(1+\theta)\varepsilon < x_n < (1-\theta)\varepsilon$. If $x_n \geq -\theta\varepsilon$, then

$$\begin{aligned}\{y' : (y', \psi(y')) \in B_{M\varepsilon}(x))\} &\supset \{y' : (y', -\theta\varepsilon) \in B_{M\varepsilon}(x)\} \\ &= B'_{((M\varepsilon)^2-(x_n+\theta\varepsilon)^2)^{1/2}}(x') \\ &\supset B'_{(\varepsilon^2-(x_n+\theta\varepsilon)^2)^{1/2}}(x')\end{aligned}$$

because $M > 1$. If $-(1+\theta)\varepsilon < x_n < -\theta\varepsilon$, then

$$\begin{aligned}\{y' : (y', \psi(y')) \in B_{M\varepsilon}(x))\} &\supset \{y' : (y', \theta\varepsilon) \in B_{M\varepsilon}(x)\} \\ &= B'_{((M\varepsilon)^2-(x_n-\theta\varepsilon)^2)^{1/2}}(x')\end{aligned}$$

and

$$\begin{aligned}&(M\varepsilon)^2 - (x_n - \theta\varepsilon)^2 - (\varepsilon^2 - (x_n + \theta\varepsilon)^2) \\ &\quad = (1+2\theta)^2\varepsilon^2 - (x_n^2 - 2\theta\varepsilon x_n + (\theta\varepsilon)^2) - \varepsilon^2 + (x_n^2 + 2\theta\varepsilon x_n + (\theta\varepsilon)^2) \\ &\quad = 4\theta\varepsilon^2 + 4\theta^2\varepsilon^2 + 4\theta\varepsilon x_n \geq 0.\end{aligned}$$

Therefore,

$$B'_{((M\varepsilon)^2-(x_n-\theta\varepsilon)^2)^{1/2}}(x') \supset B'_{(\varepsilon^2-(x_n+\theta\varepsilon)^2)^{1/2}}(x'),$$

and we conclude that (2.4.2) holds. □

Proof of Theorem 2.4.2. Let $M = 1 + 2\theta > 1$. By Corollary 2.3.2 with $a = -\theta\varepsilon$, $c_0 = M^n(1+\delta)(1-\varepsilon)^{-1}$ and $B_{3/4}$ in place of B_1, there is a unique classical solution $\underline{w}$ to the flat transmission problem

$$\begin{cases}\Delta\underline{w} = M^n(1+\delta)(1-\varepsilon)^{-1}\, dH^{n-1}\big|_{T_{-\theta\varepsilon}} & \text{in } B_{3/4} \\ \underline{w} = u & \text{on } \partial B_{3/4}.\end{cases}$$

Moreover, by subtracting the harmonic function h in B_1 such that $h = u$ on ∂B_1 and applying similar arguments as in the proof of Theorem 2.2.2, it can be seen that $u \in C^{0,\gamma}(\overline{B_{3/4}})$ with

$$\|u\|_{C^{0,\gamma}(\overline{B_{3/4}})} \leq C(n, \gamma, \Gamma)(\|u\|_{L^\infty(B_1)} + \|g\|_{L^\infty(\Gamma)}) \leq C,$$

where $C = C(n,\gamma) > 0$ because Γ is ε-horizontal, $\|u\|_{L^\infty(B_1)} \leq 1$, and $\|g - 1\|_{L^\infty(\Gamma)} \leq \delta$. Hence, by Corollary 2.3.2 with $B_{3/4}$ in place of B_1, $\underline{w} \in C^\infty(B_{3/4} \setminus T_{-\theta\varepsilon}) \cap C^{0,\gamma}(\overline{B_{3/4}})$ with

$$\|\underline{w}\|_{C^{0,\gamma}(\overline{B_{3/4}})} \leq C(n,\gamma)(c_0 + \|u\|_{C^{0,\gamma}(\overline{B_{3/4}})}) \leq C,$$

where $C = C(n,\gamma) > 0$.

Define the averages

$$u_\varepsilon(x) = \frac{1}{|B_\varepsilon|} \int_{B_\varepsilon(x)} u(y)\, dy \qquad \text{for } x \in B_{3/4-\varepsilon} \subset B_{3/4}$$

and

$$\underline{w}_{M\varepsilon}(x) = \frac{1}{|B_{M\varepsilon}|} \int_{B_{M\varepsilon}(x)} \underline{w}(y)\, dy \qquad \text{for } x \in B_{3/4-M\varepsilon} \subset B_{3/4}.$$

By Proposition 2.4.1(iii), $\Delta u_\varepsilon(x) = g_\varepsilon(x)$ for every $x \in B_{3/4-\varepsilon}$, and

$$\Delta \underline{w}_{M\varepsilon}(x) = \frac{1}{|B_{M\varepsilon}|} \int_{B_{M\varepsilon}(x) \cap T_{-\theta\varepsilon}} M^n (1+\delta)(1-\varepsilon)^{-1}\, dH^{n-1} \qquad \text{for } x \in B_{3/4-M\varepsilon}.$$

In addition, notice that

$$\operatorname{supp}(\Delta u_\varepsilon) \subset \{x \in B_{3/4-\varepsilon} : \operatorname{dist}(x,\Gamma) < \varepsilon\}$$

and

$$\operatorname{supp}(\Delta \underline{w}_{M\varepsilon}) \subset \{x \in B_{3/4-M\varepsilon} : |x_n| < M\varepsilon\}.$$

Since Γ is $\theta\varepsilon$-flat in B_1 and $M = 1 + 2\theta$ it follows that

$$\operatorname{supp}(\Delta u_\varepsilon) \subset \operatorname{supp}(\Delta \underline{w}_{M\varepsilon}).$$

Let us first show that

$$\Delta \underline{w}_{M\varepsilon} \geq \Delta u_\varepsilon \qquad \text{in } B_{3/4-M\varepsilon}.$$

If $x \notin \operatorname{supp}(g_\varepsilon)$ there is nothing to prove because $\Delta \underline{w}_{M\varepsilon} \geq 0$ in $B_{3/4-M\varepsilon}$. Let us then take $x \in B_{3/4-M\varepsilon}$ such that $\operatorname{dist}(x, \Gamma) < \varepsilon$. Using that $0 < g \leq 1+\delta$, Γ is ε-horizontal and (2.4.1) in Lemma 2.4.4, we get

$$\begin{aligned}
\Delta \underline{w}_{M\varepsilon}(x) &= \frac{1}{M^n |B_\varepsilon|} \int_{B_{M\varepsilon}(x) \cap T_{-\theta\varepsilon}} M^n (1+\delta)(1-\varepsilon)^{-1} \, dH^{n-1} \\
&\geq \frac{1}{|B_\varepsilon|} \int_{\{y' : (y', -\theta\varepsilon) \in B_{M\varepsilon}(x)\}} g(y', \psi(y')) \sqrt{1 + |\nabla' \psi(y')|^2} \, dy' \\
&\geq \frac{1}{|B_\varepsilon|} \int_{\{y' : (y', \psi(y')) \in B_\varepsilon(x)\}} g(y', \psi(y')) \sqrt{1 + |\nabla' \psi(y')|^2} \, dy' \\
&= \frac{1}{|B_\varepsilon|} \int_{B_\varepsilon(x) \cap \Gamma} g \, dH^{n-1} = \Delta u_\varepsilon(x).
\end{aligned}$$

We also have

$$\underline{w}_{M\varepsilon} \leq u_\varepsilon + C\varepsilon^\gamma \qquad \text{on } \partial B_{3/4-M\varepsilon},$$

for some $C = C(n,\gamma) > 0$. Indeed, fix any $x \in \partial B_{3/4-M\varepsilon}$, and let $z \in \partial B_{3/4}$ be such that $\operatorname{dist}(x, \partial B_{3/4}) = |x - z| = M\varepsilon$. By using that $\underline{w}, u \in C^{0,\gamma}(\overline{B_{3/4}})$ and $\underline{w} = u$ on $\partial B_{3/4}$,

$$\begin{aligned}
\underline{w}_{M\varepsilon}(x) - u_\varepsilon(x) &= (\underline{w}_{M\varepsilon}(x) - \underline{w}(x)) + (\underline{w}(x) - \underline{w}(z)) \\
&\quad + (u(z) - u(x)) + (u(x) - u_\varepsilon(x)) \\
&\leq \frac{1}{|B_{M\varepsilon}|} \int_{B_{M\varepsilon}(x)} |\underline{w}(y) - \underline{w}(x)| \, dy \\
&\quad + \big([\underline{w}]_{C^{0,\gamma}(\overline{B_{3/4}})} + [u]_{C^{0,\gamma}(\overline{B_{3/4}})} \big) |x - z|^\gamma \\
&\quad + \frac{1}{|B_\varepsilon|} \int_{B_\varepsilon(x)} |u(y) - u(x)| \, dy \leq C\varepsilon^\gamma,
\end{aligned} \tag{2.4.3}$$

where $C = C(n,\gamma) > 0$. Hence, by the maximum principle, $\underline{w}_{M\varepsilon} - u_\varepsilon \leq C\varepsilon^\gamma$ in $B_{3/4-M\varepsilon}$. Consequently, by arguing similarly as in (2.4.3), it follows that, for some $C = C(n,\gamma) > 0$,

$$\underline{w} - u \le C\varepsilon^\gamma \qquad \text{in } B_{3/4-M\varepsilon}. \tag{2.4.4}$$

Secondly, consider the classical solution $\bar{w}$ to the flat transmission problem

$$\begin{cases} \Delta \bar{w} = M^{-n}(1-\delta)\, dH^{n-1}\big|_{T_{-\theta\varepsilon}} & \text{in } B_{3/4} \\ \bar{w} = u & \text{on } \partial B_{3/4}, \end{cases}$$

and the corresponding averages $\bar{w}_\varepsilon$ and $u_{M\varepsilon}$ of $\bar{w}$ and u, respectively. Since $g \ge 1-\delta$, by (2.4.2) in Lemma 2.4.4 we find that

$$\begin{aligned} \Delta \bar{w}_\varepsilon(x) &= \frac{1}{|B_\varepsilon|}\int_{B_\varepsilon(x)\cap T_{-\theta\varepsilon}} M^{-n}(1-\delta)\, dH^{n-1} \\ &\le \frac{1}{|B_{M\varepsilon}|}\int_{\{y':(y',-\theta\varepsilon)\in B_\varepsilon(x)\}} g(y',\psi(y'))\sqrt{1+|\nabla'\psi(y')|^2}\, dy' \\ &\le \frac{1}{|B_{M\varepsilon}|}\int_{\{y':(y',\psi(y'))\in B_{M\varepsilon}(x)\}} g(y',\psi(y'))\sqrt{1+|\nabla'\psi(y')|^2}\, dy' \\ &= \frac{1}{|B_{M\varepsilon}|}\int_{B_{M\varepsilon}(x)\cap\Gamma} g\, dH^{n-1} = \Delta u_{M\varepsilon}(x). \end{aligned}$$

By using parallel arguments to those in (2.4.3) we also get that

$$u - \bar{w} \le C\varepsilon^\gamma \qquad \text{in } B_{3/4-M\varepsilon}. \tag{2.4.5}$$

for some $C = C(n,\gamma) > 0$. Define $w = \frac{\underline{w}+\bar{w}}{2}$. By (2.4.4) and (2.4.5),

$$u - w \le \bar{w} + C\varepsilon^\gamma - \frac{\underline{w}+\bar{w}}{2} = \frac{\bar{w}-\underline{w}}{2} + C\varepsilon^\gamma$$

and

$$u - w \le \underline{w} - C\varepsilon^\gamma - \frac{\underline{w}+\bar{w}}{2} = \frac{\underline{w}-\bar{w}}{2} - C\varepsilon^\gamma.$$

Hence,

$$\|u-w\|_{L^\infty(B_{1/2})} \le \frac{1}{2}\|\bar{w}-\underline{w}\|_{L^\infty(B_{1/2})} + C\varepsilon^\gamma,$$

where $C = C(n,\gamma) > 0$. Since

$$\begin{cases} \Delta(\bar{w} - \underline{w}) = [M^{-n}(1-\delta) - M^n(1+\delta)(1-\varepsilon)^{-1}]\, dH^{n-1}\big|_{T_{-\theta\varepsilon}} & \text{in } B_{3/4} \\ \bar{w} - \underline{w} = 0 & \text{on } \partial B_{3/4}, \end{cases}$$

by Theorem 2.3.1,

$$\|\bar{w} - \underline{w}\|_{L^\infty(B_{1/2})} \leq C|M^n(1+\delta)(1-\varepsilon)^{-1} - M^{-n}(1-\delta)| \leq C(\theta + \delta + \varepsilon),$$

for some $C = C(n) > 0$. Therefore,

$$\|u - w\|_{L^\infty(B_{1/2})} \leq C(\theta + \delta + \varepsilon^\gamma) \tag{2.4.6}$$

for some $C = C(n,\gamma) > 0$. Also, $\Delta w = (1+\eta)\, dH^{n-1}\big|_{T_{-\theta\varepsilon}}$, where

$$1 + \eta = \frac{M^n(1+\delta)(1-\varepsilon)^{-1} + M^{-n}(1-\delta)}{2}.$$

Observe that, since $0 < \theta, \varepsilon < 1/2$, $0 < \delta < 1$, it follows that

$$\begin{aligned} |\eta| &= \frac{|M^{2n}(1+\delta) + (1-\delta)(1-\varepsilon) - 2(1-\varepsilon)M^n|}{2(1-\varepsilon)M^n} \\ &\leq C\big(|(1+2\theta)^{2n} + 1 - 2(1+2\theta)^n| + \delta + \varepsilon\big) \leq C(\theta + \delta + \varepsilon), \end{aligned} \tag{2.4.7}$$

where $C = C(n) > 0$.

Let $v \in C^\infty(B_{3/4} \setminus T_{-\theta\varepsilon}) \cap C^{0,\gamma}(\overline{B_{3/4}})$ be the solution to

$$\begin{cases} \Delta v = dH^{n-1}\big|_{T_{-\theta\varepsilon}} & \text{in } B_{3/4} \\ v = u & \text{on } \partial B_{3/4} \end{cases}$$

(see Corollary 2.3.2). Then $v - w$ solves

$$\begin{cases} \Delta(v - w) = \eta\, dH^{n-1}\big|_{T_{-\theta\varepsilon}} & \text{in } B_{3/4} \\ v - w = 0 & \text{on } \partial B_{3/4}. \end{cases}$$

Therefore, by (2.4.7),

$$\|v-w\|_{L^\infty(B_{3/4})} \leq C|\eta| \leq C(\theta+\delta+\varepsilon), \tag{2.4.8}$$

where $C = C(n) > 0$. From (2.4.6) and (2.4.8) the estimate on the statement is proved. $\square$

Remark 2.4.5 (Divergence form equations)**.** Recall that our proof of Theorem 2.4.2 is self-contained, based on the mean value property for harmonic functions and the maximum principle. In view of recently developed mean value formulas for solutions to divergence form elliptic equations by Blank–Hao [4], the natural question of extending our geometric techniques to transmission problems for divergence form elliptic equations arise. For this case, our maximum principle techniques must be replaced by energy methods. More importantly, not much is known about the geometry of the mean value sets from [4], so it is not clear at all how to mimic geometric arguments such as those in Lemma 2.4.4.

Remark 2.4.6 (Nondivergence form equations)**.** The second natural question would be to extend our methods to transmission problems with nondivergence form elliptic equations, where the maximum principle is a more adequate tool. In this situation, not only there are no useful mean value formulas available, but also the notion of distributional solution we consider in this work does not apply anymore. We approach this problem in Chapters 3 and 4, including more general equations.

2.5 Pointwise $C^{1,\alpha}$ boundary estimates

Throughout this section, Γ is an interface in B_1 given by the graph of a function $x_n = \psi(x') : T \to \mathbb{R}$. Thus, we can write $B_1 = \Omega_1 \cup \Gamma \cup \Omega_2$, where $\Omega_1 = \{x = (x', x_n) \in B_1 : x_n > \psi(x')\}$. We also assume that $0 \in \Gamma$.

2.5.1 Preliminary lemmas

Lemma 2.5.1. *Let $\Gamma = \{(y', \psi(y')) : y' \in B_1'\}$, where ψ is a Lipschitz function. Given $0 < \alpha, \gamma < 1$, there exist constants $C_0 > 0$, $0 < \lambda < 1/2$, $0 < \theta, \delta, \varepsilon < \lambda$ depending only on n, α and γ, such that for any $u \in C(\overline{B_1})$ satisfying*

$$\begin{cases} \Delta u = g\, dH^{n-1}\big|_{\Gamma} & \text{in } B_1 \\ |u| \le 1 & \text{in } B_1 \\ |g - 1| \le \delta & \text{on } \Gamma, \end{cases}$$

if Γ is $\theta\varepsilon$-flat and ε-horizontal in B_1, then there are linear polynomials $P_1(x) = A\cdot x + B$ and $Q_1(x) = C\cdot x + B$, with $A, C \in \mathbb{R}^n$, $B \in \mathbb{R}$, and $|A| + |B| + |C| \le C_0$, such that

$$|u_1(x) - P_1(x)| \le \lambda^{1+\alpha} \qquad \text{for all } x \in \Omega_1 \cap B_\lambda,$$

$$|u_2(x) - Q_1(x)| \le \lambda^{1+\alpha} \qquad \text{for all } x \in \Omega_2 \cap B_\lambda.$$

Moreover, $\nabla' P_1 = \nabla' Q_1$ and $(P_1)_{x_n} - (Q_1)_{x_n} = 1$.

Proof. Fix $0 < \theta, \delta, \varepsilon < \lambda < 1/2$ to be chosen later. Consider the solutions

$$\underline{v} = \underline{v}^+ \chi_{\overline{B_{3/4}} \cap T^+_{-\theta\varepsilon}} + \underline{v}^- \chi_{\overline{B_{3/4}} \cap T^-_{-\theta\varepsilon}},$$

$$\bar{v} = \bar{v}^+ \chi_{\overline{B_{3/4}} \cap T^+_{\theta\varepsilon}} + \bar{v}^- \chi_{\overline{B_{3/4}} \cap T^-_{\theta\varepsilon}},$$

to the flat transmission problems given in Theorem 2.4.2, and Remark 2.4.3, respectively. By Corollary 2.3.2 with $k = 2$,

$$\|\underline{v}^+\|_{C^{2,\alpha}(\overline{B_{1/2}}\cap T^+_{-\theta\varepsilon})} + \|\bar{v}^-\|_{C^{2,\alpha}(\overline{B_{1/2}}\cap T^-_{\theta\varepsilon})} \leq C\big(1 + \|u\|_{L^\infty(B_1)}\big) \leq C_0,$$

for some $C_0 = C_0(n, \alpha) > 0$. In particular,

$$|\underline{v}(0)| + |\nabla \underline{v}(0)| + |\bar{v}(0)| + |\nabla \bar{v}(0)| \leq C_0.$$

Let h be the harmonic function in $B_{3/4}$ such that $h = u$ on $\partial B_{3/4}$. Define

$$P_1(x) = \underline{v}(0) + \nabla \underline{v}(0) \cdot x + \big[\tfrac{1}{2} - \underline{v}_{x_n}(0) + h_{x_n}(0)\big] x_n,$$
$$Q_1(x) = \bar{v}(0) + \nabla \bar{v}(0) \cdot x + \big[-\tfrac{1}{2} - \bar{v}_{x_n}(0) + h_{x_n}(0)\big] x_n.$$

Then P_1 and Q_1 are small perturbations of the linear parts of $\underline{v}$ and $\bar{v}$ at the origin, respectively. To see this, first note that the functions $\underline{v}(x', x_n) - h(x', x_n)$ and $\bar{v}(x', -x_n) - h(x', -x_n)$ satisfy the same transmission problem on $T_{-\theta\varepsilon}$ with zero data on $\partial B_{3/4}$. By uniqueness,

$$\underline{v}(x', x_n) - h(x', x_n) = \bar{v}(x', -x_n) - h(x', -x_n) \qquad \text{for all } x \in \overline{B_{3/4}}.$$

In particular, $\underline{v}(x', 0) = \bar{v}(x', 0)$, $\nabla' \underline{v}(x', 0) = \nabla' \bar{v}(x', 0)$, and thus, $P_1(0) = Q_1(0)$, and $\nabla' P_1 = \nabla' \underline{v}(0) = \nabla' \bar{v}(0) = \nabla' Q_1$. Clearly, $(P_1)_{x_n} - (Q_1)_{x_n} = 1$. Moreover,

$$\underline{v}_{x_n}(x', 0) - h_{x_n}(x', 0) = -\bar{v}_{x_n}(x', 0) + h_{x_n}(x', 0),$$

and thus, $\big|\tfrac{1}{2} - \underline{v}_{x_n}(0) + h_{x_n}(0)\big| = \big|-\tfrac{1}{2} - \bar{v}_{x_n}(0) + h_{x_n}(0)\big|$. Let us show that

$$\big|\tfrac{1}{2} - \underline{v}_{x_n}(0) + h_{x_n}(0)\big| \leq D(\theta\varepsilon)^\gamma, \tag{2.5.1}$$

for some $D = D(n) > 0$. Recall that by the construction of $\underline{v}$ in Corollary 2.3.2, we can write $\underline{v} = w - H$, where $w \in C^\infty(B_{4,-\theta\varepsilon} \setminus T_{-\theta\varepsilon}) \cap C^{0,\gamma}(\overline{B_{4,-\theta\varepsilon}})$ is the harmonic function in $B_{4,-\theta\varepsilon}$ such that $w = 0$ on $\partial B_{4,-\theta\varepsilon}$, and H is the harmonic function in B_1, with $H = w - u$ on $\partial B_{3/4}$. Then

$$\left|\tfrac{1}{2} - \underline{v}_{x_n}(0) + h_{x_n}(0)\right| \leq \left|w_{x_n}(0) - \tfrac{1}{2}\right| + |(H+h)_{x_n}(0)|.$$

In particular, $w_{x_n}(0) = w^+_{x_n}(0)$, where w^+ is the harmonic function in $B^+_{4,-\theta\varepsilon}$ such that $w = 0$ on $\partial B^+_{4,-\theta\varepsilon} \setminus T_{-\theta\varepsilon}$, and $w^+_{x_n} = \frac{1}{2}$ on $T_{-\theta\varepsilon}$. By the mean value theorem,

$$w_{x_n}(0) - \tfrac{1}{2} = w^+_{x_n}(0', 0) - w^+_{x_n}(0', -\theta\varepsilon) = w^+_{x_n x_n}(0', \xi)\theta\varepsilon,$$

for some $-\theta\varepsilon \leq \xi \leq 0$. Moreover, by Theorem 2.3.1, $\|w^+\|_{C^{2,\alpha}(\overline{B^+_{2,-\theta\varepsilon}})} \leq D_0$, for some constant $D_0 = D_0(n) > 0$. Hence,

$$|w_{x_n}(0) - \tfrac{1}{2}| \leq D_0\theta\varepsilon.$$

Next, note that $H + h$ is harmonic in B_1, and $H + h = w$ on $\partial B_{3/4}$. Consider the harmonic function ϕ in $B_{3/4-\theta\varepsilon,-\theta\varepsilon}$ such that $\phi = w$ on $B_{3/4-\theta\varepsilon,-\theta\varepsilon}$. Observe that $B_{3/4-\theta\varepsilon,-\theta\varepsilon} \subset B_{3/4}$. Since w is symmetric with respect to the plane $T_{-\theta\varepsilon}$, it follows that $\phi_{x_n}(x', -\theta\varepsilon) = 0$ for any $(x', -\theta\varepsilon) \in B_{3/4-\theta\varepsilon,-\theta\varepsilon}$. Therefore, $|\phi_{x_n}(0)| \leq D_0\theta\varepsilon$. By interior estimates, the maximum principle, and the facts that $w \in C^{0,\gamma}(\overline{B_{3/4}})$ and $\operatorname{dist}(\partial B_{3/4}, \partial B_{3/4-\theta\varepsilon,-\theta\varepsilon}) \leq 2\theta\varepsilon$,

$$|(H+h)_{x_n}(0) - \phi_{x_n}(0)| \leq D_1\|(H+h) - w\|_{L^\infty(\partial B_{3/4-\theta\varepsilon,-\theta\varepsilon})} \leq D_1(\theta\varepsilon)^\gamma,$$

for some $D_1 = D_1(n) > 0$, and thus,

$$|(H+h)_{x_n}(0)| \leq D_1(\theta\varepsilon)^\gamma + |\phi_{x_n}(0)| \leq D_1(\theta\varepsilon)^\gamma + D_0\theta\varepsilon \leq D(\theta\varepsilon)^\gamma,$$

for some $D = D(n) > 0$. Therefore, (2.5.1) holds.

If $x \in \Omega_1 \cap B_{1/2}$, by Theorem 2.4.2 and (2.5.1), there are constants $C, D > 0$, depending only on n, such that

$$\begin{aligned}
|u_1(x) - P_1(x)| &\leq |u(x) - \underline{v}(x)| + |\underline{v}(x) - P_1(x)| \\
&\leq |u(x) - \underline{v}(x)| + |\underline{v}(x) - \underline{v}(0) - \nabla \underline{v}(0)| \\
&\qquad + \left|\tfrac{1}{2} - \underline{v}_{x_n}(0) + h_{x_n}(0)\right||x_n| \\
&\leq C(\theta + \delta + \varepsilon^\gamma) + \|D^2 \underline{v}\|_{L^\infty(\Omega_1 \cap B_{1/2})}|x|^2 + D(\theta\varepsilon)^\gamma |x_n| \\
&\leq C(\theta + \delta + \varepsilon^\gamma) + C_0|x|^2 + D(\theta\varepsilon)^\gamma |x_n|.
\end{aligned}$$

Similarly, if $x \in \Omega_2 \cap B_{1/2}$,

$$|u_2(x) - Q_1(x)| \leq C(\theta + \delta + \varepsilon^\gamma) + C_0|x|^2 + D(\theta\varepsilon)^\gamma |x_n|.$$

First, choose $0 < \lambda < 1/2$ such that

$$C_0|x|^2 \leq \frac{\lambda^{1+\alpha}}{2} \qquad \text{for all } x \in B_\lambda.$$

Then, choose $0 < \theta, \delta, \varepsilon < \lambda$ such that

$$C(\theta + \delta + \varepsilon^\gamma) + D(\theta\varepsilon)^\gamma \lambda \leq \frac{\lambda^{1+\alpha}}{2}.$$

$\square$

Lemma 2.5.2. *Let $\Gamma = \{(y', \psi(y')) : y' \in B_1'\}$, where ψ is a Lipschitz function. Given $0 < \alpha < 1$, there exist $C_0 > 0$, $0 < \lambda < 1/2$, and $0 < \delta < 1$, depending only on n and α, such that for a distributional solution $u \in C(\overline{B_1})$ to*

$$\begin{cases} \Delta u = g\, dH^{n-1}\big|_\Gamma & \text{in } B_1 \\ |u| \leq 1 & \text{in } B_1 \\ |g| \leq \delta & \text{on } \Gamma \cap B_{3/4}, \end{cases}$$

there is a linear polynomial $P(x) = A \cdot x + B$, with $A \in \mathbb{R}^n$, $B \in \mathbb{R}$ and $|A| + |B| \le C_0$, such that

$$|u(x) - P(x)| \le \lambda^{1+\alpha} \qquad \textit{for all } x \in B_\lambda.$$

Proof. Fix $\lambda, \delta > 0$ to be determined. Let v be the harmonic function in $B_{3/4}$ such that $v = u$ on $\partial B_{3/4}$. Then, the difference $w = u - v$ is the distributional solution to

$$\begin{cases} \Delta w = g \, dH^{n-1}\big|_\Gamma & \text{in } B_{3/4} \\ w = 0 & \text{on } \partial B_{3/4}. \end{cases}$$

Moreover, $\|w\|_{L^\infty(B_{3/4})} \le C\|g\|_{L^\infty(\Gamma \cap B_{3/4})} \le C\delta$, where $C = C(n, \Gamma) > 0$. Define $P(x) = v(0) + \nabla v(0) \cdot x$. By interior estimates and the maximum principle, we have

$$\|D^j v\|_{L^\infty(B_{1/2})} \le C_0 \|v\|_{L^\infty(B_{3/4})} \le C_0 \qquad \text{for all } j \ge 0,$$

where $C_0 = C_0(n, j) > 0$. Hence, for $x \in B_\lambda$, with $0 < \lambda < 1/2$, we get

$$\begin{aligned} |u(x) - P(x)| &\le |u(x) - v(x)| + |v(x) - P(x)| \\ &\le C\delta + \|D^2 v\|_{L^\infty(B_{1/2})} |x|^2 \\ &\le C\delta + C_0 \lambda^2. \end{aligned}$$

First, choose $0 < \lambda < 1/2$, such that $C_0\lambda^2 \le \lambda^{1+\alpha}/2$. Then choose $0 < \delta < 1$ such that $C\delta \le \lambda^{1+\alpha}/2$. □

2.5.2 Proof of Theorem 2.1.2

Fix $0 < \alpha, \gamma < 1$. Let $C_0, \lambda, \theta, \varepsilon, \delta > 0$ be the minimum of the constants given in Lemma 2.5.1 and Lemma 2.5.2. Let $0 < \delta_0 < \min\left\{\delta, \theta\varepsilon, \frac{\lambda^{1+\alpha}}{2}\right\}$. First, we normalize the problem. Recall that we are assuming that $0 \in \Gamma$, that is, $\psi(0') = 0$.

(*i*) By rotation, we can assume that $\nu(0) = e_n$. In particular, $\nabla'\psi(0') = 0'$.

(*ii*) If $g(0) \neq 0$, we can suppose that $g(0) = 1$. Indeed, we consider $v = u/g(0)$. The case $g(0) = 0$ will be addressed at the end.

(*iii*) Assume that $\|u\|_{L^\infty(B_1)} \leq 1$, and that

$$[g]_{C^{0,\alpha}(0)} = \sup_{x \in \Gamma \cap B_1,\, x \neq 0} \frac{|g(x) - g(0)|}{|x|^\alpha} \leq \delta_0.$$

Indeed, one can consider

$$v = \delta_0 \, \frac{u}{\|u\|_{L^\infty(B_1)} + [g]_{C^{0,\alpha}(0)}}.$$

(*iv*) Also, we let $[\psi]_{C^{1,\alpha}(0)} \leq [\psi]_{C^{1,\alpha}(B_1')} \leq \delta_0$. Recall that

$$[\psi]_{C^{1,\alpha}(0)} = \sup_{x' \in B_1',\, x' \neq 0'} \frac{|\nabla'\psi(x') - \nabla'\psi(0')|}{|x'|^\alpha} = \sup_{x' \in B_1',\, x' \neq 0'} \frac{|\nabla'\psi(x')|}{|x'|^\alpha}.$$

Then, for this normalization one can take

$$\phi = \delta_0 \, \frac{\psi}{[\psi]_{C^{1,\alpha}(B_1')}}.$$

We make an abuse of notation and call the solution, the interface, the parametrization and the right hand side as in the statement, namely, u, Γ, ψ, and g, respectively.

It is enough to prove the following.

Claim. *For all* $k \geq 1$*, there exist linear polynomials* $P_k = A_k \cdot x + B_k$ *and* $Q_k = C_k \cdot x + B_k$ *such that*

$$\lambda^k |A_{k+1} - A_k| + \lambda^k |C_{k+1} - C_k| + |B_{k+1} - B_k| \leq C_0 \lambda^{k(1+\alpha)},$$

where $C_0 = C_0(n, \alpha) > 0$*, and such that*

$$|u_1(x) - P_k(x)| \leq \lambda^{k(1+\alpha)} \qquad \textit{for all } x \in \Omega_1 \cap B_{\lambda^k},$$

$$|u_2(x) - Q_k(x)| \leq \lambda^{k(1+\alpha)} \qquad \textit{for all } x \in \Omega_2 \cap B_{\lambda^k}.$$

Moreover, $\nabla' P_k = \nabla' Q_k$ *and* $(P_k)_{x_n} - (Q_k)_{x_n} = 1$*.*

We prove the claim by induction. Let us start with the case $k = 1$. By the normalization, u, Γ and g satisfy the assumptions on Lemma 2.5.1. Indeed, by (i) and (iv), for any $(x', x_n) \in \Gamma$,

$$|x_n| = |\psi(x')| = |\psi(x') - \psi(0') - \nabla'\psi(0') \cdot x'| \leq [\psi]_{C^{1,\alpha}(0)} \leq \delta_0 \leq \theta\varepsilon.$$

Also, $1 \leq (1 + |\nabla'\psi(x')|^2)^{1/2} \leq (1 + \delta_0^2)^{1/2} \leq (1 - \varepsilon)^{-1}$. Moreover, by (iii),

$$|g(x) - 1| = |g(x) - g(0)| \leq [g]_{C^{0,\alpha}(0)} |x|^\alpha \leq \delta_0 \leq \delta \qquad \text{for any } x \in \Gamma.$$

Hence, by Lemma 2.5.1, there are linear polynomials $P_1(x) = A_1 \cdot x + B_1$, and $Q_1(x) = C_1 \cdot x + B_1$, with $A_1, C_1 \in \mathbb{R}^n$, $B_1 \in \mathbb{R}$, and $|A_1| + |B_1| + |C_1| \leq C_0$, such that

$$|u_1(x) - P_1(x)| \leq \lambda^{1+\alpha} \qquad \text{for all } x \in \Omega_1 \cap B_\lambda,$$

$$|u_2(x) - Q_1(x)| \leq \lambda^{1+\alpha} \qquad \text{for all } x \in \Omega_2 \cap B_\lambda.$$

Moreover, $\nabla' P_1 = \nabla' Q_1$, and $(P_1)_{x_n} - (Q_1)_{x_n} = 1$.

For the induction step, assume that the claim holds for some $k \geq 1$, and let P_k and Q_k be such polynomials. Denote by

$$\begin{aligned} \Omega_{i,\lambda^k} &= \{x \in B_1 : \lambda^k x \in \Omega_i\} \quad \text{for } \; i = 1, 2, \\ \Gamma_{\lambda^k} &= \{x \in B_1 : \lambda^k x \in \Gamma\}. \end{aligned}$$

Note that if ψ_{λ^k} is a parametrization of Γ_{λ^k} in B_1', then $\psi_{\lambda^k}(x') = \lambda^{-k}\psi(\lambda^k x')$. In particular, $\nabla'\psi_{\lambda^k}(x') = \nabla'\psi(\lambda^k x)$, and thus, for $x \in \Gamma_{\lambda^k}$, if $\nu_{\lambda^k}(x)$ is the normal vector on x pointing at $\Omega_{\lambda^k,1}$, then $\nu_{\lambda^k}(x) = \nu(\lambda^k x)$. Define $\mathcal{P}_k = P_k \chi_{\Omega_1} + Q_k \chi_{\Omega_2}$. Consider the rescaled function

$$w(x) = \frac{u(\lambda^k x) - \mathcal{P}_k(\lambda^k x)}{\lambda^{k(1+\alpha)}} \qquad \text{for } x \in \overline{B_1}. \tag{2.5.2}$$

By the induction hypothesis, $\|w\|_{L^\infty(B_1)} \leq 1$. Notice that w is a piecewise continuous function with a jump discontinuity on Γ_{λ^k}. In fact, if

$$w_1 = w\big|_{\overline{\Omega}_{1,\lambda^k}}, \quad w_2 = w\big|_{\overline{\Omega}_{2,\lambda^k}},$$

then for $x \in \Gamma_{\lambda^k}$, by the normalization (iv), and the induction hypothesis, we have

$$\begin{aligned} |(w_1 - w_2)(x)| = \frac{|Q_k(\lambda^k x) - P_k(\lambda^k x)|}{\lambda^{k(1+\alpha)}} &= \lambda^{-k\alpha}|x_n| \qquad (2.5.3) \\ &\leq \lambda^{-k\alpha} \sup_{x \in \Gamma_{\lambda^k}} |x_n| \\ &\leq \sup_{x' \in B_1'} \frac{|\psi_{\lambda^k}(x')|}{\lambda^{k\alpha}} \leq [\psi]_{C^{1,\alpha}(0)} \leq \delta_0. \end{aligned}$$

Let $v = v_1 \chi_{\overline{\Omega}_{1,\lambda^k}} + v_2 \chi_{\overline{\Omega}_{2,\lambda^k}}$, where v_1 and v_2 are the solutions to

$$\begin{cases} \Delta v_i = 0 & \text{in } \Omega_{i,\lambda^k} \\ v_i = w_i & \text{on } \partial\Omega_{i,\lambda^k} \setminus \Gamma_{\lambda^k} \\ v_i = \frac{w_1+w_2}{2} & \text{on } \Gamma_{\lambda^k}, \end{cases}$$

for $i = 1, 2$. Then $v \in C(\overline{B_1})$ and, by the maximum principle, $\|v\|_{L^\infty(B_1)} \leq \|w\|_{L^\infty(B_1)} \leq 1$. Moreover,

$$\begin{cases} \Delta(v_i - w_i) = 0 & \text{in } \Omega_{i,\lambda^k} \\ v_i - w_i = 0 & \text{on } \partial\Omega_{i,\lambda^k} \setminus \Gamma_{\lambda^k} \\ v_i - w_i = (-1)^i \frac{w_1-w_2}{2} & \text{on } \Gamma_{\lambda^k}. \end{cases} \tag{2.5.4}$$

By the maximum principle and (2.5.3) it follows that

$$\begin{aligned} \|v - w\|_{L^\infty(B_1)} &\leq \|v_1 - w_1\|_{L^\infty(\Omega_{1,\lambda^k})} + \|v_2 - w_2\|_{L^\infty(\Omega_{2,\lambda^k})} \\ &= \|w_1 - w_2\|_{L^\infty(\Gamma_{\lambda^k})} \leq \delta_0. \end{aligned} \tag{2.5.5}$$

We compute the distributional Laplacian of v and estimate its size. For any $\varphi \in C_c^\infty(B_1)$,

$$\begin{aligned} \Delta v(\varphi) &= \int_{B_1} v(x)\Delta\varphi(x)\,dx \\ &= \int_{\Omega_{1,\lambda^k}} v_1(x)\Delta\varphi(x)\,dx + \int_{\Omega_{2,\lambda^k}} v_2(x)\Delta\varphi(x)\,dx \\ &= \int_{\Omega_{1,\lambda^k}} (v_1 - w_1)(x)\Delta\varphi(x)\,dx + \int_{\Omega_{2,\lambda^k}} (v_2 - w_2)(x)\Delta\varphi(x)\,dx \\ &\qquad + \int_{B_1} w(x)\Delta\varphi(x)\,dx \\ &\equiv I_1 + I_2 + I_3. \end{aligned}$$

For $i = 1, 2$, by Green's formula,

$$I_i = \frac{1}{2} \int_{\Gamma_{\lambda^k}} (w_1 - w_2)(x) \varphi_{\nu_{\lambda^k}}(x) \, dH^{n-1} \\ + (-1)^{i+1} \int_{\Gamma_{\lambda^k}} (v_i - w_i)_{\nu_{\lambda^k}}(x) \varphi(x) \, dH^{n-1},$$

where we recall that ν_{λ^k} is the unit normal vector on Γ_{λ^k} pointing at Ω_{1,λ^k}. Note that

$$I_3 = \Delta w(\varphi) = \Delta \left(\frac{u(\lambda^k x)}{\lambda^{k(1+\alpha)}} \right) (\varphi) - \Delta \left(\frac{\mathcal{P}_k(\lambda^k x)}{\lambda^{k(1+\alpha)}} \right) (\varphi).$$

Since u is a distributional solution, by doing a change of variables, we get

$$\begin{aligned} \Delta(u(\lambda^k x))(\varphi) &= \int_{B_1} u(\lambda^k x) \Delta \varphi(x) \, dx \\ &= \lambda^{k(2-n)} \int_{B_{\lambda^k}} u(y) \Delta_y \varphi(\lambda^{-k} y) \, dy \\ &= \lambda^{k(2-n)} \int_{\Gamma \cap B_{\lambda^k}} g(y) \varphi(\lambda^{-k} y) \, dH_y^{n-1} \\ &= \lambda^k \int_{\Gamma_{\lambda^k}} g(\lambda^k x) \varphi(x) \, dH^{n-1}. \end{aligned}$$

Also, by Green's formula, the induction hypothesis and (2.5.3),

$$\begin{aligned} \Delta(\mathcal{P}_k(\lambda^k x))(\varphi) &= \lambda^k \int_{\Gamma_{\lambda^k}} \left[\nabla P_k(\lambda^k x) - \nabla Q_k(\lambda^k x) \right] \cdot \nu_{\lambda^k}(x) \varphi(x) \, dH^{n-1} \\ &\quad + \int_{\Gamma_{\lambda^k}} \left[Q_k(\lambda^k x) - P_k(\lambda^k x) \right] \varphi_{\nu_{\lambda^k}}(x) \, dH^{n-1} \\ &= \lambda^k \int_{\Gamma_{\lambda^k}} \nu_n(\lambda^k x) \varphi(x) \, dH^{n-1} \\ &\quad + \lambda^{k(1+\alpha)} \int_{\Gamma_{\lambda^k}} (w_1 - w_2)(x) \varphi_{\nu_{\lambda^k}}(x) \, dH^{n-1}. \end{aligned}$$

Then

$$I_3 = \int_{\Gamma_{\lambda^k}} \tilde{g}(x)\varphi(x)\, dH^{n-1} - \int_{\Gamma_{\lambda^k}} (w_1 - w_2)(x)\varphi_{\nu_{\lambda^k}}(x)\, dH^{n-1},$$

where

$$\tilde{g}(x) = \frac{g(\lambda^k x) - \nu_n(\lambda^k x)}{\lambda^{k\alpha}}.$$

Therefore, for any $\varphi \in C_c^\infty(B_1)$,

$$\begin{aligned}\Delta v(\varphi) &= \int_{\Gamma_{\lambda^k}} \Big[(v_1 - w_1)_{\nu_{\lambda^k}}(x) - (v_2 - w_2)_{\nu_{\lambda^k}}(x) + \tilde{g}(x)\Big]\varphi(x)\, dH^{n-1} \\ &\equiv \int_{\Gamma_{\lambda^k}} \hat{g}\varphi\, dH^{n-1}.\end{aligned}$$

By $C^{1,\alpha}$ boundary estimates for harmonic functions applied to (2.5.4) and, by taking into account (2.5.5) and the first line of (2.5.3), we get

$$\|v_i - w_i\|_{C^{1,\alpha}(\overline{\Omega_{i,\lambda^k} \cap B_{3/4}})} \le C\|w_2 - w_1\|_{C^{1,\alpha}(\Gamma_{\lambda^k})} = C\lambda^{-k\alpha}\|\psi_{\lambda^k}\|_{C^{1,\alpha}(\overline{B_1'})}.$$

Using the normalization of ψ, we find that

$$\lambda^{-k\alpha}\|\psi_{\lambda^k}\|_{L^\infty(B_1')} = \sup_{x' \in B_1'} \frac{|\psi(\lambda^k x')|}{\lambda^{k(1+\alpha)}} \le [\psi]_{C^{1,\alpha}(0)} \le \delta_0,$$

$$\lambda^{-k\alpha}\|\nabla'\psi_{\lambda^k}\|_{L^\infty(B_1')} = \sup_{x' \in B_1'} \frac{|\nabla'\psi(\lambda^k x')|}{\lambda^{k\alpha}} \le [\psi]_{C^{1,\alpha}(0)} \le \delta_0,$$

and

$$\lambda^{-k\alpha}[\nabla'\psi_{\lambda^k}]_{C^{0,\alpha}(\overline{B_1'})} = \sup_{\substack{x',y' \in \overline{B_1'} \\ x' \ne y'}} \frac{|\nabla'\psi(\lambda^k x') - \nabla'\psi(\lambda^k y')|}{\lambda^{k\alpha}|x' - y'|^\alpha} \le [\psi]_{C^{1,\alpha}(\overline{B_1'})} \le \delta_0.$$

In particular, it follows that

$$\|(v_i - w_i)_{\nu_{\lambda^k}}\|_{L^\infty(\Gamma_{\lambda^k} \cap B_{3/4})} \le C\delta_0.$$

Moreover, for $x \in \Gamma_{\lambda^k}$,

$$
\begin{aligned}
|\tilde{g}(x)| &\leq \frac{|g(\lambda^k x) - 1|}{\lambda^{k\alpha}} + \frac{|1 - \nu_n(\lambda^k x)|}{\lambda^{k\alpha}} \\
&\leq [g]_{C^{0,\alpha}(0)} + [\nu_n]_{C^{0,\alpha}(0)} \leq \delta_0 + \delta_0 = 2\delta_0.
\end{aligned}
$$

Hence, choosing δ_0 sufficiently small, we see that

$$
\begin{aligned}
\|\hat{g}\|_{L^\infty(\Gamma_{\lambda^k} \cap B_{3/4})} &\leq \|(v_1 - w_1)_{\nu_{\lambda^k}}\|_{L^\infty(\Gamma_{\lambda^k} \cap B_{3/4})} + \|(v_2 - w_2)_{\nu_{\lambda^k}}\|_{L^\infty(\Gamma_{\lambda^k} \cap B_{3/4})} \\
&\quad + \|\tilde{g}\|_{L^\infty(\Gamma_{\lambda^k})} \leq C\delta_0 + C\delta_0 + 2\delta_0 = 2(C+1)\delta_0 \leq \delta.
\end{aligned}
$$

We have proved that $v \in C(\overline{B_1})$ satisfies

$$
\begin{cases}
\Delta v = \hat{g}\, dH^{n-1}\big|_{\Gamma_{\lambda^k}} & \text{in } B_1 \\
|v| \leq 1 & \text{in } B_1 \\
|\hat{g}| \leq \delta & \text{on } \Gamma_{\lambda^k} \cap B_{3/4}.
\end{cases}
$$

Therefore, we can apply Lemma 2.5.2 to v to find a linear polynomial $P(x) = A \cdot x + B$, with $A \in \mathbb{R}^n$, $B \in \mathbb{R}$ and $|A| + |B| \leq C_0$, such that

$$
|v(x) - P(x)| \leq \frac{\lambda^{1+\alpha}}{2} \qquad \text{for all } x \in B_\lambda.
$$

Hence, for any $x \in B_\lambda$, by the estimate above and (2.5.5),

$$
|w(x) - P(x)| \leq |w(x) - v(x)| + |v(x) - P(x)| \leq \delta_0 + \frac{\lambda^{1+\alpha}}{2} \leq \lambda^{1+\alpha},
$$

since $\delta_0 \leq \lambda^{1+\alpha}/2$. According to (2.5.2),

$$
\left| \frac{u(\lambda^k x) - \mathcal{P}_k(\lambda^k x)}{\lambda^{k(1+\alpha)}} - P(x) \right| \leq \lambda^{1+\alpha} \qquad \text{for all } x \in B_\lambda,
$$

or equivalently, for $y = \lambda^k x$,

$$
|u(y) - \mathcal{P}_k(y) - \lambda^{k(1+\alpha)} P(y/\lambda^k)| \leq \lambda^{(k+1)(1+\alpha)} \qquad \text{for all } y \in B_{\lambda^{k+1}}.
$$

Define the polynomials P_{k+1} and Q_{k+1} as

$$P_{k+1}(y) = P_k(y) + \lambda^{k(1+\alpha)} P(y/\lambda^k), \qquad Q_{k+1}(y) = Q_k(y) + \lambda^{k(1+\alpha)} P(y/\lambda^k).$$

From the previous estimate, it follows that

$$\begin{aligned} |u_1(y) - P_{k+1}(y)| &\le \lambda^{(k+1)(1+\alpha)} && \text{for all } y \in \Omega_1 \cap B_{\lambda^{k+1}}, \\ |u_2(y) - Q_{k+1}(y)| &\le \lambda^{(k+1)(1+\alpha)} && \text{for all } y \in \Omega_2 \cap B_{\lambda^{k+1}}. \end{aligned}$$

Moreover, since $P_k(0) = Q_k(0)$, and $\nabla' P_k = \nabla' Q_k$, it is clear that $P_{k+1}(0) = Q_{k+1}(0)$, and $\nabla' P_{k+1} = \nabla' Q_{k+1}$. Also, $(P_{k+1})_{x_n} - (Q_{k+1})_{x_n} = (P_k)_{x_n} - (Q_k)_{x_n} = 1$. If $P_{k+1}(y) = A_{k+1} \cdot y + B_{k+1}$ and $Q_{k+1}(y) = C_{k+1} \cdot y + B_{k+1}$ then

$$A_{k+1} = A_k + \lambda^{k\alpha} A, \quad B_{k+1} = B_k + \lambda^{k(1+\alpha)} B, \quad C_{k+1} = C_k + \lambda^{k\alpha} A.$$

By the estimate $|A| + |B| \le C_0$, we conclude

$$\lambda^k |A_{k+1} - A_k| + \lambda^k |C_{k+1} - C_k| + |B_{k+1} - B_k| \le C_0 \lambda^{k(1+\alpha)}.$$

The proof of the claim is completed.

Finally, we consider the case $g(0) = 0$. As before, it is enough to prove the following.

Claim. *For all $k \ge 1$, there exists a linear polynomial $P_k = A_k \cdot x + B_k$ such that*

$$\lambda^k |A_{k+1} - A_k| + |B_{k+1} - B_k| \le C_0 \lambda^{k(1+\alpha)},$$

where $C_0 = C_0(n, \alpha) > 0$, and such that

$$|u(x) - P_k(x)| \le \lambda^{k(1+\alpha)} \qquad \text{for all } x \in \Omega \cap B_{\lambda^k}.$$

The proof is by induction. For $k = 1$, since $\|u\|_{L^\infty(B_1)} \le 1$, and

$$\|g\|_{L^\infty(\Gamma)} = \sup_{x \in \Gamma} |g(x) - g(0)| \le \delta_0,$$

we can apply Lemma 2.5.2 to u. Then we find a linear polynomial $P_1(x) = A_1 \cdot x + B_1$, with $A_1 \in \mathbb{R}^n$, $B_1 \in \mathbb{R}$, and $|A_1| + |B_1| \le C_0$, such that

$$|u(x) - P_1(x)| \le \lambda^{1+\alpha} \qquad \text{for all } x \in B_\lambda.$$

Assume the claim holds for $k \ge 1$. Define

$$w(x) = \frac{u(\lambda^k x) - P_k(\lambda^k x)}{\lambda^{k(1+\alpha)}} \qquad \text{for } x \in \overline{B_1}.$$

Then, for any $\varphi \in C_c^\infty(B_1)$,

$$\Delta w(\varphi) = \frac{\Delta(u(\lambda^k x))(\varphi)}{\lambda^{k(1+\alpha)}} = \int_{\Gamma_{\lambda^k}} \frac{g(\lambda^k x)}{\lambda^{k\alpha}} \, \varphi(x) \, dH^{n-1}.$$

Also, for any $x \in \Gamma_{\lambda^k}$,

$$\frac{|g(\lambda^k x)|}{\lambda^{k\alpha}} = \frac{|g(\lambda^k x) - g(0)|}{\lambda^{k\alpha}} \le [g]_{C^{0,\alpha}(0)} \le \delta_0.$$

Then the claim follows for $k + 1$ by applying again Lemma 2.5.2. $\square$

2.6 Proof of main result: Theorem 2.1.1

To prove Theorem 2.1.1 we need Campanato's characterization of $C^{1,\alpha}$ spaces [19] and a technical result that patches the interior and boundary estimates together. We believe that the latter belongs to the folklore (see, for example, [45]) but, for the sake of completeness, we will give a proof.

Theorem 2.6.1 (Campanato)**.** *Let u be a measurable function defined on a bounded $C^{1,\alpha}$ domain Ω. Then $u \in C^{1,\alpha}(\overline{\Omega})$ if and only if there exists $C_0 > 0$ such that for any $x \in \overline{\Omega}$, there exists a linear polynomial $Q_x(z)$ such that*

$$|u(z) - Q_x(z)| \leq C_0|x - z|^{1+\alpha},$$

for all $z \in B_1(x) \cap \Omega$. In this case, if C_* denotes the least constant $C_0 > 0$ for which the property above holds, then

$$\|u\|_{C^{1,\alpha}(\overline{\Omega})} \sim C_* + \sup_{x \in \overline{\Omega}} |Q_x|,$$

where $|Q_x|$ denotes the sum of the coefficients of the polynomial $Q_x(z)$.

Proposition 2.6.2. *Let S be a collection of measurable functions defined on a bounded $C^{1,\alpha}$ domain Ω. For $x \in \Omega$, we let $d_x = \operatorname{dist}(x, \partial\Omega)$. Fix $u \in S$, and suppose the following hold.*

(*i*) *(Interior estimates). There exist $A, C, D > 0$ such that for any $x \in \Omega$ there exists a linear polynomial $P_x(z)$ such that*

$$\|P_x\|_{L^\infty(B)} + d_x\|\nabla P_x\|_{L^\infty(B)} \leq C\|u\|_{L^\infty(B)}$$

and

$$|u(z) - P_x(z)| \leq \left(A\frac{\|u\|_{L^\infty(B)}}{d_x^{1+\alpha}} + D\right)|z - x|^{1+\alpha}$$

for all $z \in B \equiv B_{d_x/2}(x)$.

(ii) *(Boundary estimates). There exists $E > 0$ such that for any $y \in \partial\Omega$, there is a linear polynomial $P_y(z)$ such that*

$$\|P_y\|_{L^\infty(\Omega)} + \|\nabla P_y\|_{L^\infty(\Omega)} \leq E$$

and

$$|u(z) - P_y(z)| \leq E|z - y|^{1+\alpha},$$

for all $z \in \overline{\Omega}$.

(iii) *(Invariance property). For any $u \in S$, and any $y \in \partial\Omega$, with corresponding linear polynomial P_y as in (ii), the function $v = u - P_y$ also satisfies the estimates of (i).*

Then $S \subset C^{1,\alpha}(\overline{\Omega})$ and there exists $M > 0$, depending only on A, C, D, E such that

$$\|u\|_{C^{1,\alpha}(\overline{\Omega})} \leq M\|u\|_{L^\infty(\Omega)}.$$

Proof. We need to show that any $u \in S$ satisfies the Campanato characterization from Theorem 2.6.1. Let us pick any point $x \in \overline{\Omega}$. If $x \in \partial\Omega$ then the polynomial $Q_x(z) \equiv P_x(z)$, where $P_x(z)$ is as in assumption (ii), satisfies the Campanato condition with $C_0 = E$.

Suppose next that $x \in \Omega$. Let $y \in \partial\Omega$ be a boundary point that realizes the distance from x to the boundary, namely, $d_x = |x - y|$. Let $P_y(z)$ be the linear polynomial that satisfies (ii). Consider the function $v(z) = u(z) - P_y(z)$. By (iii), there is a linear polynomial $P_x(z)$ such that the conditions in (i) are met for v in place of u. We claim that the polynomial Q_x for the Campanato condition is

$$Q_x(z) \equiv P_y(z) + P_x(z).$$

To show this, we split the argument into two cases.

Case 1. Suppose that $|z - x| < d_x/2$. This is the case when we can apply (i) for $v - P_x$:

$$\begin{aligned} |u(z) - Q_x(z)| &= |u(z) - P_y(z) - P_x(z)| = |v(z) - P_x(z)| \\ &\leq \left(A \frac{\|v\|_{L^\infty(B_{d_x/2}(x))}}{d_x^{1+\alpha}} + D \right) |z - x|^{1+\alpha} \\ &= \left(A \frac{\|u - P_y\|_{L^\infty(B_{d_x/2}(x))}}{d_x^{1+\alpha}} + D \right) |z - x|^{1+\alpha}. \end{aligned}$$

Now, we notice that, by (ii), by the choice of y, and the fact that $|z-x| < d_x/2$,

$$|u(z) - P_y(z)| \leq E|z - y|^{1+\alpha} \leq E(3/2d_x)^{1+\alpha} \leq 2^{1+\alpha} E d_x^{1+\alpha}.$$

Hence,

$$|u(z) - Q_x(z)| \leq (2^{1+\alpha} AE + D)|z - x|^{1+\alpha}$$

and $C_0 = 2^{1+\alpha} AE + D$.

Case 2. Suppose that $|z - x| \geq d_x/2$. By the estimate in (i) for $P_x(z)$, we get

$$\begin{aligned} |P_x(z)| &= |P_x(x) + \nabla P_x(x) \cdot (z - x)| \\ &\leq C\|u - P_y\|_{L^\infty(B)} + C d_x^{-1} \|u - P_y\|_{L^\infty(B)} |z - x|. \end{aligned}$$

Also, by the boundary estimate in (ii),

$$\|u - P_y\|_{L^\infty(B)} \leq (3/2)^{1+\alpha} E d_x^{1+\alpha}.$$

Hence,

$$\begin{aligned}
|u(z) - Q_x(z)| &\le |u(z) - P_y(z)| + |P_x(z)| \\
&\le E|z-y|^{1+\alpha} + C\|u - P_y\|_{L^\infty(B)} \\
&\qquad + Cd_x^{-1}\|u - P_y\|_{L^\infty(B)}|z-x| \\
&\le 3^{1+\alpha}E|z-x|^{1+\alpha} + 3^{1+\alpha}CEd_x^{1+\alpha} \\
&\qquad + Cd_x^{-1}(3/2)^{1+\alpha}Ed_x^{1+\alpha}|z-x| \\
&\le 3^{1+\alpha}E(1+2C)|z-x|^{1+\alpha}.
\end{aligned}$$

Thus, in this case, the Campanato constant is $C_0 = 3^{1+\alpha}E(1+2C)$. □

Proof of Theorem 2.1.1. Let $u \in \mathrm{LogLip}(\overline{\Omega})$ be the solution given by Theorem 2.2.2. We will show the statement for the function $u_2 : \overline{\Omega}_2 \to \mathbb{R}$, and we can argue similarly for $u_1 : \overline{\Omega}_1 \to \mathbb{R}$. The following holds.

(i) *(Interior estimates).* For any $x \in \Omega_2$, there exists a linear polynomial $P_x(z)$ such that

$$\|P_x\|_{L^\infty(B)} + d_x\|\nabla P_x\|_{L^\infty(B)} \le (1+2n)\|u_2\|_{L^\infty(B)}$$

and

$$|u_2(z) - P_x(z)| \le 2^{\alpha-1}n\frac{\|u\|_{L^\infty(B)}}{d_x^{1+\alpha}}|z-x|^{1+\alpha},$$

for all $z \in B \equiv B_{d_x/2}(x)$.

Indeed, fix $x \in \Omega_2$. Since u_2 is harmonic, it is smooth in Ω_2, so we can define

$$P_x(z) = u_2(x) + \nabla u_2(x) \cdot (z - x).$$

Then, by classical interior estimates for harmonic functions,

$$
\begin{aligned}
\|P_x\|_{L^\infty(B)} + d_x \|\nabla P_x\|_{L^\infty(B)} &\leq \|u_2\|_{L^\infty(B)} + d_x \|\nabla u_2\|_{L^\infty(B)} \\
&\quad + d_x \|\nabla u_2\|_{L^\infty(B)} \\
&\leq \|u_2\|_{L^\infty(B)} + 2n \|u_2\|_{L^\infty(B)} \\
&\leq (1+2n) \|\nabla u_2\|_{L^\infty(B)}.
\end{aligned}
$$

Moreover,

$$
\begin{aligned}
|u_2(z) - P_x(z)| &\leq \|D^2 u_2\|_{L^\infty(B)} |z-x|^2 \\
&\leq n \frac{\|u_2\|_{L^\infty(B)}}{d_x^2} |z-x|^2 \\
&\leq 2^{\alpha-1} n \frac{\|u_2\|_{L^\infty(B)}}{d_x^{1+\alpha}} |z-x|^{1+\alpha}.
\end{aligned}
$$

(*ii*) *(Boundary estimates).* Consider $\partial\Omega_2 = \Gamma \cup \partial\Omega$.

If $y \in \Gamma$, by Theorem 2.1.2, there exists a linear polynomial $P_y(z)$ such that

$$\|P_y\|_{L^\infty(\Omega_2)} + \|\nabla P_y\|_{L^\infty(\Omega_2)} \leq E$$

and

$$|u_2(z) - P_y(z)| \leq E|z-y|^{1+\alpha},$$

for all $z \in \overline{\Omega}_2$, with $E \leq C_0 \|\psi\|_{C^{1,\alpha}(B_1')} \|g\|_{C^{0,\alpha}(\Gamma)}$, and $C_0 = C_0(n,\alpha) > 0$.

If $y \in \partial\Omega \in C^\infty$, since $u_2 = 0$, then, by classical boundary regularity for harmonic functions, $u_2 \in C^{1,\alpha}(\overline{B \cap \Omega})$, with $B \equiv B_r(y)$, for some $r > 0$ sufficiently small. By Theorem 2.6.1, there exists a linear polynomial

$P_y(z)$ such that

$$|u_2(z) - P_y(z)| \le C_0|z - y|^{1+\alpha},$$

for all $z \in \overline{\Omega}_2$, for some $C_0(n, \alpha) > 0$.

(*iii*) *(Invariance property).* Fix $y \in \partial\Omega_2$, and let P_y be the corresponding linear polynomial given in (*ii*). Clearly, the function $v = u_2 - P_y$ is harmonic in Ω_2, so it satisfies the interior estimates in (*i*).

Therefore, by Theorem 2.6.2, we have $u_2 \in C^{1,\alpha}(\overline{\Omega}_2)$, and there exists a constant $C > 0$, depending only on n, α and Γ such that

$$\|u_2\|_{C^{1,\alpha}(\overline{\Omega}_2)} \le C\|g\|_{C^{0,\alpha}(\Gamma)}.$$

$\square$

2.7 Appendix

A special Lipschitz domain Ω in $\mathbb{R}^n$ is a set of the form

$$\Omega = \{(x', x_n) \in \mathbb{R}^n : x' \in \mathbb{R}^{n-1},\, x_n > \psi(x')\}$$

where $\psi \in \mathrm{Lip}(\mathbb{R}^{n-1})$, that is, there exists $M > 0$ such that

$$|\psi(x') - \psi(y')| \le M|x' - y'| \qquad \text{for all } x', y' \in \mathbb{R}^{n-1}.$$

In other words, Ω is the set of points lying above the graph of a Lipschitz function ψ. Then, by Rademacher's Theorem, ψ is Fréchet differentiable almost everywhere with $\|\nabla\psi\|_{L^\infty(\mathbb{R}^{n-1})} \le M$. On $\partial\Omega$ we thus have

$$dH^{n-1}\big|_{\partial\Omega} = \sqrt{1 + |\nabla\psi(x')|^2}\, dx' \quad \text{and} \quad \nu(x', \psi(x')) = \frac{(\nabla\psi(x'), -1)}{\sqrt{1 + |\nabla\psi(x')|^2}},$$

where $x = (x', \psi(x')) \in \partial\Omega$. For a measurable function f on $\partial\Omega$, we have

$$\int_{\partial\Omega} f(x)\, dH^{n-1} = \int_{\mathbb{R}^{n-1}} f(x', \psi(x'))\sqrt{1 + |\nabla\psi(x')|^2}\, dx'.$$

For more details see [23, 41].

A bounded Lipschitz domain in $\mathbb{R}^n$ is a bounded domain Ω such that the boundary $\partial\Omega$ can be covered by finitely many open balls B_j in $\mathbb{R}^n$, $j = 1, \ldots, J$, centered at $\partial\Omega$, such that

$$B_j \cap \Omega = B_j \cap \Omega_j, \quad j = 1, \ldots, J,$$

where Ω_j are rotations of suitable special Lipschitz domains given by Lipschitz functions ψ_j. One may then assume that $\partial\Omega \cap B_j$ can be represented in local coordinates by $x_n = \psi_j(x')$, where ψ_j is a Lipschitz function on $\mathbb{R}^{n-1}$ with $\psi_j(0') = 0$. Recall also that if ψ is a Lipschitz function defined on an set $A \subset \mathbb{R}^{n-1}$, with Lipschitz constant M, then there exists an extension $\overline{\psi} : \mathbb{R}^{n-1} \to \mathbb{R}$ of ψ such that $\overline{\psi} = \psi$ on A and the Lipschitz constant of $\overline{\psi}$ does not exceed M, see [23].

Let $\Omega_0 = \Omega \cap \left(\bigcup_{j=1}^J B_j\right)^c$. A partition of unity $\{\xi_j\}_{j=0}^J$ subordinated to $\{\Omega_0, B_1, \ldots, B_J\}$ is a family of nonnegative smooth functions ξ_j on $\mathbb{R}^n$ such that $\xi_0 \in C_c^\infty(\Omega_0)$, $\xi_j \in C_c^\infty(B_j)$ for all $j = 1, \ldots, J$, and

$$\sum_{j=0}^J \xi_j(x) = 1 \quad \text{for all } x \in \overline{\Omega}.$$

It follows that $0 \leq \xi_j \leq 1$, $j = 0, 1, \ldots, J$. Obviously the family $\{\xi_j\}_{j=1}^J$ is a partition of unity subordinated to the open cover $\{B_1, \ldots, B_J\}$ of $\partial\Omega$ and $\sum_{j=1}^J \xi_j(x) = 1$ for every $x \in \partial\Omega$.

Let $f : \Gamma \to \mathbb{R}$ be a measurable function, where $\Gamma = \partial\Omega$ is the boundary of a bounded Lipschitz domain Ω. Consider the balls B_j, $j = 1, \dots, J$, that cover Γ as above, and the corresponding Lipschitz functions $\psi_j : \mathbb{R}^{n-1} \to \mathbb{R}$. Let $\{\xi_j\}_{j=1}^J$ be a smooth partition of unity subordinated to the open cover $\{B_j\}_{j=1}^J$ of Γ. Then

$$\int_\Gamma f\, dH^{n-1} = \sum_{j=1}^J \int_\Gamma \xi_j f\, dH^{n-1} = \sum_{j=1}^J \int_{B_j\cap\Gamma} \xi_j f\, dH^{n-1}.$$

Let us consider each one of the terms in the sum above separately. We study the following situation: let B be a ball and let $\bar{f} : B \cap \Gamma \to \mathbb{R}$ of compact support in $B \cap \Gamma$. Let $\psi : \mathbb{R}^{n-1} \to \mathbb{R}$ be a Lipschitz function such that $\psi(B_1') = B \cap \Gamma$. Then, by extending trivially $\bar{f}$ to the rest of the graph of ψ and using the coarea formula [23, 41],

$$\begin{aligned}
\int_{B\cap\Gamma} \bar{f}\, dH^{n-1} &= \int_{\psi(B_1')} \bar{f}\, dH^{n-1} = \int_{\psi(\mathbb{R}^{n-1})} \bar{f}\, dH^{n-1} \\
&= \int_{\mathbb{R}^{n-1}} \bar{f}(y', \psi(y')) \sqrt{1 + |\nabla\psi(y')|^2}\, dy' \\
&= \int_{B_1'} \bar{f}(y', \psi(y')) \sqrt{1 + |\nabla\psi(y')|^2}\, dy'.
\end{aligned}$$

Chapter 3

Transmission problems for fully nonlinear equations and flat interfaces

3.1 Introduction

We study the following fully nonlinear transmission problem in B_1:

$$\begin{cases} F^+(D^2u^+) = f^+ & \text{in } B_1^+ = B_1 \cap \{x_n > 0\} \\ F^-(D^2u^-) = f^- & \text{in } B_1^- = B_1 \cap \{x_n < 0\} \\ u^+_{x_n} - u^-_{x_n} = g & \text{on } T = B_1 \cap \{x_n = 0\}, \end{cases} \tag{3.1.1}$$

where $D^2u^\pm$ denotes the Hessian of $u^\pm$ and $u^\pm_{x_n}$ denotes the normal derivative of $u^\pm$. We assume that $F^\pm : \mathcal{S}^n \to \mathbb{R}$ are fully nonlinear uniformly elliptic operators, with ellipticity constants $0 < \lambda \leq \Lambda$, and $F^\pm(0) = 0$. That is, for every $M, N \in \mathcal{S}^n$, with $N \geq 0$, we have

$$\lambda\|N\| \leq F^\pm(M+N) - F^\pm(M) \leq \Lambda\|N\|.$$

We denote by $\mathcal{S}^n$ the set of square $n \times n$ symmetric matrices and by $\mathcal{E}(\lambda, \Lambda)$ this class of uniformly elliptic fully nonlinear operators.

Transmission problems of the form (3.1.1) may be understood as two Neumann problems that have been attached at the flat interface T. Indeed, if we know the function $u^-_{x_n}$ on T a priori, then the transmission condition in

(3.1.1) becomes a Neumann condition for u^+ (and viceversa). In [45], E. Milakis and L. Silvestre consider the Neumann problem,

$$\begin{cases} F(D^2u) = f & \text{in } B_1^+ \\ u_{x_n} = g & \text{on } T, \end{cases}$$

where $F \in \mathcal{E}(\lambda, \Lambda)$, and prove that if $f \in L^p$, with $p > n$, and $g \in C^{0,\alpha}$, then viscosity solutions are $C^{1,\alpha}$ up to the flat boundary in $B_{1/2}^+$.

A similar problem to (3.1.1) was considered by D. De Silva, F. Ferrari, and S. Salsa in [58], where the transmission condition is replaced by $u_{x_n}^+ - \mu u_{x_n}^- = 0$, for some $\mu > 0$. In fact, these transmission problems play a key role in the regularity theory of solutions to two-phase free boundary problems with distributed sources, studied by the same authors in [55–57]. In their paper, they assume that the functions $f^\pm$ are Lipschitz continuous, and they show that u^+ and u^- are $C^{1,\alpha}$ up to the flat interface in $B_{1/2}$. Their approach is to consider incremental quotients in the x'-direction and prove that they belong to a Pucci class of functions (see Definition 3.1.3), which are known to be Hölder continuous in the interior. In particular, tangential derivatives will be Hölder continuous, and thus, the $C^{1,\alpha}$ boundary regularity of solutions follows by the well-known results for the Dirichlet problem. We point out that the existence and uniqueness of viscosity solutions is left as an open problem. In fact, the proof of this result is one of the main novelties of our work.

This chapter is organized as follows. First, we introduce the notion of viscosity solution of (3.1.1) and present the main results. In Section 3.2, we prove a new ABP estimate (Theorem 3.2.1) for viscosity supersolutions of

(3.1.1) and obtain the maximum principle as an immediate consequence. Using this tool, we show an oscillation decay lemma in Section 3.3, which implies the $C^{0,\alpha}$ regularity of viscosity solutions across the interface T (Theorem 3.1.8). In Section 3.4, we define a family of regularizations in the x'-direction, also known as ε-envelopes, that play the same role as the Jensen's approximations in the classical theory, and derive some properties that will be useful in future proofs. We prove existence and uniqueness of viscosity solutions to (3.1.1) with prescribed boundary values in Section 3.5 (Theorem 3.1.7). For this, we show that the comparison principle for viscosity subsolutions and supersolutions holds (Theorem 3.5.6), and we carry out the usual procedure for Perron's method. Finally, in Section 3.6, we derive $C^{1,\alpha}$ estimates for u^+ and u^- up to the interface (Theorem 3.1.9). The latter follows by a standard perturbation argument, using the results in [58].

3.1.1 Preliminaries

Definition 3.1.1. We say that a continuous function φ touches u by above at x_0 in B_1 if there exists $\delta > 0$ such that the following holds:

$$\begin{aligned}&\varphi(x_0) = u(x_0)\\ &\varphi(x) \geq u(x) \quad \text{for all } x \in B_\delta(x_0) \subset B_1.\end{aligned}$$

Similarly, we say that φ touches u from below at x_0 in B_1 if the same conditions hold with the inequality reversed.

We denote by $\mathrm{USC}(B_1)$ the space of upper semicontinuous functions on B_1. Similarly, LSC(B_1) is the space of lower semicontinuous functions on B_1. In the sequel, we write $F^\pm(D^2u) = f^\pm$ to denote both interior equations in (3.1.1). It is clear that $u^\pm = u|_{\overline{B_1^\pm}}$.

Definition 3.1.2. For $M \in \mathcal{S}^n$, we define the Pucci's extremal operators as

$$\mathcal{M}^-_{\lambda,\Lambda}(M) = \lambda\,\mathrm{tr}(M^+) - \Lambda\,\mathrm{tr}(M^-) \quad \text{and} \quad \mathcal{M}^+_{\lambda,\Lambda}(M) = \Lambda\,\mathrm{tr}(M^+) - \lambda\,\mathrm{tr}(M^-),$$

where $M = M^+ - M^-$ and $M^+, M^- \geq 0$.

Definition 3.1.3. We denote by $\underline{S}_{\lambda,\Lambda}(f^\pm)$ the class of upper semicontinuous functions on B_1 such that $\mathcal{M}^+_{\lambda,\Lambda}(D^2u) \geq f^\pm$ in $B_1^\pm$ in the viscosity sense. Analogously, we denote by $\overline{S}_{\lambda,\Lambda}(f^\pm)$ the class of lower semicontinuous functions u on B_1 such that $\mathcal{M}^-_{\lambda,\Lambda}(D^2u) \leq f^\pm$ in $B_1^\pm$ in the viscosity sense. We define

$$S_{\lambda,\Lambda}(f^\pm) = \overline{S}_{\lambda,\Lambda}(f^\pm) \cap \underline{S}_{\lambda,\Lambda}(f^\pm) \quad \text{and} \quad S^*_{\lambda,\Lambda}(f^\pm) = \underline{S}_{\lambda,\Lambda}(-|f^\pm|) \cap \overline{S}_{\lambda,\Lambda}(|f^\pm|).$$

Definition 3.1.4. We say that a function $u \in \mathrm{USC}(B_1)$ is a viscosity subsolution of (3.1.1) in B_1 if for any φ touching u by above at x_0 in B_1, the following holds:

(*i*) If $x_0 \in B_1^\pm$ and $\varphi \in C^2(B_\delta(x_0))$, then

$$F^\pm(D^2\varphi(x_0)) \geq f^\pm(x_0).$$

(*ii*) If $x_0 \in T$ and $\varphi \in C^2(\overline{B_\delta^+(x_0)}) \cap C^2(\overline{B_\delta^-(x_0)})$, then

$$\varphi^+_{x_n}(x_0) - \varphi^-_{x_n}(x_0) \geq g(x_0),$$

where $\varphi^\pm = \varphi|_{\overline{B_\delta^\pm(x_0)}}$.

Similarly, a function $u \in \mathrm{LSC}(B_1)$ is a viscosity supersolution of (3.1.1) in B_1 if whenever a test function φ touches u from below at x_0 in B_1, then it satisfies conditions (i) and (ii), where all inequalities are reversed. Finally, a function $u \in C^0(B_1)$ is a viscosity solution of (3.1.1) in B_1 if it is a viscosity subsolution and a viscosity supersolution.

Remark 3.1.5. The following condition given in [58] is equivalent to (ii):

(ii') Let $x_0 \in T$ and let

$$\varphi(x) = P(x') + p^+ x_n^+ - p^- x_n^-$$

where P is a quadratic polynomial, $p^\pm \in \mathbb{R}$, $x_n^+ = \max\{0, x_n\}$, and $x_n^- = -\min\{0, x_n\}$. If φ touches u by above at x_0, then

$$p^+ - p^- \geq g(x_0).$$

Indeed, we may suppose that $x_0 = 0$. If $\varphi \in C^2(\overline{B_\delta^+}) \cap C^2(\overline{B_\delta^-})$ touches u from above at 0, then by the Taylor's expansion, we have that

$$\varphi(0) + A' \cdot x' + \varphi_{x_n}^+(0) x_n^+ - \varphi_{x_n}^-(0) x_n^- + B|x'|^2 + B x_n^2$$

touches u from above at 0, possibly in a smaller neighborhood $B_{\delta'}$, where $\|\nabla' \varphi\|_{L^\infty(B_{\delta'})} \leq A'$ and $\sup_{y \in B_{\delta'} \cap \{y_n \neq 0\}} \|D^2 \varphi(y)\| \leq 2B$. Let $p^\pm = \varphi_{x_n}^\pm(0)$. Then for $\varepsilon > 0$ small, we get

$$\varphi(0) + A' \cdot x' + (p^+ + \varepsilon) x_n^+ - (p^- - \varepsilon) x_n^-$$

also touches u from above at 0 in B'_δ, provided we choose $\delta' > 0$ small enough so that $\varepsilon|x_n| - Bx_n^2 \geq 0$, for any $x \in B_{\delta'}$. By (ii'), it follows that

$$(p^+ + \varepsilon) - (p^- - \varepsilon) \geq g(0).$$

Letting $\varepsilon \to 0$, we conclude the desired inequality.

Lemma 3.1.6. *Definition 3.1.4 is equivalent to replacing* (ii) *by the following statement: if* $x_0 \in T$ *and* $\varphi \in C^2(\overline{B_\delta^+(x_0)}) \cap C^2(\overline{B_\delta^-(x_0)})$ *touches* u *by above at* x_0*, then either*

$$F^\pm(D^2\varphi^\pm(x_0)) \geq f^\pm(x_0) \quad \textit{or} \quad \varphi^+_{x_n}(x_0) - \varphi^-_{x_n}(x_0) \geq g(x_0).$$

Proof. If u is a viscosity subsolution of (3.1.1), then it is clear that the statement is true. To prove the converse, let $x_0 \in T$ and assume that $\varphi \in C^2(\overline{B_\delta^+(x_0)}) \cap C^2(\overline{B_\delta^-(x_0)})$ touches u by above at x_0. Suppose by means of contradiction that

$$\varphi^+_{x_n}(x_0) - \varphi^-_{x_n}(x_0) < g(x_0). \tag{3.1.2}$$

Define the function

$$\psi(x) = \varphi(x) + \eta|x_n| - C|x_n|^2 \quad \text{for } x \in B_\tau,$$

where $\eta, \tau, C > 0$ are constants to be determined.

For η small, and C large fixed, we choose $\tau < r$ such that

$$\eta|x_n| - C|x_n|^2 \geq 0 \quad \text{in } B_\tau.$$

In particular, $\psi \in C^2(\overline{B^+_\tau(x_0)}) \cap C^2(\overline{B^-_\tau(x_0)})$ and

$$\begin{cases} \psi(x_0) = \varphi(x_0) = u(x_0) \\ \psi(x) \geq \varphi(x) \geq u(x) \qquad \text{for } x \in B_\tau(x_0). \end{cases}$$

Then ψ is a test function touching u by above at x_0, and thus, either

$$F^{\pm}(D^2\psi(x_0)) \geq f^{\pm}(x_0) \quad \text{or} \quad \psi^+_{x_n}(x_0) - \psi^-_{x_n}(x_0) \geq g(x_0). \tag{3.1.3}$$

We will see that both of these conditions cannot happen, hence reaching a contradiction. Indeed, by (3.1.2), and choosing η sufficiently small, we get

$$\psi^+_{x_n}(x_0) - \psi^-_{x_n}(x_0) = \varphi^+_{x_n}(x_0) - \varphi^-_{x_n}(x_0) + 2\eta < g(x_0).$$

Therefore, the first inequality in (3.1.3) must hold. Call $E_n = e_n e_n^T \in \mathcal{S}^n$. Then

$$\begin{aligned} \mathcal{M}^+_{\lambda/n,\Lambda}(D^2\psi(x_0)) &= \mathcal{M}^+_{\lambda/n,\Lambda}(D^2\varphi(x_0) - 2CE_n) \\ &\leq \mathcal{M}^+_{\lambda/n,\Lambda}(D^2\varphi(x_0)) - 2C\lambda/n \\ &< f^+(x_0), \end{aligned}$$

choosing C sufficiently large. This is a contradiction since by uniform ellipticity, we have

$$\begin{aligned} f^+(x_0) \leq F^+(D^2\psi(x_0)) &\leq \Lambda\|[D^2\psi(x_0)]^+\| - \lambda\|[D^2\psi(x_0)]^-\| \\ &\leq \mathcal{M}^+_{\lambda/n,\Lambda}(D^2\psi(x_0)). \end{aligned}$$

We conclude that

$$\varphi^+_{x_n}(x_0) - \varphi^-_{x_n}(x_0) \geq g(x_0).$$

$\square$

3.1.2 Main results

Our main theorems are the following.

Theorem 3.1.7 (Existence and uniqueness)**.** *Let* $f^{\pm} \in C^0(B_1^{\pm} \cup T)$*,* $g \in C^0(T)$*, and* $\phi \in C^0(\partial B_1)$*. Then there exists a unique viscosity solution* $u \in C^0(\overline{B_1})$ *of* (3.1.1) *in* B_1 *with* $u = \phi$ *on* ∂B_1*.*

Theorem 3.1.8 ($C^{0,\alpha}$ regularity)**.** *Let* u *satisfy*

$$\begin{cases} u \in S^*_{\lambda,\Lambda}(f^{\pm}) & \text{in } B_1 \\ u^+_{x_n} - u^-_{x_n} = g & \text{on } T, \end{cases}$$

with $f^{\pm} \in C^0(B_1^{\pm}) \cap L^{\infty}(B_1^{\pm})$ *and* $g \in L^{\infty}(T)$*. Then* $u \in C^{0,\alpha_1}(\overline{B_{1/2}})$ *with*

$$\|u\|_{C^{0,\alpha_1}(\overline{B_{1/2}})} \le C\big(\|u\|_{L^{\infty}(B_1)} + \|g\|_{L^{\infty}(T)} + \|f^-\|_{L^n(B_1^-)} + \|f^+\|_{L^n(B_1^+)}\big),$$

where $0 < \alpha_1 < 1$ *and* $C > 0$ *depend only on* $n, \lambda,$ *and* Λ*.*

Theorem 3.1.9 ($C^{1,\alpha}$ regularity)**.** *Let* $0 < \alpha < \bar{\alpha}$*, where* $\bar{\alpha} < 1$ *is given in Theorem 3.6.1. Assume that* $g \in C^{0,\alpha}(T)$ *and* $f^{\pm} \in C^0(B_1^{\pm} \cup T)$ *satisfies*

$$\Big(\fint_{B_r(x_0) \cap B_1^{\pm}} |f^{\pm}|^n \, dx\Big)^{1/n} \le C_{f^{\pm}} r^{\alpha - 1}, \tag{3.1.4}$$

for all $x_0 \in B_1^{\pm} \cup T$ *and* $r < 1$*. Let* u *be a bounded viscosity solution of* (3.1.1) *in* B_1*. Then*

$$u^{\pm} \in C^{1,\alpha}(\overline{B_{1/2}^{\pm}}),$$

and the following estimate holds:

$$\|u^{\pm}\|_{C^{1,\alpha}(\overline{B_{1/2}^{\pm}})} \le C\big(\|u\|_{L^{\infty}(B_1)} + \|g\|_{C^{0,\alpha}(T)} + C_{f^-} + C_{f^+}\big), \tag{3.1.5}$$

where $C > 0$ *depends only on* n*,* λ*,* Λ*, and* α*.*

Remark 3.1.10. Note that the condition given in (3.1.4) is weaker than asking Hölder continuity. For instance, (3.1.4) holds if $f \in L^p(B_1)$ with $p > n$ and $0 < \alpha \le 1 - n/p$. Indeed, for any $B_r(x_0) \subset B_1$, by Hölder's inequality with $1/p' = n/p$ and $1/q' = 1 - n/p$, we get

$$\begin{aligned}
\Big(\fint_{B_r(x_0)} |f|^n \, dx \Big)^{1/n} &= \frac{1}{|B_r|^{1/n}} \Big(\int_{B_r(x_0)} |f|^n \, dx \Big)^{1/n} \\
&\le \frac{C}{r} \Big(\| |f|^n \|_{L^{p'}(B_r(x_0))} |B_r|^{1/q'} \Big)^{1/n} \\
&= \frac{C}{r} \|f\|_{L^p(B_r(x_0))} r^{1-n/p} \\
&\le C \|f\|_{L^p(B_1)} r^{\alpha - 1}
\end{aligned}$$

where $C > 0$ depends only on n.

Remark 3.1.11. L. Caffarelli and X. Cabré prove in [15, Theorem 8.3] that if u is a viscosity solution of $F(D^2 u) = f$ in B_1, where $F \in \mathcal{E}(\lambda, \Lambda)$ and f satisfies (3.1.4), then $u \in C^{1,\bar{\alpha}}_{loc}(B_1)$, for some $\bar{\alpha} < 1$ depending only on n, λ, and Λ. Hence, to prove Theorem 3.1.9, it is enough to derive pointwise $C^{1,\alpha}$ estimates at the interface. Furthermore, (3.1.5) follows by a standard argument of patching the interior and boundary estimates.

3.2 ABP estimate

A key tool in the regularity theory of viscosity solutions is the Alexandroff – Bakelman–Pucci estimate, also known as the ABP estimate. In particular, for any supersolution u in B_1, we give a bound for the infimum of u in B_1, in terms of the infimum of u on ∂B_1, the supremum of g on T and the L^n-norm of $f^{\pm}$. More precisely:

Theorem 3.2.1 (ABP estimate)**.** *Let u satisfy*

$$\begin{cases} u \in \overline{S}_{\lambda,\Lambda}(f^{\pm}) & \text{in } B_1^{\pm} \\ u_{x_n}^{+} - u_{x_n}^{-} \le g & \text{on } T \end{cases} \tag{3.2.1}$$

with $f^{\pm} \in C^0(B_1^{\pm}) \cap L^{\infty}(B_1^{\pm})$ and $g \in L^{\infty}(T)$. Then

$$\sup_{B_1} u_- \le \sup_{\partial B_1} u_- + C\big(\max_T g_+ + \|f_+^-\|_{L^n(B_1^- \cap \{u=\Gamma_u\})} + \|f_+^+\|_{L^n(B_1^+ \cap \{u=\Gamma_u\})}\big)$$

where $C > 0$ depends only on n, λ, and Λ. We denote by $u_- = -\min\{0, u\}$, $g_+ = \max\{0, g\}$, $f_+^{\pm} = \max\{0, f^{\pm}\}$, and Γ_u is the convex envelope of $-u_-$ on $\overline{B_2}$ with $u \equiv 0$ on $\overline{B_2} \setminus B_1$.

Remark 3.2.2. To prove Theorem 3.2.1, we will proceed similarly as in the classical approach (see [15, Chapter 3]). A key step in that proof is to show that the convex envelope is $C^{1,1}$ at the contact points. In general, this is not the case for functions satisfying (3.2.1) since the transmission condition $u_{x_n}^{+} - u_{x_n}^{-} \le g$ prescribes an angle on the graph of u on T. Now, the convex envelope may touch u at T, and thus, it will be singular at those points. To overcome this difficulty, we consider an auxiliary function that makes the angle concave, in some sense. Hence, the convex envelope will not touch this function on T and, in particular, it will be $C^{1,1}$. We show this in the next proof.

First, we state a couple of lemmas from [15, Chapter 3].

Lemma 3.2.3. *Let $u \in \overline{S}_{\lambda,\Lambda}(f)$ in B_δ. Assume that f is bounded and φ is a convex function such that $0 \le \varphi \le u$ in B_δ and $\varphi(0) = u(0) = 0$. Then*

$$\varphi(x) \le C\Big(\sup_{B_\delta} f_+\Big)|x|^2 \quad \text{for all } x \in B_{\gamma\delta},$$

where $0 < \gamma < 1$ and $C > 0$ are constants.

Lemma 3.2.4. *Let $u \in \mathrm{LSC}(\overline{B_1})$ such that $u \geq 0$ on ∂B_1 and let Γ_u be defined as in Theorem 3.2.1. Let $K > 0$ and $0 < r \leq 1$ be constants. Assume that for any $x_0 \in \overline{B_1} \cap \{u = \Gamma_u\}$, there exists a convex paraboloid of opening K that touches Γ_u by above at x_0 in $B_r(x_0)$. Then $\Gamma_u \in C^{1,1}(\overline{B_1})$, and thus, there exists a set $E \subset B_1$ such that $|B_1 \backslash E| = 0$, and Γ_u is second order differentiable at any $x \in E$. Moreover,*

$$\sup_{B_1} u_- \leq C\Big(\int_{E\cap\{u=\Gamma_u\}} \det D^2\Gamma_u(x)\,dx\Big)^{1/n},$$

where $C > 0$ is a constant depending only on n.

Proof of Theorem 3.2.1. Fix $\varepsilon > 0$ small and consider in B_1 the function

$$v = u - \tfrac{1}{2}\big(\max_T g_+ + \varepsilon\big)|x_n|.$$

By [15, Lemma 2.12], we have that $v \in \overline{S}_{\lambda,\Lambda}(f^{\pm})$ in $B_1^{\pm}$. Also,

$$v_{x_n}^+ - v_{x_n}^- \leq g - \big(\max_T g_+ + \varepsilon\big) \leq g_+ - \max_T g_+ - \varepsilon \leq -\varepsilon \quad \text{on } T,$$

in the viscosity sense. Without loss of generality, we may assume that $v \geq 0$ on ∂B_1. Otherwise, we consider $v - \inf_{\partial B_1} v$. Assume that $v_- \not\equiv 0$, and let Γ_v be the convex envelope of $-v_-$ on $\overline{B_2}$, where we have extended v by zero outside of B_1. Clearly, by definition of Γ_v, we have that $\partial B_1 \cap \{v = \Gamma_v\} = \varnothing$. Also, we claim that $T \cap \{v = \Gamma_v\} = \varnothing$. Indeed, if $A \cdot x + b$ touches v from below at $x_0 \in T$, for some $A \in \mathbb{R}^n$ and $b \in \mathbb{R}$, then

$$-\varepsilon \geq A \cdot e_n - A \cdot e_n = 0,$$

which is a contradiction. Moreover, there exists $\delta > 0$ such that for any $x_0 \in T$, we have $B_\delta(x_0) \cap \{v = \Gamma_v\} = \varnothing$. If not, for any $k \geq 1$, there exist $x_k \in T$ such that there is some $y_k \in B_{1/k}(x_k) \cap \{v = \Gamma_v\}$. Then, up to a subsequence, it follows that $x_k, y_k \to y_0$ for some $y_0 \in \overline{T \cap \{v = \Gamma_v\}}$, which is a contradiction.

Next, we show that $\Gamma_v \in C^{1,1}(\overline{B_1})$. By Lemma 3.2.4, it is enough to see that there are constants $K > 0$ and $0 < r \leq 1$ such that for any $x_0 \in \overline{B_1} \cap \{v = \Gamma_v\}$, there exists a convex paraboloid of opening K that touches Γ_v by above at x_0 in $B_r(x_0)$. Namely,

$$\Gamma_v(x) \leq l(x) + \tfrac{K}{2}|x - x_0|^2 \quad \text{for all } x \in B_r(x_0), \tag{3.2.2}$$

for some $K, r > 0$ (independent of x_0) and some linear function l such that $l(x_0) = \Gamma_v(x_0)$. Indeed, fix $x_0 \in \overline{B_1} \cap \{v = \Gamma_v\}$. Since $x_0 \notin \partial B_1 \cup T$, we may assume that $x_0 \in B_1^+ \cap \{v = \Gamma_v\}$. Furthermore, $B_\delta(x_0) \subset B_1^+$, for δ small enough. Let l be a supporting plane of Γ_v at x_0. Then $0 \leq \Gamma_v - l \leq -v_- - l$ in $B_\delta(x_0)$ and $\Gamma_v(x_0) - l(x_0) = -v_-(x_0) - l(x_0) = 0$. By [15, Proposition 2.8], we know that $-v_- - l \in \overline{S}_{\lambda,\Lambda}(f^+)$. Applying Lemma 3.2.3 to $-v_- - l$ in $B_\delta(x_0)$ and $\varphi = \Gamma_v - l$, we get

$$\Gamma_v(x) \leq l(x) + C^+\Big(\sup_{B_\delta(x_0)} f_+^+\Big)|x - x_0|^2 \quad \text{for all } x \in B_{\delta\gamma^+}(x_0) \tag{3.2.3}$$

where $\gamma^+ < 1$ and C^+ are universal constants. If $x_0 \in B_1^- \cap \{v = \Gamma_v\}$, the proof is analogous. Hence, choosing $K = 2\max\{C^+\|f_+^+\|_\infty, C^-\|f_+^-\|_\infty\}$ and $r = \delta\min\{\gamma^+, \gamma^-\}$, we get (3.2.2).

By Lemma 3.2.4, there exists a set $E \subset B_1$ such that $|B_1 \setminus E| = 0$, and

Γ_v is second order differentiable at any $x \in E$. Moreover, we have that

$$\sup_{B_1} v_- \leq C\Big(\int_{E\cap\{v=\Gamma_v\}} \det D^2\Gamma_v(x)\,dx\Big)^{1/n},$$

where $C > 0$ is a constant depending only on n. Since $f^+ \in C^0(B_1^+)$, then letting $\delta \to 0$ in (3.2.3), we see that $\det D^2\Gamma_v(x_0) \leq C f_+^+(x_0)^n$ for almost every $x_0 \in B_1^+ \cap \{v = \Gamma_v\}$. Therefore,

$$\int_{E\cap\{v=\Gamma_v\}} \det D^2\Gamma_v(x)\,dx \leq \int_{B_1^-\cap\{v=\Gamma_v\}} f_+^-(x)^n\,dx + \int_{B_1^+\cap\{v=\Gamma_v\}} f_+^+(x)^n\,dx.$$

Combining the two previous estimates, it follows that

$$\sup_{B_1} v_- \leq \sup_{\partial B_1} v_- + C\Big(\|f_+^-\|_{L^n(B_1^-\cap\{v=\Gamma_v\})} + \|f_+^+\|_{L^n(B_1^+\cap\{v=\Gamma_v\})}\Big).$$

From the definition of v, we have that

$$\sup_{B_1} u_- \leq \sup_{B_1} v_- \quad \text{and} \quad \sup_{\partial B_1} v_- \leq \sup_{\partial B_1} u_- + \tfrac{1}{2}\big(\max_T g_+ + \varepsilon\big)|x_n|.$$

Hence, letting $\varepsilon \to 0$, we get

$$\sup_{B_1} u_- \leq \sup_{\partial B_1} u_- + C\big(\max_T g_+ + \|f_+^-\|_{L^n(B_1^-\cap\{u=\Gamma_u\})} + \|f_+^+\|_{L^n(B_1^+\cap\{u=\Gamma_u\})}\big),$$

where $C > 0$ depends only on n, λ, and Λ. Note that $\{v = \Gamma_v\} \subseteq \{u = \Gamma_u\}$, since $\Gamma_v + \frac{1}{2}\big(\max_T g_+ + \varepsilon\big)|x_n|$ is convex, and Γ_u is the largest convex function that lies below u. $\square$

An immediate consequence of the ABP estimate is the maximum principle.

Corollary 3.2.5 (Maximum principle)**.** *Let u satisfy*

$$\begin{cases} u \in \overline{S}_{\lambda,\Lambda}(0) & \text{in } B_1^{\pm} \\ u_{x_n}^+ - u_{x_n}^- \geq 0 & \text{on } T. \end{cases}$$

If $u \geq 0$ on ∂B_1, then $u \geq 0$ in B_1.

Remark 3.2.6. Replacing u by $-u$, we get the maximum principle for sub-solutions.

3.3 Hölder regularity across interface

In this section, we prove Hölder regularity of viscosity solutions across T. In particular, Theorem 3.1.8 follows by a standard argument (e.g., see [26, Lemma 8.23]) from the next oscillation lemma, which gives a geometric decay of the oscillation.

Lemma 3.3.1 (Oscillation lemma)**.** *Let u satisfy*

$$\begin{cases} u \in S^*_{\lambda,\Lambda}(f^{\pm}) & \text{in } B_1 \\ u_{x_n}^+ - u_{x_n}^- = g & \text{on } T, \end{cases}$$

with $f^{\pm} \in C^0(B_1^{\pm}) \cap L^\infty(B_1^{\pm})$ and $g \in L^\infty(T)$. Then

$$\operatorname*{osc}_{B_{1/3}} u \leq \mu \operatorname*{osc}_{B_1} u + C\big(\|g\|_{L^\infty(T)} + \|f^-\|_{L^n(B_1^-)} + \|f^+\|_{L^n(B_1^+)}\big),$$

where $0 < \mu < 1$ and C depend only on n, λ, and Λ.

The oscillation lemma will be a consequence of the next result. The ideas are inspired by [58].

Lemma 3.3.2. *Let u, $f^{\pm}$, and g be as in Lemma 3.3.1. Assume further that $\|u\|_{L^\infty(B_1)} \leq 1$ and $u(\bar{x}) \geq 0$ with $\bar{x} = \frac{1}{5}e_n$. There exist $0 < \varepsilon_0, c < 1$ depending on n, λ, and Λ such that if $\|g\|_{L^\infty(T)} + \|f^-\|_{L^n(B_1^-)} + \|f^+\|_{L^n(B_1^+)} \leq \varepsilon_0$, then*

$$\inf_{B_{1/3}} u \geq -1 + c.$$

Proof. Since $u + 1 \geq 0$, by the Harnack inequality in $B_{1/20}(\bar{x})$ (see [15, Theorem 4.3]), we get

$$\sup_{B_{1/20}(\bar{x})} (u+1) \leq C\Big(\inf_{B_{1/20}(\bar{x})} (u+1) + \|f^+\|_{L^n(B_1^+)} \Big),$$

where $C \geq 1$ is a universal constant. Then

$$1 \leq u(\bar{x}) + 1 \leq \sup_{B_{1/20}(\bar{x})} (u+1) \leq C\big(u(x) + 1 + \varepsilon_0\big)$$

for any $x \in B_{1/20}(\bar{x})$, and thus,

$$u \geq -1 + \tilde{c} \quad \text{in } B_{1/20}(\bar{x}), \tag{3.3.1}$$

with $\tilde{c} = 1/C - \varepsilon_0$ and $\varepsilon_0 < 1/C$. For $x \in D = B_{3/4}(\bar{x}) \setminus B_{1/20}(\bar{x})$, we define

$$v(x) = \eta\phi(r) + \varepsilon_0|x_n|, \quad \phi(r) = r^{-\gamma} - (2/3)^{-\gamma}, \quad r = |x - \bar{x}|,$$

where $\gamma > \max\big\{0, \frac{\Lambda}{\lambda}(n-1) - 1\big\}$ and $\eta > 0$ to be chosen later. We have

$$\begin{aligned} \partial_{ij}\phi(x) &= \gamma(\gamma+2)r^{-\gamma-4}(x_i - \bar{x}_i)(x_j - \bar{x}_j) \quad \text{if } i \neq j \\ \partial_{ii}\phi(x) &= \gamma r^{-\gamma-2}\big((\gamma+2)r^{-2}(x_i - \bar{x}_i)^2 - 1\big). \end{aligned}$$

If $1/20 \leq r \leq 3/4$, then at the point $x = re_1 + \bar{x}$,

$$\partial_{ij}\phi(x) = 0 \quad \text{if } i \neq j$$
$$\partial_{11}\phi(x) = \gamma(\gamma+1)r^{-\gamma-2}$$
$$\partial_{ii}\phi(x) = -\gamma r^{-\gamma-2} \quad \text{if } i > 1.$$

By rotational symmetry, for any $x \in D^{\pm}$ we get

$$\mathcal{M}^{-}_{\lambda,\Lambda}(D^2 v^{\pm}(x)) = \eta\mathcal{M}^{-}_{\lambda,\Lambda}(D^2\phi(x)) = \eta\gamma r^{-\gamma-2}\big(\lambda(\gamma+1) - \Lambda(n-1)\big) > 0$$

by the choice of γ. For $x \in T \cap D$, it follows that

$$v^{+}_{x_n}(x) - v^{-}_{x_n}(x) = 2\varepsilon_0 \geq \|g\|_{L^\infty(T)} \geq g(x).$$

We will choose η and ε_0 so that $v \leq \tilde{c}$ on $\partial B_{1/20}(\bar{x})$ and $v \leq 0$ on $\partial B_{3/4}(\bar{x})$. Note that $\phi(r) \geq 0$ if $0 < r \leq 2/3$, and $\phi(r) \leq 0$ if $r \geq 2/3$. First, choose η such that $\eta \leq \frac{\tilde{c}}{2\phi(1/20)}$. Then choose ε_0 such that $\varepsilon_0 \leq \min\big\{\tilde{c}/2, -\eta\phi(3/4)\big\}$. By (3.3.1) we obtain that

$$v \leq u + 1 \quad \text{on } \partial D.$$

Since $u + 1 \in \overline{S}_{\lambda,\Lambda}(|f^{\pm}|)$ in D, $v^{\pm} \in C^2(D^{\pm})$, and $\mathcal{M}^{-}_{\lambda,\Lambda}(D^2 v^{\pm}) \geq 0$ in $D^{\pm}$, by [15, Lemma 2.12], we have that $u + 1 - v \in \overline{S}_{\lambda,\Lambda}(|f^{\pm}|)$ in D. Also,

$$(u+1-v)^{+}_{x_n} - (u+1-v)^{-}_{x_n} \leq g - (v^{+}_{x_n} - v^{-}_{x_n}) \leq g - g = 0 \quad \text{on } T \cap D,$$

in the viscosity sense. Hence, applying the ABP estimate (Theorem 3.2.1) to $u + 1 - v$ in D, with $g \equiv 0$, we see that

$$\sup_{D}(u+1-v)_{-} \leq \sup_{\partial D}(u+1-v)_{-} + C\big(\|f^{-}\|_{L^n(B_1^{-})} + \|f^{+}\|_{L^n(B_1^{+})}\big) \leq C\varepsilon_0,$$

where $C > 0$ depends only on n, λ, and Λ. Therefore, $u \geq -1 + v - C\varepsilon_0$ in D. Moreover, for any $x \in B_{1/3}(0) \setminus B_{1/20}(\bar{x})$, we have that

$$v(x) \geq \eta\phi(23/60) = c_1$$

and $c_1 > 0$ depends only on n, λ, and Λ. Choosing ε_0 such that $\varepsilon_0 \leq \frac{c_1}{2C}$, we get $u \geq -1 + \frac{c_1}{2}$ in $B_{1/3}(0) \setminus B_{1/20}(\bar{x})$. Therefore,

$$\inf_{B_{1/3}} u \geq -1 + c,$$

with $c = \min\{\tilde{c}, \frac{c_1}{2}\}$. □

Proof of Lemma 3.3.1. Let $M = \|g\|_{L^\infty(T)} + \|f^-\|_{L^n(B_1^-)} + \|f^+\|_{L^n(B_1^+)}$ and let ε_0 be as in Lemma 3.3.2. Consider the rescaled function:

$$\tilde{u} = \frac{2u - (\inf_{B_1} u + \sup_{B_1} u)}{\operatorname{osc}_{B_1} u + 2M/\varepsilon_0} \in S^*_{\lambda,\Lambda}(\tilde{f}^\pm),$$

with $\tilde{f}^\pm = 2f^\pm(\operatorname{osc}_{B_1} u + 2M/\varepsilon_0)^{-1}$. Also, $(\tilde{u}^+)_{x_n} - (\tilde{u}^-)_{x_n} \leq \tilde{g}$ on T, in the viscosity sense, with $\tilde{g} = 2g(\operatorname{osc}_{B_1} u + 2M/\varepsilon_0)^{-1}$. Note that $\|\tilde{u}\|_{L^\infty(B_1)} \leq 1$, and

$$\|\tilde{g}\|_{L^\infty(T)} + \|\tilde{f}^-\|_{L^n(B_1^-)} + \|\tilde{f}^+\|_{L^n(B_1^+)} \leq \varepsilon_0.$$

If $\tilde{u}(\bar{x}) \geq 0$, then by Lemma 3.3.2, it follows that $\inf_{B_{1/3}} \tilde{u} \geq -1+c$. Otherwise, $\tilde{u}(\bar{x}) < 0$, and applying the lemma to $-\tilde{u}$, we see that $\sup_{B_{1/3}} \tilde{u} \leq 1 - c$. In both cases, we get

$$\operatorname*{osc}_{B_{1/3}} \tilde{u} = \sup_{B_{1/3}} \tilde{u} - \inf_{B_{1/3}} \tilde{u} = \frac{2 \operatorname{osc}_{B_{1/3}} u}{\operatorname{osc}_{B_1} u + 2M/\varepsilon_0} \leq 2 - c.$$

Therefore,

$$\operatorname*{osc}_{B_{1/3}} u \le \mu \operatorname*{osc}_{B_1} u + C\big(\|g\|_{L^\infty(T)} + \|f^-\|_{L^n(B_1^-)} + \|f^+\|_{L^n(B_1^+)}\big),$$

with $\mu = \frac{2-c}{2} < 1$ and $C = \frac{2-c}{\varepsilon_0}$. $\square$

When $f^\pm \equiv 0$ and g has compact support on T, we obtain the following global Hölder estimate.

Proposition 3.3.3. *Assume $u \in C^0(\overline{B_1})$ satisfies*

$$\begin{cases} u \in S_{\lambda,\Lambda}(0) & \text{in } B_1^\pm \\ u_{x_n}^+ - u_{x_n}^- = g & \text{on } T \\ u = \varphi & \text{on } \partial B_1, \end{cases}$$

where $g \in L^\infty(T)$, $\operatorname{supp} g \subset T \cap B_{1-2\rho}$, for some $0 < \rho < 1/4$, and $\varphi \in C^{0,\alpha}(\partial B_1)$, with $0 < \alpha < 1$. Then $u \in C^{0,\beta}(\overline{B_1})$, with $0 < \beta \le \min\{\alpha_1, \alpha/2\}$, and

$$\|u\|_{C^{0,\beta}(\overline{B_1})} \le \frac{C}{\rho^\gamma}\big(\|\varphi\|_{C^{0,\alpha}(\partial B_1)} + \|g\|_{L^\infty(T)}\big),$$

where $0 < \alpha_1 < 1$ is given in Theorem 3.1.8, $\gamma = \max\{\alpha_1, \alpha\}$, and C depends only on n, λ, and Λ.

This result follows from the interior Hölder regularity (Theorem 3.1.8) and the following boundary pointwise Hölder estimate.

Lemma 3.3.4. *Assume we are under the same conditions of Proposition 3.3.3. Then $u \in C^{0,\alpha/2}(x_0)$ for any $x_0 \in \partial B_1$, with*

$$\sup_{x \in B_\rho(x_0) \cap B_1} \frac{|u(x) - u(x_0)|}{|x - x_0|^{\alpha/2}} \le \frac{2^{\alpha/2}}{\rho^\alpha}\big(\|\varphi\|_{L^\infty(\partial B_1)} + [\varphi]_{C^{0,\alpha}(x_0)} + C\|g\|_{L^\infty(T)}\big),$$

where $C > 0$ depends only on n, λ, Λ, and α.

Proof. After a translation and rotation, we can assume that $B_1 \equiv B_1(e_n)$, $x_0 = 0$, and $\varphi(x_0) = 0$. Let $M = 2^{\alpha/2}\rho^{-\alpha}\big(\|\varphi\|_{L^\infty(\partial B_1)} + [\varphi]_{C^{0,\alpha}(x_0)} + C\|g\|_{L^\infty(T)}\big)$. Note that if $x \in \partial B_1$, then $|x|^2 = 2x_n$. Therefore, for any $x \in \partial B_1$, we have

$$u(x) = \varphi(x) \le [\varphi]_{C^{0,\alpha}(0)}|x|^\alpha = 2^{\alpha/2}[\varphi]_{C^{0,\alpha}(0)}x_n^{\alpha/2} \le Mx_n^{\alpha/2}. \tag{3.3.2}$$

From the ABP estimate (Theorem 3.2.1), for any $x \in \partial B_\rho(0) \cap B_1$, we get

$$u(x) \le \|u\|_{L^\infty(B_1)} \le \|\varphi\|_{L^\infty(\partial B_1)} + C\|g\|_{L^\infty(T)} \le M2^{-\alpha/2}\rho^\alpha \le Mx_n^{\alpha/2}, \tag{3.3.3}$$

where C depends only on n, λ, and Λ.

Define $h(x) = Mx_n^{\alpha/2}$, for $x \in \overline{B_\rho(0) \cap B_1}$. By (3.3.2) and (3.3.3), we have that $u - h \le 0$ on $\partial(B_\rho(0) \cap B_1)$. Moreover, for any $x \in B_\rho(0) \cap B_1$,

$$\mathcal{M}^+(D^2h(x)) = \lambda M\frac{\alpha}{2}\Big(\frac{\alpha}{2} - 1\Big)x_n^{\alpha/2-2} < 0.$$

Also, since h is smooth, $h^+_{x_n} - h^-_{x_n} = 0$ on $T \cap B_\rho(0)$. It follows that

$$\begin{cases} u - h \in \underline{S}_{\lambda,\Lambda}(0) & \text{in } B_\rho(0) \cap B_1 \\ (u-h)^+_{x_n} - (u-h)^-_{x_n} = 0 & \text{on } T \cap B_\rho(0) \\ u - h \le 0 & \text{on } \partial(B_\rho(0) \cap B_1). \end{cases}$$

From the maximum principle (Corollary 3.2.5), we get

$$u(x) \le h(x) = Mx_n^{\alpha/2} \le M|x|^{\alpha/2} \quad \text{for any } x \in B_\rho(0) \cap B_1.$$

Applying this result to $-u$, and taking the supremum over all $x \in B_\rho(0) \cap B_1$, we get

$$\sup_{x \in B_\rho(0)\cap B_1} \frac{|u(x)|}{|x|^{\alpha/2}} \le \frac{2^{\alpha/2}}{\rho^\alpha}\big(\|\varphi\|_{L^\infty(\partial B_1)} + [\varphi]_{C^{0,\alpha}(0)} + C\|g\|_{L^\infty(T)}\big).$$

□

The following proof is in the same spirit as [15, Proposition 4.13].

Proof of Proposition 3.3.3. We need to estimate $\|u\|_{C^{0,\beta}(\overline{B_1})} = \|u\|_{L^\infty(B_1)} + [u]_{C^{0,\beta}(\overline{B_1})}$. From the ABP estimate (Theorem 3.2.1), we have that

$$\|u\|_{L^\infty(B_1)} \le C\big(\|\varphi\|_{L^\infty(\partial B_1)} + \|g\|_{L^\infty(T)}\big),$$

where C depends only on n, λ, and Λ. Hence, it remains to control $[u]_{C^{0,\beta}(\overline{B_1})}$. Fix any $x, y \in B_1$. Let $d_x = \operatorname{dist}(x, \partial B_1)$, $d_y = \operatorname{dist}(y, \partial B_1)$, and assume without loss of generality that $d_y \le d_x$. Take $x_0, y_0 \in \partial B_1$ such that $|x - x_0| = d_x$ and $|y - y_0| = d_y$.

We study three cases. First, assume that $d_x \ge d_y \ge \rho/2$. Then since $0 < \beta \le \alpha_1$, by Theorem 3.1.8 (rescaled), it follows that

$$\frac{|u(x) - u(y)|}{|x - y|^\beta} \le [u]_{C^{0,\alpha_1}(\overline{B_{1-\rho/2}})} \le \frac{C}{\rho^{\alpha_1}}\big(\|\varphi\|_{L^\infty(\partial B_1)} + \|g\|_{L^\infty(T)}\big).$$

Second, assume that $d_y \le d_x \le \rho/2$. If $|x - y| \le d_x/2$, then

$$y \in \overline{B_{d_x/2}(x)} \subset B_{d_x}(x) \subset B_\rho(x_0) \cap B_1.$$

Applying Theorem 3.1.8 (rescaled) to $u - u(x_0)$ in $B_{d_x}(x)$, we get

$$d_x^\beta \frac{|u(x) - u(y)|}{|x - y|^\beta} \le d_x^{\alpha_1} \frac{|u(x) - u(y)|}{|x - y|^{\alpha_1}} \le C \sup_{z \in B_{d_x}(x)} |u(z) - u(x_0)|.$$

By Lemma 3.3.4, we get

$$\sup_{z \in B_{d_x}(x)} |u(z) - u(x_0)| \le d_x^\beta \frac{2^{\alpha/2}}{\rho^\alpha}\big(\|\varphi\|_{L^\infty(\partial B_1)} + [\varphi]_{C^{0,\alpha}(x_0)} + C\|g\|_{L^\infty(T)}\big).$$

Combining both inequalities, we see that

$$\frac{|u(x)-u(y)|}{|x-y|^\beta} \le \frac{C}{\rho^\alpha}\big(\|\varphi\|_{C^{0,\alpha}(\partial B_1)} + \|g\|_{L^\infty(T)}\big).$$

If $|x-y| > d_x/2$, in particular, $d_y \le d_x \le 2|x-y|$, and thus, by Lemma 3.3.4,

$$\begin{aligned}|u(x)-u(y)| &\le |u(x)-u(x_0)| + |u(x_0)-u(y_0)| + |u(y_0)-u(y)| \\ &\le \frac{2^{\alpha/2}}{\rho^\alpha}(\|\varphi\|_{C^{0,\alpha}(\partial B_1)} + C\|g\|_{L^\infty(T)})\big(d_x^{\alpha/2} + |x_0-y_0|^\alpha + d_y^{\alpha/2}\big) \\ &\le \frac{C}{\rho^\alpha}(\|\varphi\|_{C^{0,\alpha}(\partial B_1)} + \|g\|_{L^\infty(T)})|x-y|^\beta,\end{aligned}$$

since $|x_0-y_0| \le d_x + |x-y| + d_y \le 5|x-y|$.

Third, assume that $d_y \le \rho/2 \le d_x$. Let z be on the intersection between $\partial B_{1-\rho/2}$ and the segment that joins the points x and y. Then $|x-z| \le |x-y|$, $|z-y| \le |x-y|$, and $d_z = \rho/2$. Hence, we can use the first case for x and z, and the second case for y and z. We see that

$$\begin{aligned}|u(x)-u(y)| &\le |u(x)-u(z)| + |u(z)-u(y)| \\ &\le \frac{C}{\rho^\gamma}\big(\|\varphi\|_{L^\infty(\partial B_1)} + \|g\|_{L^\infty(T)}\big)\big(|x-z|^\beta + |y-z|^\beta\big) \\ &\le \frac{C}{\rho^\gamma}\big(\|\varphi\|_{L^\infty(\partial B_1)} + \|g\|_{L^\infty(T)}\big)|x-y|^\beta,\end{aligned}$$

where $\gamma = \max\{\alpha_1, \alpha\}$, and the last inequality follows by concavity. $\square$

3.4 Lower and upper ε-envelopes

This section is devoted to the study of a family of regularizations in the x'-direction, which play the same role as the Jensen's approximations (see [15, Chapter 5]). The following definition was introduced in [58].

Definition 3.4.1. Given $u \in \mathrm{USC}(B_1)$, for any $\varepsilon > 0$, we define the upper ε-envelope of u in the x'-direction as

$$u^\varepsilon(y', y_n) = \sup_{x \in \overline{B_\rho} \cap \{x_n = y_n\}} \left\{ u(x', y_n) - \tfrac{1}{\varepsilon} |x' - y'|^2 \right\}$$

for $y = (y', y_n) \in \overline{B_\rho} \subset B_1$. Similarly, given $u \in \mathrm{LSC}(B_1)$, we define the lower ε-envelope of u in the x'-direction as

$$u_\varepsilon(y', y_n) = \inf_{x \in \overline{B_\rho} \cap \{x_n = y_n\}} \left\{ u(x', y_n) + \tfrac{1}{\varepsilon} |x' - y'|^2 \right\}$$

for $y = (y', y_n) \in \overline{B_\rho} \subset B_1$.

Remark 3.4.2. Note that since $u \in \mathrm{USC}(B_1)$, then the supremum in the definition of u^ε is attained. Namely, for every $y \in \overline{B_\rho}$, there exists $y_\varepsilon \in \overline{B_\rho} \cap \{x_n = y_n\}$ such that

$$u^\varepsilon(y) = u(y_\varepsilon) - \tfrac{1}{\varepsilon} |y'_\varepsilon - y'|^2. \tag{3.4.1}$$

Moreover, if u is bounded in B_1, from the previous identity, it follows that

$$|y'_\varepsilon - y'| \le \left(2\varepsilon \|u\|_\infty\right)^{1/2}.$$

Similarly for the lower ε-envelope u_ε.

Lemma 3.4.3. *The following properties hold:*

(i) $u^\varepsilon \ge u$ *in* B_ρ *and* $\limsup_{\varepsilon \to 0} u^\varepsilon = u$*.*

(ii) $u^\varepsilon \in C^{0,1}_{y'}(\overline{B_\rho})$*, with* $[u^\varepsilon]_{C^{0,1}_{y'}(\overline{B_\rho})} \le 6\rho/\varepsilon$*.*

(iii) $u^\varepsilon \in C^{1,1}_{y'}$ *by below in* B_ρ. *Hence,* u^ε *is punctually second order differentiable in the* x'*-direction almost everywhere in* B_ρ.

Proof. (i) Clearly, $u^\varepsilon \geq u$ in B_ρ.For any $y \in \overline{B_\rho}$, using (3.4.1) we have

$$0 \leq u^\varepsilon(y) - u(y) = u(y_\varepsilon) - \tfrac{1}{\varepsilon}|y'_\varepsilon - y'|^2 - u(y) \leq u(y_\varepsilon) - u(y) \tag{3.4.2}$$

with $y_\varepsilon \to y$ as $\varepsilon \to 0$. Since $u \in \mathrm{USC}(B_1)$,

$$\limsup_{\varepsilon \to 0} u(y_\varepsilon) \leq u(y).$$

Therefore, taking the $\limsup$ as $\varepsilon \to 0$ in (3.4.2), we obtain the result.

(ii) Let $y_0, y_1 \in \overline{B_\rho} \subset B_1$ such that $(y_0)_n = (y_1)_n$. Take any $y \in \overline{B_\rho} \cap \{x_n = (y_0)_n\}$. Then

$$\begin{aligned} u^\varepsilon(y_0) &\geq u(y) - \tfrac{1}{\varepsilon}|y' - y'_0|^2 \\ &\geq u(y) - \tfrac{1}{\varepsilon}|y' - y'_1|^2 - \tfrac{1}{\varepsilon}|y'_1 - y'_0|^2 - \tfrac{2}{\varepsilon}|y'_1 - y'||y'_1 - y'_0| \\ &\geq u(y) - \tfrac{1}{\varepsilon}|y'_1 - y'|^2 - \tfrac{6\rho}{\varepsilon}|y'_1 - y'_0|. \end{aligned}$$

Taking the supremum over all $y \in \overline{B_\rho} \cap \{x_n = (y_1)_n\}$ we get

$$|u^\varepsilon(y_1) - u^\varepsilon(y_0)| \leq \tfrac{6\rho}{\varepsilon}|y'_1 - y'_0|.$$

(iii) Let $y_0 \in B_\rho$. Then $u^\varepsilon(y_0) = u(y_\varepsilon) - \frac{1}{\varepsilon}|y'_\varepsilon - y'_0|^2$. In particular,

$$P(y') = u(y_\varepsilon) - \tfrac{1}{\varepsilon}|y'_\varepsilon - y'|^2 \leq u^\varepsilon(y) \qquad \text{for all } y \in B_\rho \cap \{x_n = (y_0)_n\}$$

and equality holds at y_0. Hence, P touches u^ε by below at y_0 in the y'-direction. Note that

$$u^\varepsilon(y', y_n) + \tfrac{1}{\varepsilon}|y'|^2 = \sup_{x \in \overline{B_\rho} \cap \{x_n = y_n\}} \Big\{ u(x', y_n) - \tfrac{1}{\varepsilon}|x'|^2 + \tfrac{2}{\varepsilon} x' \cdot y' \Big\}.$$

Hence, $u^\varepsilon(y', y_n) + \frac{1}{\varepsilon}|y'|^2$ is convex, since it is the supremum of affine functions. By Alexandrov theorem (see [15, Proposition 1.5]), we see that u^ε is punctually second order differentiable in the x'-direction almost everywhere in B_ρ. $\square$

Given a uniformly continuous function h, we define its modulus of continuity, ω_h, as

$$w_h(r) = \sup_{|x-y|\le r} |h(x) - h(y)| \qquad \text{for } r > 0.$$

Proposition 3.4.4. *Let $f^\pm \in C^0(B_1^\pm)$ and $g \in C^0(T)$. If u is a bounded viscosity subsolution of (3.1.1), then for any $\varepsilon > 0$ small, it holds that u^ε is a viscosity subsolution of*

$$\begin{cases} F^\pm(D^2u^\varepsilon) = f_\varepsilon^\pm & \text{in } B_r^\pm \\ (u^\varepsilon)^+_{x_n} - (u^\varepsilon)^-_{x_n} = g_\varepsilon & \text{on } T_r = B_r \cap \{x_n = 0\}, \end{cases}$$

with $r \le \rho - r_\varepsilon$, $r_\varepsilon = (2\varepsilon\|u\|_\infty)^{1/2}$, $f_\varepsilon^\pm = f - \omega_{f^\pm}(r_\varepsilon)$, and $g_\varepsilon = g - \omega_g(r_\varepsilon)$.

Remark 3.4.5. If u is a bounded viscosity supersolution of (3.1.1), then u_ε is a viscosity supersolution of the previous problem.

Proof. First, we show that

$$F^\pm(D^2u^\varepsilon) \ge f_\varepsilon^\pm \quad \text{in } B_r^\pm$$

in the viscosity sense. Let $y_0 \in B_r^+$ and let $\delta > 0$ be small enough such that $B_\delta(y_0) \subset B_r^+$. Assume that $\varphi \in C^2(B_\delta(y_0))$ touches u^ε from above at y_0. By (3.4.1), we have

$$u^\varepsilon(y_0) = u(y_\varepsilon) - \tfrac{1}{\varepsilon}|y'_\varepsilon - y'_0|^2,$$

with $|y_\varepsilon - y_0| = |y'_\varepsilon - y'_0| \le r_\varepsilon$. Consider the function

$$\phi(y) = \varphi(y + y_0 - y_\varepsilon) + \tfrac{1}{\varepsilon}|y'_\varepsilon - y'_0|^2 \qquad \text{for } y \in B_r^+ \cap B_\delta(y_\varepsilon).$$

Then $\phi(y_\varepsilon) = u(y_\varepsilon)$. Note that $y + y_0 - y_\varepsilon \in B_\rho^+$ since

$$|y + y_0 - y_\varepsilon| \le |y| + |y_0 - y_\varepsilon| \le r + r_\varepsilon \le \rho,$$

and thus, by definition of u^ε, we get

$$u(y) \le u^\varepsilon(y + y_0 - y_\varepsilon) + \tfrac{1}{\varepsilon}|y'_\varepsilon - y'_0|^2.$$

Moreover, using that $\varphi(y) \ge u^\varepsilon(y)$ for all $y \in B_\delta(y_0)$, and that $y + y_0 - y_\varepsilon \in B_\delta(y_0)$, the previous estimate yields

$$u(y) \le \varphi(y + y_0 - y_\varepsilon) + \tfrac{1}{\varepsilon}|y'_\varepsilon - y'_0|^2 = \phi(y).$$

Therefore, ϕ touches u from above at y_ε. Since u is a subsolution of (3.1.1), we see that

$$F^+(D^2\varphi(y_0)) = F^+(D^2\phi(y_\varepsilon)) \ge f^+(y_\varepsilon) \ge f^+(y_0) - \omega_{f^+}(r_\varepsilon) = f_\varepsilon^+(y_0).$$

It remains to show the transmission condition,

$$(u^\varepsilon)^+_{x_n} - (u^\varepsilon)^-_{x_n} \ge g_\varepsilon \qquad \text{on } T_r$$

in the viscosity sense. Let $y_0 = (y'_0, 0) \in T_r$ and assume that

$$\varphi(y) = P(y') + p^+ y_n^+ - p^- y_n^-,$$

with P a quadratic polynomial, touches u^ε by above at y_0. Arguing as before, it follows that

$$\phi(y) = P(y' + y_0' - y_\varepsilon') + \tfrac{1}{\varepsilon}|y_\varepsilon' - y_0'|^2 + p^+ y_n^+ - p^- y_n^-$$

touches u at y_ε from above. Therefore,

$$p^+ - p^- = (\phi_{x_n}^+ - \phi_{x_n}^-)(y_\varepsilon) \geq g(y_\varepsilon) \geq g(y_0) - \omega_g(r_\varepsilon) = g_\varepsilon(y_0).$$

□

3.5 Existence and uniqueness

To prove existence and uniqueness of viscosity solutions (Theorem 3.1.7), we will follow the usual *greatest subsolution approach,* also known as Perron's method. One of the main ingredients of this method is the comparison principle (Theorem 3.5.6). This theorem will be a consequence of a Jensen's uniqueness type result (Theorem 3.5.4) and the ABP estimate (Theorem 3.2.1).

3.5.1 Half-relaxed limits

We introduce the notion of half-relaxed limits and some of its properties that will be useful for the proof. For more details, see [21].

Definition 3.5.1. Let $\{u_k\}_{k=1}^\infty$ be a sequence of functions. For $x \in \overline{B_1}$, we define

$$\limsup{}^* u_k(x) = \lim_{j\to\infty} \sup \left\{ u_k(y) : k \geq j,\ y \in \overline{B_1},\ \text{and } |y - x| \leq \tfrac{1}{j} \right\}.$$

Similarly, for $x \in \overline{B_1}$, we define

$$\liminf_* u_k(x) = \lim_{j\to\infty} \inf \left\{ u_k(y) : k \geq j,\ y \in \overline{B_1},\ \text{and } |y-x| \leq \tfrac{1}{j} \right\}.$$

Remark 3.5.2. Observe that $\limsup^* u_k \in \mathrm{USC}(\overline{B_1})$. Indeed, it is enough to show that $O = \left\{ x \in \overline{B_1} : \limsup^* u_k(x) < r \right\}$ is open, for any $r \in \mathbb{R}$. For simplicity, given $x \in \overline{B_1}$, call

$$A_j(x) = \left\{ u_k(y) : k \geq j,\ y \in \overline{B_1},\ \text{and } |y-x| \leq \tfrac{1}{j} \right\}.$$

Notice that $A_{j+1}(x) \subseteq A_j(x)$, and thus, $\sup A_{j+1}(x) \leq \sup A_j(x)$. Fix $x_0 \in O$. Then $\limsup^* u_k(x_0) < r$ and, by monotonicity, there exists $j_0 \geq 1$ such that $\sup A_j(x_0) < r$, for all $j \geq j_0$. Furthermore, there exists $\varepsilon > 0$ small such that

$$\sup A_j(x_0) < r - \varepsilon \quad \text{for all } j \geq j_0.$$

In particular $u_k(y) < r - \varepsilon$, for all $k \geq j_0$ and $|y - x_0| \leq 1/j_0$. Let $\rho < \frac{1}{2j_0}$ and $j_1 \geq 2j_0$. Then for any $x \in B_\rho(x_0)$, we have that

$$u_k(y) < r - \varepsilon \quad \text{for all } k \geq j_1 \text{ and } |y-x| \leq \tfrac{1}{j_1},$$

since $|y - x_0| \leq 1/j_0$. Therefore,

$$\sup A_j(x) \leq r - \varepsilon < r \quad \text{for all } j \geq j_1.$$

We conclude that $B_\rho(x_0) \subset O$, and thus, O is open.

Similarly, $\liminf_* u_k \in \mathrm{LSC}(\overline{B_1})$.

Lemma 3.5.3. *Let* $\{u_k\}_{k=1}^{\infty} \subseteq \mathrm{USC}(\overline{B_1})$ *and* $u = \limsup^* u_k$*. Fix* $x_0 \in \overline{B_1}$*. If* $\varphi \in C^0$ *touches* u *from above at* x_0*, then there exist indexes* $k_j \to \infty$*, points* $x_j \in \overline{B_1}$*, and functions* $\varphi_j \in C^0$ *such that* φ_j *touches* u_{k_j} *from above at* x_j*,*

$$x_j \to x_0 \quad \textit{and} \quad u_{k_j}(x_j) \to u(x_0), \qquad \textit{as } j \to \infty.$$

Moreover, $\varphi_j(x) = \varphi(x) - \varphi(x_j) + u_{k_j}(x_j) + \delta(|x - x_0|^2 - |x_j - x_0|^2)$*, for an arbitrary* $\delta > 0$*.*

Proof. Fix $x_0 \in \overline{B_1}$. Assume that $\varphi \in C^0$ touches u from above at x_0. Then

$$u(x_0) = \varphi(x_0) \quad \text{and} \quad u(x) \leq \varphi(x) \text{ for all } x \in \overline{B_r(x_0)}.$$

Since $u_k \in \mathrm{USC}(\overline{B_1})$, by the definition of u, there exist indexes $k_j \to \infty$ and points $y_j \in \overline{B_1}$ such that

$$y_j \to x_0 \quad \text{and} \quad u_{k_j}(y_j) \to u(x_0), \qquad \text{as } j \to \infty.$$

Fix $\delta > 0$ small. For $j \geq 1$, let $x_j \in \overline{B_1}$ be a maximum point of

$$u_{k_j}(x) - \varphi(x) - \delta|x - x_0|^2 \quad \text{on } \overline{B_r(x_0)}.$$

Then for all $x \in \overline{B_r(x_0)}$, we have that

$$u_{k_j}(x) \leq u_{k_j}(x_j) + \varphi(x) - \varphi(x_j) + \delta(|x - x_0|^2 - |x_j - x_0|^2).$$

By compactness, up to a subsequence, $x_j \to y \in \overline{B_r(x_0)}$. Using the previous estimate, with $x = y_j$, and passing to the limit, we get that

$$\begin{aligned} u(x_0) = \liminf_{j\to\infty} u_{k_j}(y_j) &\leq \liminf_{j\to\infty} u_{k_j}(x_j) + \varphi(x_0) - \varphi(y) - \delta|y - x_0|^2 \\ &\leq u(y) + \varphi(x_0) - \varphi(y) - \delta|y - x_0|^2 \\ &\leq \varphi(x_0) - \delta|y - x_0|^2, \end{aligned}$$

since $u(y) \le \varphi(y)$. Therefore, $y = x_0$, since $u(x_0) = \varphi(x_0)$, and thus, $x_j \to x_0$ as $j \to \infty$. Moreover, we see that

$$\limsup_{j\to\infty} u_{k_j}(x_j) \le u(x_0) \le \liminf_{j\to\infty} u_{k_j}(x_j).$$

Hence, $u_{k_j}(x_j) \to u(x_0)$, as $j \to \infty$. Define

$$\varphi_j(x) = u_{k_j}(x_j) + \varphi(x) - \varphi(x_j) + \delta(|x - x_0|^2 - |x_j - x_0|^2).$$

Then φ_j touches u_{k_j} from above at x_j since

$$\varphi_j(x_j) = u_{k_j}(x_j) \quad \text{and} \quad \varphi_j(x) \ge u_{k_j}(x) \text{ for all } x \in \overline{B_r(x_0)}.$$

□

3.5.2 Comparison principle and uniqueness

The next theorem will be key to prove the comparison principle. As a consequence, we show uniqueness of viscosity solutions (Corollary 3.5.5). Our proof is inspired by [58, Lemma 4.2].

Theorem 3.5.4 (Jensen's uniqueness type result)**.** *Let $f_1^\pm, f_2^\pm \in C^0(B_1^\pm)$ and $g_1, g_2 \in C^0(T)$. Assume that $u \in \mathrm{USC}(\overline{B_1})$ and $v \in \mathrm{LSC}(\overline{B_1})$ are bounded functions satisfying*

$$\begin{cases} F^\pm(D^2u) \ge f_1^\pm & \text{in } B_1^\pm \\ u_{x_n}^+ - u_{x_n}^- \ge g_1 & \text{on } T \end{cases} \quad \text{and} \quad \begin{cases} F^\pm(D^2v) \le f_2^\pm & \text{in } B_1^\pm \\ v_{x_n}^+ - v_{x_n}^- \le g_2 & \text{on } T, \end{cases}$$

in the viscosity sense. Then the function $w = u - v$ satisfies

$$\begin{cases} w \in \underline{S}_{\lambda/n,\Lambda}(f_1^\pm - f_2^\pm) & \text{in } B_1^\pm \\ w_{x_n}^+ - w_{x_n}^- \ge g_1 - g_2 & \text{on } T, \end{cases}$$

in the viscosity sense.

Proof. Let u and v be as in the statement. Consider $w = u - v$. By [34, Theorem 3.1], we know that $w \in \underline{S}_{\lambda/n,\Lambda}(f_1^{\pm} - f_2^{\pm})$ in $B_1^{\pm}$. Hence, we only need to show the transmission condition. Let $x_0 = (x_0', 0) \in T$ and assume that

$$P(x') + p^+ x_n^+ - p^- x_n^-$$

touches w by above at x_0, where P is a quadratic polynomial and $p^{\pm} \in \mathbb{R}$.

We need to show that

$$p^+ - p^- \geq g_1(x_0) - g_2(x_0). \tag{3.5.1}$$

Fix $\tau > 0$ and $C > 0$ large to be chosen. Then

$$\varphi(x) = P(x') + (p^+ + \tau)x_n^+ - (p^- - \tau)x_n^- - Cx_n^2$$

touches w strictly by above at x_0, possibly in a smaller neighborhood where $\tau|x_n| - Cx_n^2 \geq 0$.

For $\varepsilon > 0$, consider $w_\varepsilon = u^\varepsilon - v_\varepsilon$, where u^ε and v_ε are the upper and lower ε-envelopes of u and v, respectively, given in Definition 3.4.1. By (i) in Lemma 3.4.3, we have that $\limsup_{\varepsilon\to 0} w_\varepsilon = w$. By Lemma 3.5.3, up to a subsequence, there exist points $x_\varepsilon \in B_1$, with $x_\varepsilon \to x_0$, and functions φ_ε given by

$$\varphi_\varepsilon(x) = \varphi(x) - \varphi(x_\varepsilon) + w_\varepsilon(x_\varepsilon) + |x - x_0|^2 - |x_\varepsilon - x_0|^2$$

such that φ_ε touches w_ε strictly by above at x_ε. In particular, given $\delta > 0$ small, there exists $\eta > 0$ such that

$$\varphi_\varepsilon - w_\varepsilon \geq \eta > 0 \quad \text{on } \partial B_\delta(x_\varepsilon).$$

By Proposition 3.4.4, Remark 3.4.5, and [34, Theorem 3.1], we have

$$\mathcal{M}^+_{\lambda/n,\Lambda}(D^2 w_\varepsilon) \geq (f_1^\pm)_\varepsilon - (f_2^\pm)_\varepsilon \quad \text{in } B_\rho^\pm, \tag{3.5.2}$$

in the viscosity sense, for some $0 < \rho < 1$ such that $\overline{B_\delta(x_\varepsilon)} \subset B_\rho$.

Choose C large enough so that

$$\begin{aligned} \mathcal{M}^+_{\lambda/n,\Lambda}(D^2 \varphi_\varepsilon) &\leq \Lambda \|D^2_{x'} P\| + 2\Lambda - 2\lambda(C-1) \\ &< \inf_{B_\rho^\pm} \left\{ (f_1^\pm)_\varepsilon - (f_2^\pm)_\varepsilon \right\} \quad \text{in } B_\rho^\pm. \end{aligned} \tag{3.5.3}$$

Since φ_ε touches w_ε by above at x_ε, it immediately follows that $x_\varepsilon \in T$. Otherwise,

$$\mathcal{M}^+_{\lambda/n,\Lambda}(D^2 \varphi_\varepsilon(x_\varepsilon)) < (f_1^\pm)_\varepsilon(x_\varepsilon) - (f_2^\pm)_\varepsilon(x_\varepsilon),$$

which contradicts (3.5.2). Define

$$\psi = \varphi_\varepsilon - w_\varepsilon - \eta/2. \tag{3.5.4}$$

Then $\psi \geq \eta/2 > 0$ on $\partial B_\delta(x_\varepsilon)$ and $\psi(x_\varepsilon) = -\eta/2 < 0$. Let Γ_ψ be the convex envelope of $-\psi_-$ in $B'_{2\delta}(x_\varepsilon) = B_{2\delta}(x_\varepsilon) \cap \{x_n = 0\}$, where we have extended $-\psi_- \equiv 0$ outside of $\overline{B'_\delta(x_\varepsilon)}$. By (iii) in Lemma 3.4.3, we know that $\psi \in C^{1,1}_{x'}$ by above in B_ρ. Hence, for any $x'_0 \in \overline{B'_\delta(x_\varepsilon)}$, there exists a convex paraboloid $P(x')$ with uniform opening that touches $\psi(x', 0)$ by above at x'_0. Using [15, Lemma 3.5], we see that $\Gamma_\psi \in C^{1,1}_{x'}(\overline{B'_\delta(x_\varepsilon)})$, and for $\gamma > 0$ sufficiently small,

$$|D_\gamma| \equiv \left| \left\{ x' \in \overline{B'_\delta(x_\varepsilon)} : \Gamma_\psi(x') = \psi(x', 0) \text{ and } |\nabla' \Gamma_\psi(x')| \leq \gamma \right\} \right| > 0,$$

since $\Gamma_\psi(x'_\varepsilon) = \psi(x'_\varepsilon, 0)$ and $0' \in \nabla' \Gamma_\psi(x'_\varepsilon)$. Hence, choosing $\gamma \leq \frac{\eta}{4\delta}$, there exists $y'_\varepsilon \in D_\gamma$ such that both u^ε and v_ε are punctually second order differentiable at $y_\varepsilon = (y'_\varepsilon, 0)$ in the x'-direction, and such that

$$l(x') = \nabla'\Gamma_\psi(y'_\varepsilon) \cdot (x' - y'_\varepsilon) + \psi(y_\varepsilon)$$

touches ψ from below at y_ε on $\overline{B_\delta(x_\varepsilon)}$. Indeed, $l(x') \le \psi(x', 0)$, for all $x' \in B'_\delta(x_\varepsilon)$, and

$$l(x') \le |\nabla'\Gamma_\psi(y'_\varepsilon)||x' - y'_\varepsilon| \le \tfrac{\eta}{4\delta}(2\delta) = \eta/2 \le \psi(x) \quad \text{for all } x \in \partial B_\delta(x_\varepsilon).$$

Therefore, $l \le \psi$ on $\partial B_\delta^\pm(x_\varepsilon)$. In particular, by (3.5.4), we see that

$$w_\varepsilon \le \varphi_\varepsilon - l - \eta/2 \quad \text{on } \partial B_\delta^\pm(x_\varepsilon).$$

Moreover, in view of (3.5.2) and (3.5.3), we get

$$\mathcal{M}^+_{\lambda/n,\Lambda}(D^2 w_\varepsilon) > \mathcal{M}^+_{\lambda/n,\Lambda}(D^2(\varphi_\varepsilon - l - \eta/2)) \quad \text{in } B_\delta^\pm(x_\varepsilon).$$

Hence, by the comparison principle, it follows that $w_\varepsilon \le \varphi_\varepsilon - l - \eta/2$ on $\overline{B_\delta(x_\varepsilon)}$. Define

$$\bar\varphi = \varphi_\varepsilon - l - \eta/2.$$

Consider the viscosity solutions $\bar u^\varepsilon$ and $\bar v_\varepsilon$ to the Dirichlet problems,

$$\begin{cases} F^\pm(D^2\bar u^\varepsilon) = (f_1^\pm)_\varepsilon & \text{in } B_\delta^\pm(x_\varepsilon) \\ \bar u^\varepsilon = u^\varepsilon & \text{on } \partial B_\delta^\pm(x_\varepsilon) \end{cases}$$

and

$$\begin{cases} F^\pm(D^2\bar v_\varepsilon) = (f_2^\pm)_\varepsilon & \text{in } B_\delta^\pm(x_\varepsilon) \\ \bar v_\varepsilon = v_\varepsilon & \text{on } \partial B_\delta^\pm(x_\varepsilon). \end{cases}$$

By the comparison principle, $\bar u_\varepsilon \ge u_\varepsilon$ and $\bar v_\varepsilon \le v_\varepsilon$ in $B_\delta(x_\varepsilon)$, and thus,

$$(\bar u^\varepsilon)^+_{x_n} - (\bar u^\varepsilon)^-_{x_n} \ge (g_1)_\varepsilon \quad \text{and} \quad (\bar v_\varepsilon)^+_{x_n} - (\bar v_\varepsilon)^-_{x_n} \le (g_2)_\varepsilon \tag{3.5.5}$$

on $B_\delta(x_\varepsilon) \cap \{x_n = 0\}$, in the viscosity sense, where $(g_1)_\varepsilon$ and $(g_2)_\varepsilon$ are given in Proposition 3.4.4. More precisely,

$$(g_1)_\varepsilon = g_1 - \omega_{g_1}\big((2\varepsilon\|u\|_\infty)^{1/2}\big) \quad \text{and} \quad (g_2)_\varepsilon = g_2 - \omega_{g_2}\big((2\varepsilon\|v\|_\infty)^{1/2}\big).$$

Recall that $u^\varepsilon, v_\varepsilon \in C^{1,\alpha}_{x'}(y_\varepsilon)$, and thus, by pointwise $C^{1,\alpha}$-estimates (see [38, Theorem 1.6]), there exists $r_0 > 0$ and linear polynomials $l_u^\pm$ and $l_v^\pm$ such that

$$\|\bar{u}_\varepsilon^\pm - l_u^\pm\|_{L^\infty(B_r^\pm(y_\varepsilon))} \le Cr^{1+\alpha} \qquad \text{for all } 0 < r < r_0.$$

For simplicity, call $p_u^\pm = \nabla l_u^\pm \cdot e_n$ and $p_v^\pm = \nabla l_v^\pm \cdot e_n$. Then by similar arguments as in [58, Lemma 4.3], we see that (3.5.5) holds pointwise at y_ε. Namely,

$$p_u^+ - p_u^- \ge (g_1)_\varepsilon(y_\varepsilon) \quad \text{and} \quad p_v^+ - p_v^- \le (g_2)_\varepsilon(y_\varepsilon). \tag{3.5.6}$$

Let $\bar{w}_\varepsilon = \bar{u}^\varepsilon - \bar{v}_\varepsilon$. Then by previous computations, $\bar{w}_\varepsilon$ satisfies

$$\mathcal{M}^+_{\lambda/n,\Lambda}(D^2\bar{w}_\varepsilon) \ge \mathcal{M}^+_{\lambda/n,\Lambda}(D^2\bar{\varphi}) \quad \text{in } B_\delta^\pm(x_\varepsilon) \quad \text{and} \quad \bar{\varphi} \ge \bar{w}_\varepsilon \quad \text{on } \partial B_\delta^\pm(x_\varepsilon).$$

It follows that $\bar{\varphi} \ge \bar{w}_\varepsilon$ in $B_\delta(x_\varepsilon)$ and $\bar{\varphi}(y_\varepsilon) = \bar{w}_\varepsilon(y_\varepsilon)$. Since $\bar{w}_\varepsilon \in C^{1,\alpha}(y_\varepsilon)$, we have that

$$\begin{aligned} p^+ + \tau &= \bar{\varphi}^+_{x_n}(y_\varepsilon) \ge (\bar{w}_\varepsilon)^+_{x_n}(y_\varepsilon) = p_u^+ - p_v^+, \\ p^- - \tau &= \bar{\varphi}^-_{x_n}(y_\varepsilon) \le (\bar{w}_\varepsilon)^-_{x_n}(y_\varepsilon) = p_u^- - p_v^-. \end{aligned}$$

Therefore, combining the previous estimates with (3.5.6), we get

$$\begin{aligned} p^+ - p^- + 2\tau &\ge (g_1)_\varepsilon(y_\varepsilon) - (g_2)_\varepsilon(y_\varepsilon) \\ &= (g_1 - g_2)(y_\varepsilon) + \omega_{g_1}\big((2\varepsilon\|u\|_\infty)^{1/2}\big) - \omega_{g_2}\big((2\varepsilon\|v\|_\infty)^{1/2}\big). \end{aligned}$$

Recall that $y_\varepsilon \in B_\delta(x_\varepsilon)$ and $x_\varepsilon \to x_0$, as $\varepsilon \to 0$. Hence, letting $\tau \to 0$, then $\delta \to 0$, so that $y_\varepsilon \to x_\varepsilon$, and finally, $\varepsilon \to 0$, we obtain (3.5.1). $\square$

Corollary 3.5.5 (Uniqueness)**.** *The transmission problem*

$$\begin{cases} F^{\pm}(D^2u^{\pm}) = f^{\pm} & \text{in } B_1^{\pm} \\ u_{x_n}^+ - u_{x_n}^- = g & \text{on } T \\ u = \phi & \text{on } \partial B_1 \end{cases} \tag{3.5.7}$$

has at most one viscosity solution.

Proof. Assume by contradiction that there are two distinct solutions u and v. Then, by Theorem 3.5.4, $w = u - v$ satisfies

$$\begin{cases} w \in S_{\lambda/n,\Lambda}(0) & \text{in } B_1^{\pm} \\ w_{x_n}^+ - w_{x_n}^- = 0 & \text{on } T \\ w = 0 & \text{on } \partial B_1, \end{cases}$$

in the viscosity sense. By the maximum principle (Corollary 3.2.5), it follows that $w \equiv 0$ in B_1. This is a contradiction with $u \neq v$. Therefore, there exists at most one viscosity solution to the transmission problem (3.5.7). $\square$

Theorem 3.5.6 (Comparison principle)**.** *Let $u, v : \overline{B_1} \to \mathbb{R}$ be a bounded viscosity subsolution and a bounded viscosity supersolution of* (3.1.1)*, respectively. If $u \leq v$ on ∂B_1, then*

$$u \leq v \quad \text{in } B_1.$$

Proof. Let $w = u - v$. By Theorem 3.5.4, w satisfies

$$\begin{cases} w \in \underline{S}_{\lambda/n,\Lambda}(0) & \text{in } B_1^{\pm} \\ w_{x_n}^+ - w_{x_n}^- \geq 0 & \text{on } T. \end{cases}$$

From the ABP estimate (Theorem 3.2.1), it follows that

$$0 \leq \sup_{B_1} w_- \leq \sup_{\partial B_1} w_- = 0.$$

Therefore, $w \geq 0$ in B_1, which implies the result. $\square$

Lemma 3.5.7 (Barriers)**.** *There exist functions* $\underline{u}, \overline{u} \in C^2(B_1 \setminus T) \cap C^0(\overline{B_1})$ *such that* $\underline{u}$ *is a viscosity subsolution and* $\overline{u}$ *is a viscosity supersolution of* (3.1.1)*, respectively, with*

$$\underline{u} \leq \overline{u} \text{ in } B_1 \quad \text{and} \quad \underline{u} = \overline{u} = \phi \text{ on } \partial B_1.$$

Proof. Let $f = f^+\chi_{B_1^+} + f^-\chi_{B_1^-}$. Consider the Dirichlet problems

$$\begin{cases} \mathcal{M}^-_{\lambda,\Lambda}(D^2\underline{\psi}) = \|f\|_\infty & \text{in } B_1 \\ \underline{\psi} = \phi - \frac{1}{2}\|g\|_\infty |x_n| & \text{on } \partial B_1 \end{cases}$$

and

$$\begin{cases} \mathcal{M}^+_{\lambda,\Lambda}(D^2\overline{\psi}) = -\|f\|_\infty & \text{in } B_1 \\ \overline{\psi} = \phi + \frac{1}{2}\|g\|_\infty |x_n| & \text{on } \partial B_1. \end{cases}$$

By [15, Proposition 9.8], there exist unique solutions $\underline{\psi}, \overline{\psi} \in C^2(B_1) \cap C^0(\overline{B_1})$.

Define the functions $\underline{u}, \overline{u} \in C^2(B_1 \setminus T) \cap C^0(\overline{B_1})$ as

$$\underline{u} = \underline{\psi} + \tfrac{1}{2}\|g\|_\infty |x_n| \quad \text{and} \quad \overline{u} = \overline{\psi} - \tfrac{1}{2}\|g\|_\infty |x_n|.$$

Then $\underline{u} = \overline{u} = \phi$ on ∂B_1. By construction, we have that

$$\begin{cases} F^\pm(D^2\underline{u}) \geq \mathcal{M}^-_{\lambda,\Lambda}(D^2\underline{\psi}) = \|f\|_\infty \geq f^\pm & \text{in } B_1^\pm \\ (\underline{u}^+)_{x_n} - (\underline{u}^-)_{x_n} = \|g\|_\infty \geq g & \text{on } T. \end{cases}$$

Hence, $\underline{u}$ is a subsolution of (3.1.1). Arguing similarly, we see that $\overline{u}$ is a supersolution of (3.1.1). By the comparison principle (Theorem 3.5.6), we conclude that $\underline{u} \leq \overline{u}$ in B_1. $\square$

3.5.3 Existence via Perron's method

Define the set of admissible subsolutions as

$$\mathcal{A} = \Big\{ v \in \mathrm{USC}(\overline{B_1}) : \underline{u} \le v \le \overline{u} \text{ and } v \text{ viscosity subsolution of (3.1.1)} \Big\}.$$

Note that $\underline{u} \in \mathcal{A}$, so $\mathcal{A} \neq \varnothing$. Set

$$u(x) = \sup_{v \in \mathcal{A}} v(x).$$

For $x \in \overline{B_1}$, we define the lower and upper semicontinuous envelopes of u as

$$\begin{aligned} u_*(x) &= \lim_{r \to 0} \inf \big\{ u(y) : y \in \overline{B_1} \text{ and } |y - x| \le r \big\} \\ u^*(x) &= \lim_{r \to 0} \sup \big\{ u(y) : y \in \overline{B_1} \text{ and } |y - x| \le r \big\}. \end{aligned}$$

Remark 3.5.8. Observe that $u_* \in \mathrm{LSC}(\overline{B_1})$ and $u^* \in \mathrm{USC}(\overline{B_1})$. Clearly, $u_* \le u \le u^*$.

Lemma 3.5.9. *If $\{v_k\}_{k=1}^{\infty} \subset \mathcal{A}$, then $v = \limsup^* v_k \in \mathcal{A}$.*

Proof. Let $\{v_k\}_{k=1}^{\infty} \subset \mathcal{A}$ and $v = \limsup^* v_k$. It is clear that $\underline{u} \le v \le \overline{u}$. Hence, we only need to show that v is a subsolution of (3.1.1). Fix $x_0 \in B_1$ and assume that $\varphi \in C^2$ touches v by above at x_0. Then by Lemma 3.5.3, there exist indexes $k_j \to \infty$, points $x_j \in \overline{B_1}$, and functions $\varphi_j \in C^2$ such that $D^2\varphi_j = D^2\varphi + 2\delta I$, φ_j touches v_{k_j} from above at x_j, and

$$x_j \to x_0 \quad \text{and} \quad v_{k_j}(x_j) \to v(x_0), \quad \text{as } j \to \infty.$$

If $x_0 \in B_1^{\pm}$, then for j sufficiently large we may assume that $x_j \in B_1^{\pm}$. Since $v_{k_j} \in \mathcal{A}$, and φ touches v_{k_j} by above at x_j, it follows that

$$F^{\pm}(D^2\varphi(x_j) + 2\delta I) = F^{\pm}(D^2\varphi_j(x_j)) \ge f^{\pm}(x_j).$$

Letting $j \to \infty$ and $\delta \to 0$, by continuity of $F, D^2\varphi$, and $f^{\pm}$, we obtain the result.

If $x_0 \in T$, then either there exists $j_0 \geq 1$ such that for all $j \geq j_0$, we have $x_j \in B_1^{\pm}$, or for all $j_0 \geq 1$, there exists $j \geq j_0$ such that $x_j \in T$. In the first case, by the previous argument, we get

$$F^{\pm}(D^2\varphi(x_0)) \geq f^{\pm}(x_0).$$

In the second case, we have that

$$(\varphi_j^+)_{x_n}(x_j) - (\varphi_j^-)_{x_n}(x_j) \geq g(x_j).$$

Passing to the limit, we obtain the desired estimate.

Therefore, by Lemma 3.1.6, we conclude that v is a viscosity subsolution of (3.1.1).

$\square$

We divide the proof of Theorem 3.1.7 into two steps.

Lemma 3.5.10 (Step 1). *The function u^* is a subsolution of* (3.1.1). *In particular, $u^* \in \mathcal{A}$.*

Proof. Let $x_0 \in B_1$. By the construction of u^*, there exist points $\{x_k\}_{k=1}^{\infty}$ and functions $\{v_k\}_{k=1}^{\infty} \subset \mathcal{A}$ such that $x_k \to x_0$, and

$$u^*(x_0) = \lim_{k \to \infty} v_k(x_k).$$

Hence, $\limsup^* v_k(x_0) \geq u^*(x_0)$. On the other hand, for all $x \in B_1$, we have

$$u^*(x) \geq \limsup{}^* v_j(x)$$

for any $\{v_j\}_{j=1}^{\infty} \subset \mathcal{A}$. Therefore, $\limsup^* v_k(x_0) = u^*(x_0)$. In particular, if $\varphi \in C^2$ touches u^* by above at x_0, then the same holds for $\limsup^* v_k$. Using Lemma 3.5.9, we know that $\limsup^* v_k \in \mathcal{A}$, which implies that $u^* \in \mathcal{A}$. $\square$

Remark 3.5.11. By the previous lemma, it follows that $u^* \leq u$, but by definition, $u^* \geq u$. Therefore, $u^* = u$ on $\overline{B_1}$, and thus, $u \in \mathcal{A}$.

Lemma 3.5.12 (Step 2)**.** *The function u_* is a supersolution of* (3.1.1).

Proof. Assume by means of contradiction that there exists $x_0 \in B_1$ and some test function φ that touches u_* from below at x_0 such that the following holds:

(i) If $x_0 \in B_1^{\pm}$, then

$$F^{\pm}(D^2\varphi(x_0)) > f^{\pm}(x_0).$$

(ii) If $x_0 \in T$, then

$$\varphi^+_{x_n}(x_0) - \varphi^-_{x_n}(x_0) > g(x_0).$$

Without loss of generality, we may assume that φ touches u_* strictly from below. Otherwise, take $\varphi - \varepsilon|x - x_0|^2$, for some $\varepsilon > 0$ small. By continuity, the strict inequalities in (i) and (ii) hold in a neighborhood of x_0. Indeed, let $x_0 \in B_1^+$. For $\delta > 0$ sufficiently small, it holds that

$$F^+(D^2\varphi(x_0)) - \delta > f^+(x_0).$$

Since $f^+ \in C^0(B_1^+ \cup T)$, there exists $r > 0$, depending on δ, such that

$$f^+(x_0) > f^+(x) - 2\delta \qquad \text{for all } x \in B_r(x_0).$$

We choose r sufficiently small so that $B_r(x_0) \subset B_1^+$, and

$$\|D^2\varphi(x) - D^2\varphi(x_0)\| \leq \delta/(2\Lambda) \qquad \text{for all } x \in B_r(x_0).$$

This is possible since $\varphi \in C^2$ in a neighborhood of x_0. By the uniform ellipticity of F^+,

$$F^+(D^2\varphi(x_0)) \leq F^+(D^2\varphi(x)) + \Lambda\|D^2\varphi(x) - D^2\varphi(x_0)\|.$$

Therefore, combining the previous estimates, we get

$$F^+(D^2\varphi(x)) > f^+(x) \qquad \text{for all } x \in B_r(x_0).$$

The proof is analogous for $x \in B_1^-$. If $x_0 \in T$, then possibly taking δ smaller, we see that

$$\varphi^+_{x_n}(x_0) - \varphi^-_{x_n}(x_0) - 3\delta > g(x_0).$$

Since $g \in C^0(T)$ and $\varphi \in C^2(\overline{B_r^-(x_0)}) \cap C^2(\overline{B_r^+(x_0)})$, we get

$$\varphi^+_{x_n}(x_0) - \delta < \varphi^+_{x_n}(x), \quad \varphi^-_{x_n}(x_0) + \delta > \varphi^-_{x_n}(x), \quad \text{and} \quad g(x_0) > g(x) - \delta$$

for all $x \in T \cap B_r(x_0)$. Hence, φ is a classical strict subsolution in $B_r(x_0)$.

Consider $\varphi_\delta = \varphi - \delta|x - x_0|^2 + \delta r^2/2$. Then

$$\varphi_\delta(x_0) > u_*(x_0) \quad \text{and} \quad \varphi_\delta < u_* \leq u \text{ on } \partial B_r(x_0).$$

Hence, there exists some $x_1 \in B_r(x_0)$ such that $u(x_1) < \varphi_\delta(x_1)$. Define

$$\bar{u} = \begin{cases} \max\{u, \varphi_\delta\} & \text{in } B_r(x_0) \\ u & \text{on } \overline{B_1} \setminus B_r(x_0). \end{cases}$$

Then $\bar{u} \in \mathcal{A}$, since $u \in \mathcal{A}$ and φ_δ is a subsolution. This is a contradiction with

$$u(x_1) = \sup_{v \in \mathcal{A}} v(x_1) \geq \bar{u}(x_1) = \varphi(x_1) > u(x_1).$$

Therefore, u_* is a supersolution of (3.1.1). $\square$

Remark 3.5.13. By definition, $u_* \leq u$. Since u_* is a supersolution and u is a subsolution of (3.1.1), and $u_* = u$ on ∂B_1, then by the comparison principle (Theorem 3.5.6), we get $u_* = u$ on $\overline{B_1}$. Furthermore, by Remark 3.5.11, we conclude that

$$u_* = u = u^* \qquad \text{on } \overline{B_1}.$$

In particular, by Corollary 3.5.5, $u \in C^0(\overline{B_1})$ is the unique viscosity solution of (3.1.1) with $u = \phi$ on ∂B_1. This concludes the proof of Theorem 3.1.7.

3.6 Pointwise $C^{1,\alpha}$ estimates up to the interface

In this section, we derive $C^{1,\alpha}$ estimates for viscosity solutions of

$$\begin{cases} F^\pm(D^2u) = f^\pm & \text{in } B_1^\pm \\ u^+_{x_n} - u^-_{x_n} = g & \text{on } T. \end{cases} \tag{3.6.1}$$

Our main goal is to show Theorem 3.1.9.

3.6.1 Homogeneous problem

For the homogeneous problem, we can use the results in [58]. In particular, solutions will be differentiable across T.

Theorem 3.6.1. *Suppose that v is a bounded viscosity solution of* (3.6.1) *with $f^\pm \equiv g \equiv 0$. Then $v \in C^{1,\bar\alpha}(\overline{B_{1/2}})$, with*

$$\|v\|_{C^{1,\bar\alpha}(\overline{B_{1/2}})} \leq C\|v\|_{L^\infty(B_1)}$$

where $0 < \bar\alpha < 1$ and $C > 0$ are constants depending only on n, λ, and Λ.

Proof. We apply [58, Theorem 1.2] with $a = b = 1$. Then $v^\pm \in C^{1,\bar\alpha}(\overline{B^\pm_{1/2}})$, and the following estimate holds:

$$\|v^\pm\|_{C^{1,\bar\alpha}(\overline{B^\pm_{1/2}})} \leq C\|v\|_{L^\infty(B_1)}.$$

In particular, v satisfies the transmission condition in the classical sense, and thus, v is differentiable in $B_{1/2}$, and the estimate holds for v in all of $\overline{B_{1/2}}$. $\square$

Corollary 3.6.2. *Let $0 < r \leq 1$. Suppose that v is a bounded viscosity solution of*

$$\begin{cases} F^\pm(D^2 v) = 0 & \text{in } B_r^\pm \\ v^+_{x_n} - v^-_{x_n} = 0 & \text{on } B_r \cap \{x_n = 0\}. \end{cases}$$

Then for any $0 < \rho \leq r/2$, we have that $v \in C^{1,\bar\alpha}(\overline{B_\rho})$, with

$$\operatorname*{osc}_{B_\rho}\big(v - \nabla v(0)\cdot x\big) \leq C\Big(\frac{\rho}{r}\Big)^{1+\bar\alpha} \operatorname*{osc}_{\overline{B_r}} v,$$
$$|\nabla v(0)| \leq C\frac{1}{r}\operatorname*{osc}_{\overline{B_r}} v,$$

where $\bar\alpha$ is given by Theorem 3.6.1 and $C > 0$ is a constant depending only on n, λ, and Λ.

Proof. Fix $\rho \le r/2$. Applying Theorem 3.6.1 to $v(rx)$, for $x \in B_1$, it follows that $v \in C^{1,\bar{\alpha}}(\overline{B_\rho})$. Moreover, since $v - \nabla v(0) \cdot x$ is a continuous function on $\overline{B_\rho}$, it follows that it attains its maximum and minimum values at some points x_1 and x_2 on $\overline{B_\rho}$, respectively. By the mean value theorem, we have that

$$\begin{aligned}\operatorname*{osc}_{B_\rho} \big(v - \nabla v(0) \cdot x\big) &= \big(v(x_1) - \nabla v(0) \cdot x_1\big) - \big(v(x_2) - \nabla v(0) \cdot x_2\big) \\ &\le |\nabla v(x_3) - \nabla v(0)||x_1 - x_2| \le [\nabla v]_{C^{0,\bar{\alpha}}(\overline{B_\rho})}|x_1 - x_2|^{1+\bar{\alpha}} \\ &\le (2\rho)^{1+\bar{\alpha}}[\nabla v]_{C^{0,\bar{\alpha}}(\overline{B_\rho})}\end{aligned}$$

for some x_3 that belongs to the segment joining x_1 and x_2. To estimate $[\nabla v]_{C^{0,\bar{\alpha}}(\overline{B_\rho})}$, we consider $w(x) = v(rx) - v(0)$, for $x \in B_1$. By Theorem 3.6.1, we get

$$\|w\|_{C^{1,\bar{\alpha}}(\overline{B_{1/2}})} \le C\|w\|_{L^\infty(B_1)}.$$

Since $\rho/r \le 1/2$, using the previous estimate, we see that

$$\begin{aligned}r^{1+\bar{\alpha}}[\nabla v]_{C^{0,\bar{\alpha}}(\overline{B_\rho})} = [\nabla w]_{C^{0,\bar{\alpha}}(\overline{B_{\rho/r}})} &\le C\|w\|_{L^\infty(B_1)} \\ &= C\|v - v(0)\|_{L^\infty(B_r)} \le C\operatorname*{osc}_{\overline{B_r}} v.\end{aligned}$$

Therefore, the first estimate follows. Moreover,

$$r|\nabla v(0)| = |\nabla w(0)| \le \|\nabla w\|_{L^\infty(B_{1/2})} \le C\|w\|_{L^\infty(B_1)} \le C\operatorname*{osc}_{\overline{B_r}} v.$$

□

3.6.2 Nonhomogeneous problem

The proof of Theorem 3.1.9 is based on a perturbation of the homogeneous case. The ideas are motivated by [45].

Proof of Theorem 3.1.9. By interior estimates, it is enough to prove (3.1.5) for the points on $T \cap \overline{B_{1/2}}$. In fact, it is enough to get a universal estimate at the origin, and then apply it to rescalings and translations of u. Without loss of generality we assume that $u(0) = 0$ and $g(0) = 0$. Otherwise, we may consider $u - u(0) - \frac{g(0)}{2}|x_n|$. Let $M = \|u\|_{L^\infty(B_1)} + \|g\|_{C^{0,\alpha}(T)} + C_{f^-} + C_{f^+}$.

We will show that there exist $0 < \gamma < 1$ and $C_0, C_1 > 0$, depending only on n, λ, Λ, and α, and a sequence of vectors $\{A_k\}_{k=0}^\infty$ such that

$$\operatorname*{osc}_{B_{\gamma^k}} (u - A_k \cdot x) \le C_0 M \gamma^{k(1+\alpha)}, \tag{3.6.2}$$

$$|A_k - A_{k-1}| \le C_1 M \gamma^{(k-1)\alpha}, \tag{3.6.3}$$

for any $k \ge 0$, where $A_{-1} = 0$. If this holds, then $A_k \to A_\infty$, as $k \to \infty$, and

$$\begin{aligned}
\operatorname*{osc}_{B_{\gamma^k}} (u - A_\infty \cdot x) &\le \operatorname*{osc}_{B_{\gamma^k}} (u(x) - A_k \cdot x) + 2\gamma^k |A_k - A_\infty| \\
&\le C_0 M \gamma^{k(1+\alpha)} + 2C_1 M \gamma^k \sum_{j=k}^{\infty} \gamma^{j\alpha} \\
&\le C_0 M \gamma^{k(1+\alpha)} + 2C_1 M \gamma^k \frac{\gamma^{k\alpha}}{1-\gamma^\alpha} \le C M \gamma^{k(1+\alpha)}.
\end{aligned}$$

By a standard argument, we see that $u \in C^{1,\alpha}(0)$, and the following estimate holds:

$$|u(x) - A_\infty \cdot x| \le CM|x|^{1+\alpha}.$$

Therefore, it remains to show (3.6.2) and (3.6.3). We prove it by induction. For $k = 0$, we set $A_0 = 0$, and choose $C_0 \ge 2$ universal such that

$$\operatorname*{osc}_{B_1} u \le 2\|u\|_{L^\infty(B_1)} \le C_0 M.$$

Assume that the estimates hold for some $k \geq 0$. We will prove that they also hold for $k+1$. Let $r = \gamma^k$ and $B = A_k$. Let $v \in C^0(\overline{B_r})$ be the viscosity solution to the following problem:

$$\begin{cases} F^{\pm}(D^2 v) = 0 & \text{in } B_r^{\pm} \\ v_{x_n}^{+} - v_{x_n}^{-} = 0 & \text{on } T \cap B_r \\ v = u - B \cdot x & \text{on } \partial B_r. \end{cases}$$

The existence is guaranteed by Theorem 3.1.7. From the ABP estimate (Theorem 3.2.1),

$$\operatorname*{osc}_{\overline{B_r}} v \leq \operatorname*{osc}_{\overline{B_r}} (u - B \cdot x). \tag{3.6.4}$$

Fix $\rho \leq r/2$ to be determined. By Corollary 3.6.2, we have $v \in C^{1,\bar{\alpha}}(\overline{B_\rho})$ with

$$\operatorname*{osc}_{B_\rho} (v - A \cdot x) \leq C \left(\frac{\rho}{r}\right)^{1+\bar{\alpha}} \operatorname*{osc}_{\overline{B_r}} v, \tag{3.6.5}$$

$$|A| \leq C \frac{1}{r} \operatorname*{osc}_{\overline{B_r}} v, \tag{3.6.6}$$

where $A = \nabla v(0)$. Let $\rho = \gamma r$ and $\varepsilon = \bar{\alpha} - \alpha > 0$. Choose $\gamma \leq 1/2$ small enough so that $C\gamma^{\varepsilon} \leq 1/2$. Combining (3.6.4), (3.6.5), and the induction hypothesis, we see that

$$\begin{aligned} \operatorname*{osc}_{B_\rho} (v - A \cdot x) &\leq C \left(\frac{\rho}{r}\right)^{1+\bar{\alpha}} \operatorname*{osc}_{\overline{B_r}} (u - B \cdot x) \\ &\leq C\gamma^{1+\alpha+\varepsilon} C_0 M \gamma^{k(1+\alpha)} \\ &\leq \tfrac{1}{2} C_0 M \gamma^{(k+1)(1+\alpha)}. \end{aligned} \tag{3.6.7}$$

Let $w = u - B \cdot x - v$. By Theorem 3.5.4 and the fact that $B \cdot x + v$ is differentiable, we have

$$\begin{cases} w \in S_{\lambda/n,\Lambda}(f^{\pm}) & \text{in } B_r^{\pm} \\ w_{x_n}^{+} - w_{x_n}^{-} = g & \text{on } T \cap B_r \\ w = 0 & \text{on } \partial B_r. \end{cases}$$

Using the rescaled ABP estimate, and the assumptions on g and $f^{\pm}$, we get

$$\begin{aligned}\|w\|_{L^\infty(B_\rho)} &\le C\rho\big(\|g\|_{L^\infty(T\cap B_\rho)} + \|f^-\|_{L^n(B_\rho^-)} + \|f^+\|_{L^n(B_\rho^+)}\big)\\ &\le C\rho^{1+\alpha}\big(\|g\|_{C^{0,\alpha}(T\cap B_\rho)} + C_{f^-} + C_{f^+}\big) \le CM\rho^{1+\alpha}.\end{aligned}$$

Choose $C_0 \ge 4C$. In view of (3.6.7) and the previous estimate, we have

$$\begin{aligned}\operatorname*{osc}_{B_{\gamma^{k+1}}}(u-(A+B)\cdot x) &= \operatorname*{osc}_{B_\rho}(u-(A+B)\cdot x) \le \operatorname*{osc}_{B_\rho} w + \operatorname*{osc}_{B_\rho}(v - A\cdot x)\\ &\le 2CM\rho^{1+\alpha} + \tfrac{1}{2}C_0M\gamma^{(k+1)(1+\alpha)} \le C_0M\gamma^{(k+1)(1+\alpha)}.\end{aligned}$$

Hence, the estimate in (3.6.2) holds for $k+1$ with $A_{k+1} = A + B$. To prove (3.6.3), we use (3.6.6), (3.6.4), and the induction hypothesis to get

$$|A_{k+1} - A_k| = |A| \le C_1 M\gamma^{k\alpha},$$

where $C_1 = CC_0$. This concludes the proof. □

Chapter 4

Transmission problems for fully nonlinear equations and $C^{1,\alpha}$ interfaces

4.1 Introduction and main results

We study the following fully nonlinear transmission problem in B_1:

$$\begin{cases} F^+(D^2u^+) = f^+ & \text{in } \Omega^+ = B_1 \cap \{x_n > \psi(x')\} \\ F^-(D^2u^-) = f^- & \text{in } \Omega^- = B_1 \cap \{x_n < \psi(x')\} \\ u_\nu^+ - u_\nu^- = g & \text{on } \Gamma = B_1 \cap \{x_n = \psi(x')\}, \end{cases} \tag{4.1.1}$$

where $\psi : \mathbb{R}^{n-1} \to \mathbb{R}$ is a given function, $D^2u^\pm$ denotes the Hessian of $u^\pm$, ν is the normal vector pointing at Ω^+, and $u_\nu^\pm$ denotes the normal derivative of $u^\pm$. Furthermore, $F^\pm : \mathcal{S}^n \to \mathbb{R}$ are fully nonlinear uniformly elliptic operators, with ellipticity constants $0 < \lambda \leq \Lambda$, and $F^\pm(0) = 0$. That is, for every $M, N \in \mathcal{S}^n$, with $N \geq 0$, we have

$$\lambda \|N\| \leq F^\pm(M + N) - F^\pm(M) \leq \Lambda \|N\|.$$

Our main assumption is that ψ is a $C^{1,\alpha}$ function, for some $0 < \alpha < 1$. Under this assumption, we say that Γ is a $C^{1,\alpha}$ interface. As we will see, this regularity condition presents several difficulties, given that the operators are of second order. For instance, the closedness lemma given in Section 4.4.1 is not available and it is not clear how to use compactness methods.

For the classical Dirichlet and Neumann problems with $C^{1,\alpha}$ boundaries, pointwise Hölder estimates at the boundary have been developed by Y. Lian and K. Zhang [38] and D. Li and K. Zhang [34], respectively. See also [59]. We point out that in our work, Γ is known a priori (ψ is given). This is in contrast to the so-called *free* transmission problems, where Γ is a free boundary, in the sense that it depends on the solution itself. For instance, E. Pimentel and M. Santos consider in [50] the model,

$$F^{+}(D^2u)\chi_{\{u>0\}} + F^{-}(D^2u)\chi_{\{u<0\}} = f.$$

In this case, the interface is the 0-level set of u, and the transmission condition arises naturally from the equation. For related works see [1, 25, 29, 51] and the references therein.

The theory of viscosity solutions of (4.1.1) when $\psi \equiv 0$ (flat interface) has been established in Chapter 3. The main purpose of this chapter is to generalize the regularity results to the case where Γ is a $C^{1,\alpha}$ interface. Recall that in Chapter 2, we studied a similar problem for $F^{\pm} = \Delta$ and $f^{\pm} = 0$. Our approach for the fully nonlinear case builds on similar ideas. In particular, we prove a stability result that will be a key tool in the study of optimal regularity of solutions of (4.1.1). The existence and regularity results from Chapter 3 will be fundamental to develop the theory for the nonflat interface problem.

Throughout this chapter, we will use the same notation as in Chapter 3. In particular, we denote by $\mathrm{USC}(B_1)$ the space of upper semicontinuous functions in B_1, and $\mathrm{LSC}(B_1)$ the space of lower semicontinuous functions in B_1.

The notion of viscosity solution is the following.

Definition 4.1.1. Assume that $\Gamma \in C^1$. We say that a function $u \in \mathrm{USC}(B_1)$ is a viscosity subsolution of (4.1.1) in B_1 if for any φ touching u by above at x_0 in B_1, the following holds:

(i) If $x_0 \in \Omega^\pm$ and $\varphi \in C^2(B_\delta(x_0) \cap \Omega^\pm)$, then

$$F^\pm(D^2\varphi(x_0)) \geq f^\pm(x_0).$$

(ii) If $x_0 \in \Gamma$ and $\varphi \in C^1(B_\delta(x_0) \cap \overline{\Omega^-}) \cap C^1(B_\delta(x_0) \cap \overline{\Omega^+})$, then

$$\varphi^+_\nu(x_0) - \varphi^-_\nu(x_0) \geq \; g(x_0),$$

where $\varphi^\pm = \varphi\big|_{B_\delta(x_0)\cap\overline{\Omega^\pm}}$.

Similarly, a function $u \in \mathrm{LSC}(B_1)$ is a viscosity supersolution of (4.1.1) in B_1 if whenever a test function φ touches u from below at x_0 in B_1, then it satisfies conditions (i) and (ii), where all inequalities are reversed. Finally, a function $u \in C^0(B_1)$ is a viscosity solution of (4.1.1) in B_1 if it is a viscosity subsolution and a viscosity supersolution.

Next we state our main results of this chapter.

Theorem 4.1.2 ($C^{0,\alpha}$ regularity)**.** *Let* $\Gamma = B_1 \cap \{x_n = \psi(x')\}$. *Let* u *satisfy*

$$\begin{cases} u \in S^*_{\lambda,\Lambda}(f^\pm) & \text{in } B_1 \\ u^+_\nu - u^-_\nu = g & \text{on } \Gamma \end{cases}$$

with $f^\pm \in C^0(\Omega^\pm) \cap L^\infty(B_1^\pm)$*,* $g \in L^\infty(\Gamma)$*, and* $\psi \in C^{1,\alpha}(\overline{B_1'})$*. Then* $u \in C^{0,\alpha_1}(\overline{B_{1/2}})$ *with*

$$\|u\|_{C^{0,\alpha_1}(\overline{B_{1/2}})} \leq C\big(\|u\|_{L^\infty(B_1)} + \|g\|_{L^\infty(\Gamma)} + \|f^-\|_{L^n(\Omega^-)} + \|f^+\|_{L^n(\Omega^+)}\big)$$

where $0 < \alpha_1 < 1$ *and* $C > 0$ *depend only on* $n, \lambda, \Lambda, \alpha$*, and* $\|\psi\|_{C^{1,\alpha}(\overline{B_1'})}$*.*

Theorem 4.1.3 ($C^{1,\alpha}$ regularity)**.** *Fix* $0 < \alpha < \bar{\alpha}$*, with* $\bar{\alpha} < 1$ *depending only on* n*,* λ*, and* Λ*. Let* $\Gamma = B_1 \cap \{x_n = \psi(x')\}$*, where* $\psi \in C^{1,\alpha}(\overline{B_1'})$*. Assume that* $g \in C^{0,\alpha}(\Gamma)$ *and* $f^\pm$ *satisfy*

$$\Big(\fint_{B_r(x_0)\cap\Omega^\pm} |f^\pm|^n \, dx\Big)^{1/n} \leq C_{f^\pm} r^{\alpha-1}$$

for all $r > 0$ *and* $x_0 \in B_1^\pm \cup \Gamma$*. We assume further that*

$$\sup_{M \in \mathbb{S}^n \setminus \{0\}} \frac{\|F^+(M) - F^-(M)\|}{\|M\|} \leq \theta \tag{4.1.2}$$

for some $0 < \theta << 1$ *depending only on* n*,* λ*,* Λ *and* α*. Suppose that* u *is a bounded viscosity solution of* (4.1.1) *in* B_1*. Let* $\Omega^\pm_{1/2} = \Omega^\pm \cap B_{1/2}$*. Then* $u^\pm \in C^{1,\alpha}(\overline{\Omega^\pm_{1/2}})$ *and the following estimate holds:*

$$\|u^\pm\|_{C^{1,\alpha}(\overline{\Omega^\pm_{1/2}})} \leq C\|\psi\|_{C^{1,\alpha}(\overline{B_1'})}\big(\|u\|_{L^\infty(B_1)} + \|g\|_{C^{0,\alpha}(\Gamma)} + C_{f^-} + C_{f^+}\big)$$

where $C > 0$ *depends only on* n*,* λ*,* Λ*, and* α*. In particular, the transmission condition is satisfied in the classical sense.*

Remark 4.1.4. The condition given in (4.1.2) may be understood as a closeness condition between the operators F^+ and F^-. For example, the linear operators given by $F^\pm(M) = \mathrm{tr}(A^\pm M)$ for some $A^\pm \in \mathbb{S}^n$, with $\lambda I \leq A^\pm \leq \Lambda I$

and $\|A^+ - A^-\| \leq \theta$, satisfy (4.1.2). This assumption is motivated by the following observation: if $g \approx 0$, then $u_\nu^+ \approx u_\nu^-$ on Γ. Hence, as we will see in Subsection 4.5.1, if the operators are close enough in the sense of (4.1.2), then we can approximate u by functions that are differentiable across Γ (see Lemma 4.5.1).

This chapter is organized as follows. In Section 4.2, we prove the ABP estimate (Theorem 4.2.1) for viscosity supersolutions of (4.1.1). For this, we construct an auxiliary function using Hopf's lemma (Lemma 4.2.6). This barrier will also be a key tool in the proof of $C^{0,\alpha}$ regularity across the interface Γ (Theorem 4.1.2) that we develop in Section 4.3. Our arguments are similar to those in Chapter 3. Section 4.4 shows that a family of viscosity solutions to transmission problems with C^2 interfaces is closed under uniform limits (Lemma 4.4.1). This result will be useful in the next section. In Section 4.5, we consider flat interface problems and discuss several approximating lemmas, including the stability result (Lemma 4.5.6). We derive the $C^{1,\alpha}$ estimates (Theorem 4.6.1) for u^+ and u^- at the intraface in Section 4.6. Our main theorem (Theorem 4.1.3) follows by a standard argument of patching the interior and boundary estimates.

4.2 ABP estimate

As we have seen in Chapter 3, the ABP estimate is a fundamental tool in the regularity theory of viscosity solutions. This result for $C^{1,\alpha}$ interfaces reads as follows.

Theorem 4.2.1 (ABP estimate)**.** *Let $\Gamma = B_1 \cap \{x_n = \psi(x')\}$. Let u satisfy*

$$\begin{cases} u \in \overline{S}_{\lambda,\Lambda}(f^{\pm}) & \text{in } \Omega^{\pm} \\ u_{\nu}^{+} - u_{\nu}^{-} \leq g & \text{on } \Gamma, \end{cases} \tag{4.2.1}$$

with $f^{\pm} \in C^0(\Omega^{\pm}) \cap L^{\infty}(B_1)$, $g \in L^{\infty}(\Gamma)$, and $\psi \in C^{1,\alpha}(\overline{B_1'})$. Then

$$\sup_{B_1} u_- \leq \sup_{\partial B_1} u_- + C\big(\max_{\Gamma} g_+ + \|f_+^-\|_{L^n(\Omega^-)} + \|f_+^+\|_{L^n(\Omega^+)}\big),$$

where $C > 0$ depends only on n, λ, Λ, α, and $[\psi]_{C^{1,\alpha}}$. We denote by $u_- = -\min\{0, u\}$, $g_+ = \max\{0, g\}$, $f_+^{\pm} = \max\{0, f^{\pm}\}$, and Γ_u is the convex envelope of $-u_-$ on $\overline{B_2}$ with $u \equiv 0$ on $\overline{B_2} \setminus B_1$.

Remark 4.2.2. The proof of Theorem 4.2.1 is similar to the one given in Theorem 3.2.1. As we discussed in Remark 3.2.2, the main difficulty that we encounter on this types of problems is that functions satisfying (4.2.1) may be singular on Γ. To avoid that, we construct an auxiliary function that removes g from the transmission condition. In the flat interface case, this function is simply a multiple of $|x_n|$. In the nonflat case, we will construct this auxiliary function with the help of a Hopf's type lemma.

First, we introduce some preliminaries.

Definition 4.2.3 (Dini function)**.** A function $\omega : [0, +\infty) \to [0, +\infty)$ is called a Dini function if ω is a nonnegative nondecreasing function and satisfies the following Dini condition for some $r_0 > 0$,

$$\int_0^{r_0} \frac{\omega(r)}{r}\, dr < +\infty.$$

Definition 4.2.4 (Interior $C^{1,\text{Dini}}$ condition)**.** We say that Ω satisfies the interior $C^{1,\text{Dini}}$ condition at $x_0 \in \partial\Omega$ if there exists $r_0 > 0$ and a system of coordinates $\{x_1, \ldots, x_n\}$ such that $x_0 = 0$ in this system, and

$$B_{r_0} \cap \left\{x_n \geq |x'|\omega(|x'|)\right\} \subset B_{r_0} \cap \Omega,$$

where ω is a Dini function.

The following lemma is proved in [37].

Lemma 4.2.5 (Hopf's lemma)**.** *Suppose that Ω satisfies the interior $C^{1,\text{Dini}}$ condition at $0 \in \partial\Omega$. Let $w \in \overline{S}_{\lambda,\Lambda}(0)$ in $\Omega \cap B_1$, with $w(0) = 0$ and $w \geq 0$ in $\Omega \cap B_1$. Then for any $l \in \mathbb{R}^n$, with $|l| = 1$ and $l_n = l \cdot e_n > 0$, we have that*

$$w(rl) \geq c l_n w(e_n/2) r,$$

for all $0 < r < r_1$, where $c > 0$, and r_1 depend only on n, λ, Λ, and ω.

Lemma 4.2.6 (Barrier)**.** *Let Ω be a $C^{1,\alpha}$ domain, with $0 < \alpha < 1$. Assume that $0 \in \partial\Omega$. Let w be the viscosity solution to*

$$\begin{cases} \mathcal{M}^-_{\lambda,\Lambda}(D^2 w) = 0 & \text{in } \Omega \cap B_2 \\ w = 0 & \text{on } \Gamma = \partial\Omega \cap B_2 \\ w = 1 & \text{on } \partial B_2 \cap \Omega. \end{cases} \tag{4.2.2}$$

Then w is a classical solution in $\Omega \cap B_1$, with $0 \leq w \leq 1$, and

$$w_\nu \geq c_0 > 0 \quad \text{on } \Gamma \cap B_1, \tag{4.2.3}$$

where ν is the interior normal to Ω, and c_0 depends on n, λ, Λ, α, and $[\Gamma]_{C^{1,\alpha}}$.

Proof. The existence and uniqueness of a viscosity solution w of (4.2.2) follows from the classical theory for fully nonlinear equations. Moreover, since $\mathcal{M}^-_{\lambda,\Lambda}$ is a concave operator, by [15, Theorem 6.6], we have that $w \in C^{2,\bar{\alpha}}(\overline{\Omega}_0)$, for any $\Omega_0 \subset\subset \Omega \cap B_2$, and some $0 < \bar{\alpha} < 1$ depending only on n, λ, and Λ. Also, from [38, Theorem 1.6], it follows that $w \in C^{1,\alpha}(\Gamma \cap B_1)$. Hence, from the interior and pointwise boundary regularity, we get that w is a classical solution in $\Omega \cap B_1$. Applying the classical ABP to w and $1 - w$, it is easy to see that $0 \leq w \leq 1$. Therefore, we only need to show (4.2.3).

Fix $x_0 \in \Gamma \cap B_1$. Without loss of generality, we may assume that $x_0 = 0$, and $\Omega \cap B_1 = B_1 \cap \{x_n > \psi(x')\}$ for some $\psi \in C^{1,\alpha}(\overline{B_1'})$ with $\nabla'\psi(0) = 0$, after a possible rotation. Also, we rescale ψ so that $[\psi]_{C^{1,\alpha}(0)} \leq 1/4$. We claim that $\Omega \cap B_1$ satisfies the interior $C^{1,\mathrm{Dini}}$ condition at 0 with $\omega_\alpha(t) = t^\alpha/2$. Indeed,

$$B_1 \cap \left\{x_n \geq |x'|\omega_\alpha(|x'|)\right\} \subseteq B_1 \cap \left\{x_n > \psi(x')\right\} = B_1 \cap \Omega,$$

since $|\psi(x')| \leq [\psi]_{C^{1,\alpha}(0)}|x'|^{1+\alpha} < |x'|\omega_\alpha(|x'|)$. Also, w satisfies the assumptions from Lemma 4.2.5. Hence, setting $l = \nu(0)$, we get

$$w(r\nu(0)) \geq cw(e_n/2)r \quad \text{for all } 0 < r < r_1$$

where c and r_1 depend only on n, λ, Λ, and ω_α. Since w is differentiable at 0, we see that

$$w_\nu(0) = \lim_{r\to 0^+} \frac{w(0 + r\nu(0)) - w(0)}{r} \geq cw(e_n/2).$$

By the interior Harnack inequality, we have

$$1 - w(e_n/2) \leq \sup_{B_{1/8}(e_n/2)} (1 - w) \leq c_1 \inf_{B_{1/8}(e_n/2)} (1 - w) \leq c_1,$$

where c_1 is a universal constant. We conclude that

$$w_\nu(0) \geq c_0 > 0,$$

with $c_0 = c(1 - c_1)$. $\square$

We are now ready to give the proof of the ABP estimate.

Proof of Theorem 4.2.1. Let $w \in C^0(B_2)$ satisfy

$$\begin{cases} \mathcal{M}^-_{\lambda,\Lambda}(D^2 w^\pm) = 0 & \text{in } \Omega^\pm \cap B_2 \\ w = 0 & \text{on } \Gamma \\ w = 1 & \text{on } \partial B_2. \end{cases}$$

By Lemma 4.2.6, we have that $0 \leq w \leq 1$, and

$$w^+_\nu \geq c^+ > 0, \quad w^-_\nu \leq -c^- < 0 \qquad \text{on } \Gamma \cap B_1,$$

where ν is the interior normal to Ω^+, and c^+, c^- depend only on n, λ, Λ, α, and $[\psi]_{C^{1,\alpha}}$. Fix $\varepsilon > 0$ small and consider in B_1 the function

$$v = u - \frac{1}{c_0}\Big(\max_\Gamma g_+ + \varepsilon\Big) w,$$

where $c_0 = c^+ + c^-$. By [15, Lemma 2.12], we have that $v \in \overline{S}_{\lambda,\Lambda}(f^\pm)$ in $B_1^\pm$. Moreover,

$$v^+_\nu - v^-_\nu \leq g - \frac{1}{c_0}\Big(\max_\Gamma g_+ - \varepsilon\Big)\big(w^+_\nu - w^-_\nu\big) \leq g_+ - \max_\Gamma g_+ - \varepsilon \leq -\varepsilon$$

on Γ, in the viscosity sense. Without loss of generality, we may assume that $v \geq 0$ on ∂B_1. Otherwise, we consider $v - \inf_{\partial B_1} v$. Assume that $v_- \not\equiv 0$, and let Γ_v be the convex envelope of $-v_-$ in B_2, where we have extended v by zero outside of B_1. Clearly, by definition of Γ_v, we have that $\partial B_1 \cap \{v = \Gamma_v\} = \varnothing$. Also, we claim that $\Gamma \cap \{v = \Gamma_v\} = \varnothing$. Indeed, if $A \cdot x + b$ touches v from below at $x_0 \in \Gamma$, for some $A \in \mathbb{R}^n$ and $b \in \mathbb{R}$, then

$$-\varepsilon \geq A \cdot \nu(x_0) - A \cdot \nu(x_0) = 0,$$

which is a contradiction. Moreover, there exists $\delta > 0$ such that for any $x_0 \in \Gamma$, we have $B_\delta(x_0) \cap \{v = \Gamma_v\} = \varnothing$. If not, for any $k \geq 1$, there exist $x_k \in \Gamma$ such that there is some $y_k \in B_{1/k}(x_k) \cap \{v = \Gamma_v\}$. Then, up to a subsequence, it follows that $x_k, y_k \to y_0$ for some $y_0 \in \overline{\Gamma \cap \{v = \Gamma_v\}}$, which is a contradiction.

Next, we show that $\Gamma_v \in C^{1,1}(\overline{B_1})$. By [15, Lemma 3.5], it is enough to see that there are $K > 0$ and $0 < r \leq 1$ such that for any $x_0 \in \overline{B_1} \cap \{v = \Gamma_v\}$ there exists a convex paraboloid of opening K that touches Γ_v by above at x_0 in $B_r(x_0)$. Indeed, fix $x_0 \in \overline{B_1} \cap \{v = \Gamma_v\}$. Since $x_0 \notin \partial B_1 \cup \Gamma$, we may assume that $x_0 \in \Omega^+ \cap \{v = \Gamma_v\}$. Furthermore, $B_\delta(x_0) \subset \Omega^+$, for δ small enough. Let l be a supporting plane of Γ_v at x_0. Then $0 \leq \Gamma_v - l \leq -v_- - l$ in $B_\delta(x_0)$ and $\Gamma_v(x_0) - l(x_0) = -v_-(x_0) - l(x_0) = 0$. By [15, Proposition 2.8], we know that $-v_- - l \in \overline{S}_{\lambda,\Lambda}(f^+)$. Applying [15, Lemma 3.3] to $-v_- - l$ in $B_\delta(x_0)$ and $\varphi = \Gamma_v - l$, we get

$$\Gamma_v(x) \leq l(x) + C^+\Big(\sup_{B_\delta(x_0)} f_+^+\Big)|x - x_0|^2 \quad \text{for all } x \in B_{\delta\gamma^+}(x_0), \tag{4.2.4}$$

where $\gamma^+ < 1$ and C^+ are universal constants. If $x_0 \in \Omega^- \cap \{v = \Gamma_v\}$, the proof is analogous. Hence, we take $K = 2\max\{C^+\|f_+^+\|_\infty, C^-\|f_+^-\|_\infty\}$ and $r = \delta \min\{\gamma^+, \gamma^-\}$.

By [15, Lemma 3.5], there exists a set $E \subset B_1$ such that $|B_1 \setminus E| = 0$ and Γ_v is second order differentiable at any $x \in E$. Moreover, we have that

$$\sup_{B_1} v_- \leq C\Big(\int_{E\cap\{v=\Gamma_v\}} \det D^2\Gamma_v(x)\,dx\Big)^{1/n},$$

where $C > 0$ is a constant depending only on n. Moreover, since $f^+ \in C^0(B_1^+)$, letting $\delta \to 0$ in (4.2.4), we see that $\det D^2\Gamma_v(x_0) \leq C f_+^+(x_0)^n$ for a.e. $x_0 \in B_1^+ \cap \{v = \Gamma_v\}$, and thus,

$$\int_{E\cap\{v=\Gamma_v\}} \det D^2\Gamma_v(x)\,dx \leq \int_{\Omega^-\cap\{v=\Gamma_v\}} f_+^-(x)^n\,dx + \int_{\Omega^+\cap\{v=\Gamma_v\}} f_+^+(x)^n\,dx.$$

Therefore,

$$\sup_{B_1} v_- \leq \sup_{\partial B_1} v_- + C\Big(\|f_+^-\|_{L^n(\Omega^-\cap\{v=\Gamma_v\})} + \|f_+^+\|_{L^n(\Omega^+\cap\{v=\Gamma_v\})}\Big).$$

From the definition of v, we have that

$$\sup_{B_1} u_- \leq \sup_{B_1} v_- \quad \text{and} \quad \sup_{\partial B_1} v_- \leq \sup_{\partial B_1} u_- + \frac{1}{c_0}\Big(\max_{\Gamma} g_+ + \varepsilon\Big).$$

Hence, letting $\varepsilon \to 0$, we see that

$$\sup_{B_1} u_- \leq \sup_{\partial B_1} u_- + C\Big(\max_{\Gamma} g_+ + \|f_+^-\|_{L^n(\Omega^-)} + \|f_+^+\|_{L^n(\Omega^+)}\Big),$$

where C depends only on n, λ, Λ, α, and $[\psi]_{C^{1,\alpha}}$. $\square$

Remark 4.2.7. In the previous proof, it is not clear how to relate the contact sets $\{v = \Gamma_v\}$ and $\{u = \Gamma_u\}$ given that not much is known about the barrier w. Hence, the latter set does not appear in the estimate. This is in contrast to the flat interface problem where w is a convex explicit function (see Theorem 3.2.1).

An immediate consequence of the ABP estimate is the maximum principle.

Corollary 4.2.8 (Maximum principle)**.** *Let u satisfy*

$$\begin{cases} u \in \overline{S}_{\lambda,\Lambda}(0) & \text{in } \Omega^{\pm} \\ u_\nu^+ - u_\nu^- \geq 0 & \text{on } \Gamma. \end{cases}$$

If $u \geq 0$ on ∂B_1, then $u \geq 0$ in B_1.

Remark 4.2.9. Replacing u by $-u$, we get the maximum principle for subsolutions.

4.3 Hölder regularity across interface

Our main goal of this section is to prove Theorem 4.1.2, that is, Hölder regularity of viscosity solutions across Γ. By [26, Lemma 8.23], it is enough to show the following oscillation lemma.

Lemma 4.3.1. *Let $\Gamma = B_1 \cap \{x_n = \psi(x')\}$. Let u satisfy*

$$\begin{cases} u \in S^*_{\lambda,\Lambda}(f^{\pm}) & \text{in } B_1 \\ u_\nu^+ - u_\nu^- = g & \text{on } \Gamma \end{cases}$$

with $f^{\pm} \in C^0(\Omega^{\pm}) \cap L^\infty(\Omega^{\pm})$, $g \in L^\infty(\Gamma)$, and $\psi \in C^{1,\alpha}(\overline{B_1'})$. Then

$$\operatorname*{osc}_{B_{1/3}} u \leq \mu \operatorname*{osc}_{B_1} u + C\big(\|g\|_{L^\infty(\Gamma)} + \|f^-\|_{L^n(\Omega^-)} + \|f^+\|_{L^n(\Omega^+)}\big)$$

where $0 < \mu, \alpha_1 < 1$ and $C > 0$ depend only on n, λ, Λ, α, and $\|\psi\|_{C^{1,\alpha}(\overline{B_1'})}$.

The oscillation lemma will be a consequence of the following result.

Lemma 4.3.2. *Let $u, f^{\pm}, g,$ and ψ be as in Lemma 4.3.1. Assume further that $\|u\|_{L^\infty(B_1)} \leq 1$, $u(\bar{x}) \geq 0$ with $\bar{x} = \frac{1}{5}e_n$, and $B_{1/20}(\bar{x}) \subset \Omega^+$. There exist $0 < \varepsilon_0, c < 1$ depending on n, λ, Λ, and $[\psi]_{C^{1,\alpha}}$ such that if $\|g\|_{L^\infty(\Gamma)} + \|f^-\|_{L^n(\Omega^-)} + \|f^+\|_{L^n(\Omega^+)} \leq \varepsilon_0$, then*

$$\inf_{B_{1/3}} u \geq -1 + c.$$

The proof is very similar to the one given in Lemma 3.3.2, so we will omit here some of the details.

Proof. By the Harnack inequality in $B_{1/20}(\bar{x})$, we get

$$1 \leq u(\bar{x}) + 1 \leq \sup_{B_{1/20}(\bar{x})} (u+1) \leq C\big(u(x) + 1 + \varepsilon_0\big),$$

for some universal $C \geq 1$. Hence,

$$u \geq -1 + \tilde{c} \quad \text{in } B_{1/20}(\bar{x}), \tag{4.3.1}$$

with $\tilde{c} = 1/C - \varepsilon_0$ and $\varepsilon_0 < 1/C$. For $x \in D = B_{3/4}(\bar{x}) \setminus B_{1/20}(\bar{x})$, we define

$$v(x) = \eta\phi(r) + \frac{\varepsilon_0}{c_0} w(x), \quad \phi(r) = r^{-\gamma} - (2/3)^{-\gamma}, \quad r = |x - \bar{x}|,$$

where w and c_0 are as in the proof of Theorem 4.2.1, $\gamma > \max\left\{0, \frac{\Lambda}{\lambda}(n-1)-1\right\}$, and $\eta > 0$ to be chosen later. For any $x \in D^{\pm} = \Omega^{\pm} \cap D$, we have

$$\begin{aligned} \mathcal{M}^-_{\lambda,\Lambda}(D^2 v^{\pm}(x)) &\geq \eta \mathcal{M}^-_{\lambda,\Lambda}(D^2\phi(x)) + \frac{\varepsilon_0}{c_0}\mathcal{M}^-_{\lambda,\Lambda}(D^2 w^{\pm}(x)) \\ &= \eta\gamma r^{-\gamma-2}\big(\lambda(\gamma+1) - \Lambda(n-1)\big) > 0, \end{aligned}$$

by the choice of γ. For $x \in \Gamma \cap D$, it follows that

$$v_\nu^+(x) - v_\nu^-(x) = \frac{\varepsilon_0}{c_0}\big(w_\nu^+(x) - w_\nu^-(x)\big) \geq 2\varepsilon_0 > \|g\|_{L^\infty(\Gamma)} \geq g(x).$$

First, choose $\eta \leq \frac{\tilde{c}}{2\phi(1/20)}$. Then choose $\varepsilon_0 \leq \frac{1}{c_0} \min\big\{\tilde{c}/2, -\eta\phi(3/4)\big\}$. By (4.3.1), we obtain that

$$v \leq u + 1 \quad \text{on } \partial D.$$

Since $u + 1 \in \overline{S}_{\lambda,\Lambda}(|f^\pm|)$ in D, $v^\pm \in C^2(D^\pm)$, and $\mathcal{M}^-_{\lambda,\Lambda}(D^2 v^\pm) \geq 0$ in $D^\pm$, by [15, Lemma 2.12], we have that $u + 1 - v \in \overline{S}_{\lambda,\Lambda}(|f^\pm|)$ in D. Also,

$$(u + 1 - v)_\nu^+ - (u + 1 - v)_\nu^- \leq g - (v_\nu^+ - v_\nu^-) \leq g - g = 0 \quad \text{on } \Gamma \cap D$$

in the viscosity sense. Hence, applying Theorem 4.2.1 to $u + 1 - v$ in D, with $g \equiv 0$, we see that

$$\sup_D (u + 1 - v)_- \leq \sup_{\partial D} (u + 1 - v)_- + C\big(\|f^-\|_{L^n(\Omega^-)} + \|f^+\|_{L^n(\Omega^+)}\big) \leq C\varepsilon_0,$$

where C depends only on n, λ, Λ, α, and $[\psi]_{C^{1,\alpha}}$. Therefore, $u \geq -1 + v - C\varepsilon_0$ in D. Moreover, for any $x \in B_{1/3}(0) \setminus B_{1/20}(\bar{x})$, we have that

$$v(x) \geq \eta\phi(23/60) = c_1$$

and $c_1 > 0$ depends only on n, λ, and Λ. Choosing ε_0 such that $\varepsilon_0 \leq \frac{c_1}{2C}$, we get $u \geq -1 + \frac{c_1}{2}$ in $B_{1/3}(0) \setminus B_{1/20}(\bar{x})$. Therefore,

$$\inf_{B_{1/3}} u \geq -1 + c,$$

with $c = \min\{\tilde{c}, \frac{c_1}{2}\}$. $\square$

Proof of Lemma 4.3.1. By choosing an appropriate system of coordinates, we assume that

$$\psi(0) = 0, \quad \nabla'\psi(0) = 0, \quad \text{and} \quad |\psi(x')| \leq |x'|.$$

Then $B_{1/20}(\bar{x}) \subset \Omega^+$, with $\bar{x} = \frac{1}{5}e_n$. Let $M = \|g\|_{L^\infty(\Gamma)} + \|f^-\|_{L^n(\Omega^-)} + \|f^+\|_{L^n(\Omega^+)}$, and let ε_0 be as in Lemma 4.3.2. Consider the rescaled function:

$$\tilde{u} = \frac{2u - \left(\inf_{B_1} u + \sup_{B_1} u\right)}{\operatorname{osc}_{B_1} u + 2M/\varepsilon_0} \in S^*_{\lambda,\Lambda}(\tilde{f}^\pm)$$

with $\tilde{f}^\pm = 2f^\pm(\operatorname{osc}_{B_1} u + 2M/\varepsilon_0)^{-1}$. Also, $(\tilde{u}^+)_\nu - (\tilde{u}^-)_\nu \leq \tilde{g}$ on Γ, in the viscosity sense, with $\tilde{g} = 2g(\operatorname{osc}_{B_1} u + 2M/\varepsilon_0)^{-1}$. Note that $\|\tilde{u}\|_{L^\infty(B_1)} \leq 1$, and

$$\max_\Gamma \tilde{g} + \|\tilde{f}^-\|_{L^n(\Omega^-)} + \|\tilde{f}^+\|_{L^n(\Omega^+)} \leq \varepsilon_0.$$

If $\tilde{u}(\bar{x}) \geq 0$, then by Lemma 4.3.2, it follows that $\inf_{B_{1/3}} \tilde{u} \geq -1+c$. Otherwise, $\tilde{u}(\bar{x}) < 0$, and applying the lemma to $-\tilde{u}$, we see that $\sup_{B_{1/3}} \tilde{u} \leq 1 - c$. In both cases, we get

$$\operatorname*{osc}_{B_{1/3}} \tilde{u} = \sup_{B_{1/3}} \tilde{u} - \inf_{B_{1/3}} \tilde{u} = \frac{2 \operatorname{osc}_{B_{1/3}} u}{\operatorname{osc}_{B_1} u + 2M/\varepsilon_0} \leq 2 - c.$$

Therefore,

$$\operatorname*{osc}_{B_{1/3}} u \leq \mu \operatorname*{osc}_{B_1} u + C\left(\|g\|_{L^\infty(\Gamma)} + \|f^-\|_{L^n(\Omega^-)} + \|f^+\|_{L^n(\Omega^+)}\right),$$

with $\mu = \frac{2-c}{2} < 1$, and $C = \frac{2-c}{\varepsilon_0}$. □

When $f^\pm \equiv 0$ and g has compact support on Γ, we obtain the following global Hölder continuity result. We omit the proof in this case since it is analogous to the one given in Proposition 4.3.3.

Proposition 4.3.3. *Assume $u \in C^0(\overline{B_1})$ satisfies*

$$\begin{cases} u \in S_{\lambda,\Lambda}(0) & \text{in } \Omega^\pm \\ u_\nu^+ - u_\nu^- = g & \text{on } \Gamma \\ u = \varphi & \text{on } \partial B_1, \end{cases}$$

where $g \in L^\infty(\Gamma)$, with $\operatorname{supp} g \subset \Gamma \cap B_{1-2\rho}$, for some $0 < \rho < 1/4$, and $\varphi \in C^{0,\alpha}(\partial B_1)$, with $0 < \alpha < 1$. Then $u \in C^{0,\beta}(\overline{B_1})$, with $0 < \beta \le \min\{\alpha_1, \alpha/2\}$, and

$$\|u\|_{C^{0,\beta}(\overline{B_1})} \le \frac{C}{\rho^\gamma}\big(\|\varphi\|_{C^{0,\alpha}(\partial B_1)} + \|g\|_{L^\infty(\Gamma)}\big),$$

where $0 < \alpha_1 < 1$ is given in Theorem 4.1.2, $\gamma = \max\{\alpha_1, \alpha\}$, and $C > 0$ depends only on n, λ, Λ, α, $[\Gamma]_{C^{1,\alpha}}$.

4.4 Closedness lemma

Next, we prove that a family of viscosity solutions to transmission problems with C^2 interfaces is closed under uniform limits. This result will be useful in the next section.

Lemma 4.4.1 (Closedness)**.** *For all $k \ge 1$, assume that $u_k \in C^0(B_1)$ satisfies*

$$\begin{cases} F_k^\pm(D^2 u_k) = \ f_k^\pm & \text{in } \Omega_k^\pm \\ (u_k^+)_\nu - (u_k^-)_\nu = g_k & \text{on } \Gamma_k, \end{cases}$$

in the viscosity sense, where $\Gamma_k = B_1 \cap \{x_n = \psi_k(x')\}$, for some $\psi_k \in C^2(B_1')$, $f_k^\pm \in C^0(\Omega_k^\pm)$, and $g_k \in C^0(\Gamma_k)$. Suppose that:

(i) $F_k \to F$ uniformly on compact subsets of $\mathcal{S}^n$.

(ii) $u_k \to u$ uniformly on compact subsets of B_1.

(iii) $\|f_k^{\pm}\|_{L^\infty(\Omega_k^{\pm})} \to 0$.

(iv) $\|g_k - g\|_{L^\infty(\Gamma_k)} = \sup_{x' \in B_1'} |g_k(x', \psi_k(x')) - g(x', 0)| \to 0$.

(v) $\Gamma_k \to T$ *in* C^2*, that is,* $\|\psi_k\|_{C^2(B_1')} \to 0$.

Then $u \in C^0(B_1)$ *is a viscosity solution of*

$$\begin{cases} F^{\pm}(D^2u) = \ 0 & \textit{in } B_1^{\pm} \\ u_{x_n}^+ - u_{x_n}^- = g & \textit{on } T. \end{cases}$$

Proof. To prove that u is a viscosity solution, we need to show that it is both a subsolution and a supersolution. Since the arguments are analogous, it suffices to see that u is a viscosity subsolution. First, we show that

$$F^{\pm}(D^2u) \geq 0 \qquad \text{in } B_1^{\pm}.$$

Suppose by contradiction that this fails. Then there is a point $x_0 \in B_1^{\pm}$ and a test function $\varphi \in C^2(B_1^{\pm})$ such that φ touches u from above at x_0, and

$$F^{\pm}(D^2\varphi(x_0)) < 0.$$

Without loss of generality, we can assume that $x_0 \in B_1^+$, and that φ touches u strictly from above. Otherwise, we can replace φ by $\varphi + \varepsilon|x - x_0|^2$, with ε small. Then since $u_k \to u$ uniformly on compact sets, there exists $\varepsilon_k > 0$ such that $\varphi + \varepsilon_k \geq u_k$ in $\overline{B_r(x_0)} \subset B_1^+$, for k large and some r small. Define

$$d_k = \inf_{B_{r_k}(x_0)} (\varphi + \varepsilon_k - u_k) \geq 0,$$

with $0 < r_k < r$ and $r_k \searrow 0$. Since $\Gamma_k \to T$, we can choose r_k such that $\overline{B_{r_k}(x_0)} \subset \Omega_k^+$, for k large. Let $x_k \in \Omega_k^+$ be a point where the infimum is attained, that is,

$$d_k = \varphi(x_k) + \varepsilon_k - u_k(x_k),$$

and define $c_k = \varepsilon_k - d_k$. Then $x_k \to x_0$, $c_k \to 0$, and $\varphi + c_k$ touches u_k from above at $x_k \in \Omega_k$, for k large. Hence, since $F_k^+(D^2 u_k(x_k)) \geq f_k^+$ in Ω_k^+, we must have

$$F_k^+(D^2\varphi(x_k)) \geq f_k^+(x_k).$$

Passing to the limit as $k \to \infty$, we get

$$F^+(D^2\varphi(x_0)) \geq 0,$$

which is a contradiction. Indeed, since $F_k^+ \to F^+$ uniformly, and $F^+ \in \mathcal{E}(\lambda, \Lambda)$,

$$\begin{aligned}
&\left|F_k^+(D^2\varphi(x_k)) - F^+(D^2\varphi(x_0))\right| \\
&\qquad \leq \left|F_k^+(D^2\varphi(x_k)) - F^+(D^2\varphi(x_k))\right| + \left|F^+(D^2\varphi(x_k)) - F^+(D^2\varphi(x_0))\right| \\
&\qquad \leq \sup_{\substack{M \in \mathcal{S}^n \\ \|M\| \leq K}} |F_k^+(M) - F^+(M)| + \Lambda \|D^2\varphi(x_k) - D^2\varphi(x_0)\| \to 0
\end{aligned}$$

where we used that $\sup_k \|D^2\varphi(x_k)\| \leq K$. Also, $|f_k^+(x_k)| \leq \|f_k^+\|_{L^\infty(\Omega_k^+)} \to 0$.

It remains to show that the transmission condition holds. If not, there exists $x_0 \in T$, $r > 0$ small, and $\varphi \in C^2(\overline{B_r^\pm(x_0)})$ such that φ touches u from above at x_0, and

$$\varphi_{x_n}^+(x_0) - \varphi_{x_n}^-(x_0) < g(x_0). \tag{4.4.1}$$

We can assume that φ touches u strictly from above at x_0, and that

$$F^{\pm}(D^2\varphi(x_0)) < 0. \tag{4.4.2}$$

If not, we can replace φ by

$$\varphi(x) + \eta|x_n| - C|x_n|^2,$$

with η small and C large such that $\eta|x_n| - C|x_n|^2 \geq 0$ in a small neighborhood of x_0. Arguing as before, there exist c_k, r_k, x_k such that $\phi(x) = \varphi(x', x_n - \psi_k(x')) + c_k$ touches u_k from above at $x_k \in B_{r_k}(x_0)$, with $c_k \to 0$, $x_k \to x_0$ and $r_k \to 0$. Then either there exists $k_0 \geq 1$ such that for every $k \geq k_0$ we have $x_k \in \Omega_k^{\pm}$, and thus,

$$F_k^{\pm}(D^2\phi(x_k)) \geq f_k^{\pm}(x_k),$$

or for every $k_0 \geq 1$ there exists $k \geq k_0$ such that $x_k \in \Gamma_k$. Hence,

$$\phi^+_{x_n k}(x_k) - \phi^-_{x_n k}(x_k) \geq g_k(x_k).$$

Passing to the limit, we get a contradiction in both cases. Indeed, let $x_k^* = x_k - e_n\psi_k(x_k')$, and compute the partial derivatives of ϕ at x_k:

$$\begin{aligned}
\phi_{x_i}(x_k) &= \varphi_{x_i}(x_k^*) - \varphi_{x_n}(x_k^*)(\psi_k)_{x_i}(x_k'), \quad i < n \\
\phi_{x_n}(x_k) &= \varphi_{x_n}(x_k^*) \\
\phi_{x_i x_j}(x_k) &= \varphi_{x_i x_j}(x_k^*) - \varphi_{x_i x_n}(x_k^*)(\psi_k)_{x_j}(x_k') \\
&\qquad - \big[\big(\varphi_{x_n x_j}(x_k^*) - \varphi_{x_n x_n}(x_k^*)(\psi_k)_{x_j}(x_k')\big)(\psi_k)_{x_i}(x_k') \\
&\qquad + \varphi_{x_n}(x_k^*)(\psi_k)_{x_i x_j}(x_k')\big], \quad i, j < n \\
\phi_{x_n x_j}(x_k) &= \varphi_{x_n x_j}(x_k^*) - \varphi_{x_n x_n}(x_k^*)(\psi_k)_{x_j}(x_k'), \quad j < n \\
\phi_{x_n x_n}(x_k) &= \varphi_{x_n x_n}(x_k^*).
\end{aligned}$$

Suppose first that $x_k \in \Gamma_k$. To get a contradiction with (4.4.1), it suffices to show that $\phi^+_{x_{nk}}(x_k) \to \varphi^+_{x_n}(x_0)$. By analogy, we will also have that $\phi^-_{x_{nk}}(x_k) \to \varphi^-_{x_n}(x_0)$. Moreover,

$$|g_k(x_k) - g(x_0)| \leq |g_k(x_k) - g(x'_k, 0)| + |g(x'_k, 0) - g(x_0)| \to 0.$$

Recall that $x_{nk}(x_k) = \frac{(-\nabla'\psi_k(x'_k),1)}{(1+|\nabla'\psi_k(x'_k)|^2)^{1/2}}$. Then:

$$\begin{aligned}\phi^+_{x_{nk}}(x_k) = &-\frac{\nabla'\varphi(x^*_k)\cdot\nabla'\psi_k(x'_k)}{(1+|\nabla'\psi_k(x'_k)|^2)^{1/2}} + \frac{\varphi_{x_n}(x^*_k)|\nabla'\psi_k(x'_k)|^2}{(1+|\nabla'\psi_k(x'_k)|^2)^{1/2}}\\ &+ \frac{\varphi_{x_n}(x^*_k)}{(1+|\nabla'\psi_k(x'_k)|^2)^{1/2}}.\end{aligned}$$

Since $\|\psi_k\|_{C^2(B'_1)} \to 0$, it follows that $\|\nabla'\psi_k\|_{L^\infty(B'_1)} \to 0$. In particular, $x^*_k \to x_0$. Therefore,

$$\begin{aligned}\left|\phi^+_{x_{nk}}(x_k) - \varphi^+_{x_n}(x_0)\right| \leq &\frac{|\nabla'\varphi(x^*_k)\cdot\nabla'\psi_k(x'_k)|}{(1+|\nabla'\psi_k(x'_k)|^2)^{1/2}} + \frac{|\varphi_{x_n}(x^*_k)||\nabla'\psi_k(x'_k)|^2}{(1+|\nabla'\psi_k(x'_k)|^2)^{1/2}}\\ &+ \left|\frac{\varphi_{x_n}(x^*_k)}{(1+|\nabla'\psi_k(x'_k)|^2)^{1/2}} - \varphi_{x_n}(x_0)\right| \equiv \mathrm{I} + \mathrm{II} + \mathrm{II}.\end{aligned}$$

Since φ is twice differentiable on $\overline{B^\pm_r(x_0)}$, we get

$$\begin{aligned}\mathrm{I} + \mathrm{II} \leq &\|\nabla'\varphi\|_{L^\infty(B_r(x_0)\setminus T)}\|\nabla'\psi_k\|_{L^\infty(B'_1)}\\ &+ \|\varphi_{x_n}\|_{L^\infty(B_r(x_0)\setminus T)}\|\nabla'\psi_k\|_{L^\infty(B'_1)} \to 0.\end{aligned}$$

Also,

$$\mathrm{III} \leq |\varphi_{x_n}(x^*_k) - \varphi_{x_n}(x_0)| + |(1+|\nabla'\psi_k(x'_k)|^2)^{1/2} - 1||\varphi_{x_n}(x_0)| \to 0.$$

Suppose now that $x_k \in \Omega^\pm_k$. To get a contradiction with (4.4.2), it suffices to show that $D^2\phi(x_k) \to D^2\varphi(x_0)$. From the previous computations,

$D^2\phi(x_k) = D^2\varphi(x_k^*) - M_k$, with

$$
\begin{aligned}
(M_k)_{i,j} &= \varphi_{x_i x_n}(x_k^*)(\psi_k)_{x_j}(x_k') + \big[\big(\varphi_{x_n x_j}(x_k^*) - \varphi_{x_n x_n}(x_k^*)(\psi_k)_{x_j}(x_k')\big)(\psi_k)_{x_i}(x_k') \\
&\qquad + \varphi_{x_n}(x_k^*)(\psi_k)_{x_i x_j}(x_k')\big], \quad i, j < n \\
(M_k)_{n,j} &= \varphi_{x_n x_n}(x_k^*)(\psi_k)_{x_j}(x_k'), \quad j < n \\
(M_k)_{n,n} &= 0.
\end{aligned}
$$

Reasoning as before, it is clear that $\|M_k\|_\infty = \sup |(M_k)_{i,j}| \to 0$. Therefore,

$$\|D^2\phi(x_k) - D^2\varphi(x_0)\|_\infty \le \|D^2\varphi(x_k^*) - D^2\varphi(x_0)\|_\infty + \|M_k\|_\infty \to 0.$$

$\square$

4.5 Approximating lemmas

Consider the nonflat interface problems given by

$$\begin{cases} F^\pm(D^2u) = f^\pm & \text{in } \Omega^\pm \\ u_\nu^+ - u_\nu^- = g & \text{on } \Gamma. \end{cases} \tag{4.5.1}$$

In this section, we will prove some approximating lemmas for viscosity solutions of (4.5.1) that will be useful to derive $C^{1,\alpha}$ estimates in the following section.

From the transmission condition in (4.5.1), we see that we need to distinguish two cases. If g is close to 0, then we will approximate u with a function that is differentiable across Γ (see Lemma 4.5.1). To prove this, it is sufficient that the operators F^+ and F^- satisfy a closeness condition. Our ideas are inspired by [51]. The most challenging case happens when g is away from 0, since u is singular at the interface. In this case, we will approximate u with solutions to flat interface problems (see Chapter 3). This is known as the stability result (see Lemma 4.5.6). We point out that, for this case, we do not require that the operators are close.

4.5.1 Case g close to 0

Lemma 4.5.1. *Let $0 < \alpha < \bar{\alpha}$, $0 < \tau < 3/4$, and $0 < \delta < 1$. Suppose that*

$$\sup_{M \in \mathcal{S}^n \setminus \{0\}} \frac{\|F^+(M) - F^-(M)\|}{\|M\|} \leq \theta, \tag{4.5.2}$$

for some $0 < \theta << 1$ depending only on n, λ, and Λ. Assume that $u \in C^0(B_1)$ is a viscosity solution to (4.5.1), with $\|u\|_{L^\infty(B_1)} \leq 1$ and $\|g\|_{L^\infty(\Gamma)} + \|f^-\|_{L^n(\Omega^-)} + \|f^+\|_{L^n(\Omega^+)} \leq \delta$. Then there exists $v \in C^{1,\alpha}_{loc}(B_{3/4}) \cap C^{0,\beta}(\overline{B_{3/4}})$ such that

$$\|u - v\|_{L^\infty(B_{3/4-\tau})} \leq C(\tau^\beta + \delta),$$

for some $C > 0$ and $0 < \beta < 1$.

Proof. Fix $0 < \rho < 1/2$ and $0 < \delta < 1$ to be determined. Given $\varepsilon > 0$ small, for $x \in B_1$, we define

$$F_\varepsilon(M, x) = h_\varepsilon(x) F^+(M) + ((1 - h_\varepsilon(x))F^-(M),$$

where $h_\varepsilon \in C^\infty(B_1)$, $0 \leq h_\varepsilon \leq 1$, and

$$h_\varepsilon(x) = \begin{cases} 1 & \text{if } x \in \{x_n > \psi(x') + \varepsilon\} \cap B_1 \\ 0 & \text{if } x \in \{x_n < \psi(x') - \varepsilon\} \cap B_1. \end{cases}$$

Note that $F_\varepsilon \in \mathcal{E}(\lambda, \Lambda)$ and $F_\varepsilon(0, x) \equiv 0$. Moreover, $F_\varepsilon \to F^\pm$ uniformly on compact subsets of $\mathcal{S}^n \times \Omega^\pm$. Indeed, let $M \in \overline{B_R} \subset \mathcal{S}^n$ and $x \in K \subset \Omega^+$, for some $R > 0$ and K compact. Then

$$\begin{aligned} |F_\varepsilon(M, x) - F^+(M)| &\leq (1 - h_\varepsilon(x))|F^+(M) - F^-(M)| \\ &\leq \sup_{x \in K}(1 - h_\varepsilon(x))\theta R \to 0, \end{aligned}$$

as $\varepsilon \to 0$, where we used (4.5.2) in the last inequality. The argument is analogous for F^-.

Let v_ε be the viscosity solution of

$$\begin{cases} F_\varepsilon(D^2 v_\varepsilon, x) = 0 & \text{in } B_{3/4} \\ v_\varepsilon = u & \text{on } \partial B_{3/4}. \end{cases}$$

For $x \in B_1$, define

$$\beta_\varepsilon(x) = \sup_{M \in \mathcal{S}^n \setminus \{0\}} \frac{|F_\varepsilon(M, x) - F_\varepsilon(M, 0)|}{\|M\|}.$$

By the previous estimate and the fact that $0 \leq h_\varepsilon \leq 1$, we have

$$\beta_\varepsilon(x) \leq (1 - h_\varepsilon(x))\theta + (1 - h_\varepsilon(0))\theta \leq 2\theta,$$

for all $x \in B_1$, where θ only depends on n, λ, and Λ. Hence, for any $0 < r \leq 1$, it follows that

$$\left(\fint_{B_r} \beta_\varepsilon^n \, dx \right)^{1/n} \leq 2\theta.$$

Choose $0 < \theta \leq \theta_0/2$, where $\theta_0 > 0$ (independent of ε) is given in [15, Theorem 8.3]. Then $v_\varepsilon \in C^{1,\bar{\alpha}}_{loc}(B_{3/4})$ and, for any $0 < \rho < 3/4$, the following estimate holds:

$$\|v_\varepsilon\|_{C^{1,\bar{\alpha}}(\overline{B_\rho})} \leq C_0 \|u\|_{L^\infty(B_1)} \leq C_0.$$

By compactness, $v_\varepsilon \to v$ in $C^{1,\alpha}_{loc}(B_{3/4})$ as $\varepsilon \to 0$, for any $0 < \alpha < \bar{\alpha}$. Moreover, by the closedness of viscosity solutions under uniform limits (see [15, Proposition 2.9]), v satisfies

$$F^{\pm}(D^2 v) = 0 \quad \text{in } \Omega^{\pm} \cap B_{3/4}, \tag{4.5.3}$$

in the viscosity sense. By Theorem 4.1.2, we have that $u \in C^{0,\alpha_1}(\overline{B_{3/4}})$, and

$$\|u\|_{C^{0,\alpha_1}(\overline{B_{3/4}})} \le C(1+\delta) \le 2C,$$

for some $0 < \alpha_1 < 1$ and C depending only on $n, \lambda,\ \Lambda,\ \alpha,$ and $\|\Gamma\|_{C^{1,\alpha}}$. By [15, Proposition 4.13], it follows that $v_\varepsilon \in C^{0,\beta}(\overline{B_{3/4}})$, with $\beta = \frac{\alpha_1}{2}$, and

$$\|v_\varepsilon\|_{C^{0,\beta}(\overline{B_{3/4}})} \le C\|u\|_{C^{0,\alpha_1}(\partial B_{3/4})} \le C_1.$$

Let $w = u - v$. Then $w \in C^{0,\beta}(\overline{B_{3/4}})$, with $w = 0$ on $\partial B_{3/4}$, and for any $0 < \tau < 1/4$, we have

$$\|w\|_{L^\infty(\partial B_{3/4-\tau})} \le [w]_{C^{0,\beta}(\overline{B_{3/4}})}\tau^\beta \le C_2\tau^\beta,$$

where $C_2 = 2C + C_1$. Since u and v satisfy (4.5.1) and (4.5.3), respectively, then

$$\begin{cases} w \in S_{\lambda/n,\Lambda}(f^{\pm}) & \text{in } \Omega^{\pm} \cap B_{3/4-\tau} \\ w^+_\nu - w^-_\nu = g & \text{on } \Gamma \cap B_{3/4-\tau}. \end{cases}$$

From the ABP estimate, and the assumptions on g and $f^{\pm}$, we get

$$\begin{aligned} \|w\|_{L^\infty(B_{3/4-\tau})} &\le \|w\|_{L^\infty(\partial B_{3/4-\tau})} + C\big(\|g\|_{L^\infty(\Gamma)} + \|f^-\|_{L^n(\Omega^-)} + \|f^+\|_{L^n(\Omega^+)}\big) \\ &\le \|w\|_{L^\infty(\partial B_{3/4-\tau})} + C\big(\|g\|_{L^\infty(\Gamma)} + C_{f^-} + C_{f^+}\big) \\ &\le C_2\tau^\beta + C\delta. \end{aligned}$$

Therefore, $\|u - v\|_{L^\infty(B_{3/4-\tau})} \le C(\tau^\beta + \delta)$. $\square$

4.5.2 Case g away from 0

Our strategy to approximate u is similar to the one in Chapter 2. For $|a| < 1/2$, define $T_a = B_1 \cap \{x_n = a\}$. We consider the flat interface transmission problem,

$$\begin{cases} F^{\pm}(D^2 v) = 0 & \text{in } B_1 \setminus T_a \\ v^+_{x_n} - v^-_{x_n} = g_a & \text{on } T_a, \end{cases} \tag{4.5.4}$$

where g_a is a mollification of χ_{T_a}, with $\operatorname{supp} g_a \subset B_{3/4} \cap T_a$. For convenience, when $a = 0$, we call the solution v_0 and the interface T. Since g_a has compact support, we can apply Proposition 4.3.3, with $\rho = 1/8$, to obtain global Hölder continuity of solutions to these flat interface problems.

Corollary 4.5.2. *Let v be a solution to (4.5.4) in B_1, with $\varphi = v|_{\partial B_1} \in C^{0,\alpha}(\partial B_1)$. Then $v \in C^{0,\beta}(\overline{B_1})$, with $0 < \beta \leq \min\{\alpha_1, \alpha/2\}$, and*

$$\|v\|_{C^{0,\beta}(\overline{B_1})} \leq C\big(1 + \|\varphi\|_{C^{0,\alpha}(\partial B_1)}\big),$$

where $0 < \alpha_1 < 1$ is given in Theorem 4.1.2, and $C > 0$ depends only on n, λ, Λ, and α.

Lemma 4.5.3. *For any $\varepsilon > 0$, there exists $0 < \delta < \min\{\varepsilon, 1/2\}$ such that if v satisfies (4.5.4), with $a = \delta$, $v = v_0$ on ∂B_1, $\sup_{\partial B_1} |v_0| \leq C_0$, and $\|g_\delta(\cdot, \delta) - g_0(\cdot, 0)\|_{L^\infty(B_1')} \leq \delta$, then*

$$\|v - v_0\|_{L^\infty(B_1)} \leq \varepsilon.$$

Fix $0 < r < 1$. In addition, if $[g_\delta(\cdot, \delta) - g_0(\cdot, 0)]_{C^{0,\alpha}(\overline{B_1'})} \leq \delta$, then

$$\|\nabla' v - \nabla' v_0\|_{C^{0,\bar{\alpha}}(\overline{B_r})} + \|(v_0^+)_{x_n} - v^-_{x_n} - 1\|_{L^\infty(D_{\delta,r})} \leq (1 - r)^{-(1+\alpha)}\varepsilon,$$

where $0 < \bar{\alpha} < \alpha$, and $D_{\delta,r} = B_r \cap \{0 < x_n < \delta\}$.

Proof. We will prove it by contradiction. Assume there exist ε_0, v_k, g_k such that

$$\begin{cases} F^{\pm}(D^2 v_k) = 0 & \text{in } B_1 \setminus T_{1/k} \\ v^+_{x_n} - v^-_{x_n} = g_k & \text{on } T_{1/k} \\ v_k = v_0 & \text{on } \partial B_1, \end{cases}$$

with $\|g_k(\cdot, 1/k) - g_0(\cdot, 0)\|_{C^{0,\alpha}(\overline{B_1'})} \leq 1/k$, and

$$\|v_k - v_0\|_{L^\infty(B_1)} > \varepsilon_0, \tag{4.5.5}$$

$$\|\nabla' v_k - \nabla' v_0\|_{C^{0,\bar{\alpha}}(\overline{B_r})} > (1-r)^{-(1+\alpha)}\varepsilon_0, \tag{4.5.6}$$

$$\|(v_0^+)_{x_n} - (v_k^-)_{x_n} - 1\|_{L^\infty(D_{1/k,r})} > (1-r)^{-(1+\alpha)}\varepsilon_0, \tag{4.5.7}$$

for all $k \geq 1$. From the ABP estimate (Theorem 4.2.1), we get

$$\|v_k\|_{L^\infty(B_1)} \leq \sup_{\partial B_1} |v_0| + C\|g_k\|_{L^\infty(T_{1/k})} \leq C_0 + 2C.$$

Hence, from the global Hölder estimate in Corollary 4.5.2, we have that

$$\|v_k\|_{C^{0,\alpha}(\overline{B_1})} \leq C\big(\|v_k\|_{L^\infty(B_1)} + \|g_k\|_{L^\infty(T_{1/k})}\big) \leq C_1.$$

By compactness, it follows that, up to a subsequence, $v_k \to v$ uniformly in B_1. Moreover, by Lemma 4.4.1, v satisfies

$$\begin{cases} F^{\pm}(D^2 v) = 0 & \text{in } B_1^{\pm} \\ v^+_{x_n} - v^-_{x_n} = g_0 & \text{on } T \\ v = v_0 & \text{on } \partial B_1. \end{cases}$$

By uniqueness of viscosity solutions (Corollary 3.5.5), we see that $v = v_0$ on B_1. This contradicts (4.5.5) for k sufficiently large. Moreover, by Theorem 3.1.9 (rescaled) we have that $v_k \in C^{1,\alpha}$ in the x'-direction in B_r, with

$$\|\nabla' v_k\|_{C^{0,\alpha}(\overline{B_r})} \leq \frac{C}{(1-r)^{1+\alpha}}\big(\|v_k\|_{L^\infty(B_1)} + \|g_k\|_{C^{0,\alpha}(T_{1/k})}\big) \leq \frac{C_2}{(1-r)^{1+\alpha}}.$$

By compactness, it follows that, up to a subsequence, $\nabla' v_k \to w$ in $C^{0,\bar{\alpha}}(\overline{B_r})$, with $0 < \bar{\alpha} < \alpha$. By uniqueness of distributional limits, we have that $w = \nabla' v_0$. This contradicts (4.5.6) for k sufficiently large. Furthermore, from the $C^{1,\alpha}$ estimate of Theorem 3.1.9, we have

$$\begin{aligned}
\|(v_k^-)_{x_n}\|_{C^{0,\alpha}\left(\overline{B_r}\cap\{x_n\leq 1/k\}\right)} &\leq \frac{C}{(1-r)^{1+\alpha}}\left(\|v_k\|_{L^\infty(B_1)} + \|g_k\|_{C^{0,\alpha}(T_{1/k})}\right) \\
&\leq \frac{C_3}{(1-r)^{1+\alpha}}.
\end{aligned}$$

By the previous argument, up to a subsequence, it follows that $(v_k^-)_{x_n} \to (v_0^-)_{x_n}$ uniformly in B_r^-. Let $x \in D_{1/k,r}$, and denote $\bar{x} = (x', 0) \in T$. Note that $|x - \bar{x}| < 1/k$. Then

$$\begin{aligned}
|(v_0^+)_{x_n}(x) - (v_k^-)_{x_n}(x) - 1| &\leq |(v_0^+)_{x_n}(x) - (v_0^+)_{x_n}(\bar{x})| \\
&\quad + |(v_0^+)_{x_n}(\bar{x}) - (v_0^-)_{x_n}(\bar{x}) - 1| \\
&\quad + |(v_0^-)_{x_n}(\bar{x}) - (v_k^-)_{x_n}(\bar{x})| \\
&\quad + |(v_k^-)_{x_n}(\bar{x}) - (v_k^-)_{x_n}(x)| \\
&\equiv \mathrm{I} + \mathrm{II} + \mathrm{III} + \mathrm{IV}.
\end{aligned}$$

By construction of v_0, $\mathrm{II} = 0$. Moreover, $\mathrm{III} \to 0$ as $k \to +\infty$, by uniform convergence, and

$$\begin{aligned}
\mathrm{I} &\leq [(v_0^+)_{x_n}]_{C^{0,\alpha}(\overline{B_r^+})}|x-\bar{x}|^\alpha \leq \frac{C}{(1-r)^{1+\alpha}}\frac{1}{k^\alpha} \to 0 \quad (r \text{ is fixed}) \\
\mathrm{IV} &\leq [(v_k^-)_{x_n}]_{C^{0,\alpha}\left(\overline{B_r}\cap\{x_n\leq 1/k\}\right)}|x-\bar{x}|^\alpha \leq \frac{C_3}{(1-r)^{1+\alpha}}\frac{1}{k^\alpha} \to 0.
\end{aligned}$$

This contradicts (4.5.7) for k sufficiently large. $\square$

Remark 4.5.4. An analogous argument shows that the result holds when $a = -\delta$.

Corollary 4.5.5. *Fix $\varepsilon > 0$. Let $\overline{v}, \underline{v} \in C^0(\overline{B_1})$ be as in Lemma 4.5.3, with $a = \delta$, and $a = -\delta$, respectively, where δ is the minimum between the two given. Fix $0 < r < 1$. Then*

$$\begin{aligned} \|\overline{v} - \underline{v}\|_{L^\infty(B_1)} &\leq \varepsilon \\ \|\nabla'\overline{v} - \nabla'\underline{v}\|_{C^{0,\bar{\alpha}}(\overline{B_r})} &\leq (1-r)^{-(1+\alpha)}\varepsilon \\ \|(\underline{v})^+_{x_n} - (\overline{v})^-_{x_n} - 1\|_{L^\infty(D_{\delta,r})} &\leq (1-r)^{-(1+\alpha)}\varepsilon \end{aligned}$$

where $0 < \bar{\alpha} < 1$, and $D_{\delta,r} = B_r \cap \{|x_n| < \delta\}$.

We denote by $\Omega^\pm_r = \Omega^\pm \cap B_r$ and $\Gamma_r = \Gamma \cap B_r$, for $0 < r < 1$.

Lemma 4.5.6 (Stability)**.** *Let $\varepsilon > 0$ be given. Assume that u, g, $f^\pm$, and Γ satisfy the assumptions from Lemma 4.6.3. Let $v = \overline{v}\chi_{\Omega^-} + \underline{v}\chi_{\Omega^+}$, where $\overline{v}$ and $\underline{v}$ are given in Corollary 4.5.5, replacing B_1 by $B_{3/4}$. If $v = u$ on $\partial B_{3/4}$, then*

$$\|u - v\|_{L^\infty(B_{1/2})} \leq C\varepsilon^{1/2},$$

where $C > 0$ depends only on n, λ, Λ, and α.

Proof. By Theorem 4.1.2, and the assumptions of u, g, $f^\pm$, we have $u \in C^{0,\alpha}(\overline{B_{3/4}})$, with

$$\|u\|_{C^{0,\alpha}(\overline{B_{3/4}})} \leq C\big(\|u\|_{L^\infty(B_1)} + \|g\|_{L^\infty(\Gamma)} + \|f^-\|_{L^n(\Omega^-)} + \|f^+\|_{L^n(\Omega^+)}\big) \leq C_1.$$

Moreover, since $v = u$ on $\partial B_{3/4}$, by Corollary 4.5.2 it follows that $v \in C^{0,\beta}(\overline{\Omega^{\pm}_{3/4}})$, with $0 < \beta \leq \min\{\alpha_1, \alpha/2\}$, and

$$\|v\|_{C^{0,\beta}(\overline{\Omega^{\pm}_{3/4}})} \leq C\big(1 + \|u\|_{C^{0,\alpha}(\partial B_{3/4})}\big)$$

for some $C > 0$ depending only on n, λ, Λ, and α. Hence,

$$\|v\|_{C^{0,\beta}(\overline{\Omega^{\pm}_{3/4}})} \leq C(1 + C_1).$$

Note that v is not continuous across $\Gamma_{3/4}$ since $\overline{v} - \underline{v} \neq 0$ on $\Gamma_{3/4}$. Let w satisfy

$$\begin{cases} F^{\pm}(D^2 w^{\pm}) = 0 & \text{in } \Omega^{\pm}_{3/4} \\ w = \frac{1}{2}(v^+ + v^-) & \text{on } \Gamma_{3/4} \\ w = v & \text{on } \partial B_{3/4}. \end{cases}$$

By Theorem 3.1.9 (rescaled), we know that $v^{\pm} \in C^{1,\bar{\alpha}}_{loc}(\Omega^{\pm}_{3/4})$, and for any $0 < \eta < 3/4$,

$$\eta \|\nabla v^{\pm}\|_{L^{\infty}(\Omega^{\pm}_{3/4-\eta})} + \eta^{1+\bar{\alpha}} [\nabla v^{\pm}]_{C^{0,\bar{\alpha}}(\overline{\Omega^{\pm}_{3/4-\eta}})} \leq C_3. \tag{4.5.8}$$

Then from [38, Theorem 1.6], we have $w^{\pm} \in C^{1,\alpha}(\overline{\Omega^{\pm}_{3/4-\eta}})$, and we will see that

$$w^+_{\nu} - w^-_{\nu} \approx 1 \quad \text{on} \quad \Gamma_{3/4-\eta}.$$

Indeed, for any $x \in \Gamma_{3/4-\eta}$, we have

$$\begin{aligned} w^+_{\nu}(x) - w^-_{\nu}(x) - 1 &= \big((w - v)^+_{\nu}(x)\big) - \big((w - v)^-_{\nu}(x)\big) + \big(v^+_{\nu}(x) - v^-_{\nu}(x) - 1\big) \\ &\equiv g_1 + g_2 + g_3. \end{aligned}$$

We will show that g_1, g_2, and g_3 are small in terms of ε and δ. For g_3, we have

$$\begin{aligned} |g_3(x)| &\leq |v^+_{\nu}(x) - v^+_{x_n}(x)| + |v^+_{x_n}(x) - v^-_{x_n}(x) - 1| + |v^-_{x_n}(x) - v^-_{\nu}(x)| \\ &\equiv \mathrm{I} + \mathrm{II} + \mathrm{III}. \end{aligned}$$

By Corollary 4.5.5, it follows that II $\leq \eta^{-(1+\alpha)}\varepsilon$. Also, since $\|\nabla'\psi\|_{L^\infty(B_1')} \leq \delta$,

$$\mathrm{I} \leq |\nabla v^+(x)||\nu(x) - e_n| \leq C\|\nabla v^+\|_{L^\infty(\Omega^+_{3/4-\eta})}|\nabla'\psi(x')| \leq \frac{C_3}{\eta}\delta.$$

Similarly for III. For g_1 and g_2 we consider $w - v$. Since $w^\pm - v^\pm \in S_{\lambda/n,\Lambda}(0)$ in $\Omega^\pm_{3/4}$, by the classical ABP, and Corollary 4.5.5, we get

$$\|w^\pm - v^\pm\|_{L^\infty(\Omega^\pm_{3/4})} \leq \|w^\pm - v^\pm\|_{L^\infty(\Gamma_{3/4})} = \|\overline{v} - \underline{v}\|_{L^\infty(\Gamma_{3/4})} \leq \varepsilon.$$

Hence, $\|w - v\|_{L^\infty(B_{3/4})} \leq \varepsilon$. Moreover, from [38, Theorem 1.6], we have

$$|g_1(x)| \leq \frac{C}{\eta}\big(\|w^+ - v^+\|_{L^\infty(\Omega^+_{3/4})} + \|\overline{v} - \underline{v}\|_{C^{1,\bar{\alpha}}(\overline{\Gamma_{3/4-\eta}})}\big),$$

where $\|v\|_{C^{1,\bar{\alpha}}(\overline{\Gamma_{3/4-\eta}})} = \|v\|_{L^\infty(\Gamma_{3/4-\eta})} + \|\nabla' v + v_{x_n}\nabla'\psi\|_{C^{0,\bar{\alpha}}(\overline{\Gamma_{3/4-\eta}})}$. By Corollary 4.5.5, and estimate (4.5.8), it follows that

$$\begin{aligned}
\|\overline{v} - \underline{v}\|_{C^{1,\bar{\alpha}}(\overline{\Gamma_{3/4-\eta}})} &\leq \|\overline{v} - \underline{v}\|_{L^\infty(B_{3/4})} + \|\nabla'\overline{v} - \nabla'\underline{v}\|_{C^{0,\bar{\alpha}}(\overline{B_{3/4-\eta}})} \\
&\quad + \|(\overline{v}_{x_n} - \underline{v}_{x_n})\nabla'\psi\|_{C^{0,\bar{\alpha}}(\overline{\Gamma_{3/4-\eta}})} \\
&\leq \varepsilon + \eta^{-(1+\alpha)}\varepsilon + 2\|\overline{v}_{x_n} - \underline{v}_{x_n}\|_{C^{0,\bar{\alpha}}(\overline{\Gamma_{3/4-\eta}})}\|\nabla'\psi\|_{C^{0,\bar{\alpha}}(\overline{B_1'})} \\
&\leq \varepsilon + \eta^{-(1+\alpha)}\varepsilon + 4C_3\eta^{-(1+\alpha)}\delta.
\end{aligned}$$

Therefore, $|g_1(x)| \leq C\eta^{-1}(\varepsilon + \eta^{-(1+\alpha)}(\varepsilon + \delta)) \leq C\eta^{-(2+\alpha)}\varepsilon$. Similarly for g_2.

Next, the function $u - w$ satisfies

$$\begin{cases}
u - w \in S_{\lambda/n,\Lambda}(f^\pm) & \text{in } \Omega^\pm_{3/4} \\
(u-w)^+_\nu - (u-w)^-_\nu = (g-1) - (g_1 + g_2 + g_3) & \text{on } \Gamma_{3/4-\eta} \\
u - w = 0 & \text{on } \partial B_{3/4}.
\end{cases}$$

Therefore, by Theorem 4.2.1 applied to $u-w$ in $B_{3/4-\eta}$, we have

$$
\begin{aligned}
\|u-w\|_{L^\infty(B_{3/4-\eta})} &\le \|u-v\|_{L^\infty(\partial B_{3/4-\eta})} + \|v-w\|_{L^\infty(\partial B_{3/4-\eta})} \\
&\qquad + C\big(\|g-1\|_{L^\infty(\Gamma_{3/4})} + \|f^-\|_{L^n(\Omega^-_{3/4})} + \|f^+\|_{L^n(\Omega^+_{3/4})} \\
&\qquad + \|g_1\|_{L^\infty(\Gamma_{3/4-\eta})} + \|g_2\|_{L^\infty(\Gamma_{3/4-\eta})} + \|g_3\|_{L^\infty(\Gamma_{3/4-\eta})}\big) \\
&\le [u-v]_{C^{0,\beta}(\overline{B_{3/4}})}\eta^\beta + \|v-w\|_{L^\infty(B_{3/4})} \\
&\qquad + C\delta + 2C\eta^{-(2+\alpha)}\varepsilon + C\eta^{-1}\delta \\
&\le (C_1+C_2)\eta^\beta + \varepsilon + \tilde{C}\eta^{-(2+\alpha)}\varepsilon.
\end{aligned}
$$

Choose $0<\eta<1/4$ such that $\eta < \min\{\varepsilon^{\frac{1}{2(2+\alpha)}}, \varepsilon^{\frac{1}{2\beta}}\}$. We conclude that

$$
\|u-v\|_{L^\infty(B_{1/2})} \le \|u-w\|_{L^\infty(B_{3/4-\eta})} + \|w-v\|_{L^\infty(B_{3/4})} \le C\varepsilon^{1/2},
$$

where $C>0$ depends only on n, λ, Λ, and α. $\square$

4.6 $C^{1,\alpha}$ regularity at the interface

In this section, we derive pointwise $C^{1,\alpha}$ boundary estimates for viscosity solutions of nonflat interface problems following a perturbation method. The approximating lemmas from Section 4.5 will be a key ingredient for this argument.

Theorem 4.6.1. *Fix $0<\alpha<\bar{\alpha}$, for some $\bar{\alpha}$ depending only on n, λ, and Λ. Assume that $0\in\Gamma$, $\psi\in C^{1,\alpha}(0)$, $\psi\neq 0$, $g\in C^{0,\alpha}(0)$, and $f^\pm$ satisfy*

$$
\left(\fint_{B_r\cap\Omega^\pm} |f^\pm|^n\,dx\right)^{1/n} \le C_{f^\pm} r^{\alpha-1} \quad \textit{for all } r>0.
$$

Assume further that

$$\sup_{M\in\mathcal{S}^n\setminus\{0\}}\frac{\|F^+(M)-F^-(M)\|}{\|M\|}\leq\theta,$$

for some $0<\theta<<1$ *depending only on* n, λ, Λ, *and* α. *Suppose that* u *is a bounded viscosity solution of* (4.5.1) *in* B_1, *with* $\|u\|_{L^\infty(B_1)}\leq 1$. *Then*

$$u^\pm\in C^{1,\alpha}(0).$$

Namely, there exist affine functions $l^\pm(x)=A^\pm\cdot x+b$ *such that*

$$|u^\pm(x)-l^\pm(x)|\leq C|x|^{1+\alpha}\qquad \text{for all } x\in\Omega^\pm_{r_0},$$

where $r_0=C_0/\|\psi\|_{C^{1,\alpha}(0)}$ *and* $C_0>0$ *depends only on* $n,\lambda,\Lambda,$ *and* α. *Moreover,*

$$|A^-|+|A^+|+|b|+|C|\leq C_0\|\psi\|_{C^{1,\alpha}(0)}\big(|g(0)|+[g]_{C^{0,\alpha}(0)}+C_{f^-}+C_{f^+}\big).$$

This theorem will follow from iterating the next two lemmas.

Lemma 4.6.2. *Given* $0<\alpha<\bar\alpha$, *there exist* $C_0>0$, $0<\delta<1$, *and* $0<\rho<1/2$, *depending only on* n, λ, Λ, *and* α, *such that for any viscosity solution* $u\in C^0(B_1)$ *of* (4.5.1) *and* (4.5.2), *with* $\|u\|_{L^\infty(B_1)}\leq 1$ *and* $\|g\|_{L^\infty(\Gamma\cap B_{3/4})}+C_{f^-}+C_{f^+}\leq\delta$, *there is an affine function* $l(x)=A\cdot x+b$, *with* $|A|+|b|\leq C_0$, *such that*

$$\|u-l\|_{L^\infty(B_\rho)}\leq\rho^{1+\alpha}.$$

Proof. Fix $0<\tau<1/4$ and $0<\delta<1$ to be chosen. Let $v\in C^{1,\bar\alpha-\varepsilon}_{loc}(B_{3/4})$ be the function given in Lemma 4.5.1, for some $\epsilon>0$ sufficiently small so that $\bar\alpha-\epsilon>\alpha$. Then

$$\|u-v\|_{L^\infty(B_{3/4-\tau})}\leq C(\tau^\beta+\delta).$$

Moreover, if $l(x) = v(0) + \nabla v(0) \cdot x$, then $|\nabla l| + |l(0)| \leq C_0$, and the following estimate holds,

$$\|v - l\|_{L^\infty(B_\rho)} \leq C_0 \rho^{1+\bar{\alpha}-\epsilon},$$

for any $0 < \rho < 1/2$. It follows that

$$\|u - l\|_{L^\infty(B_\rho)} \leq \|u - v\|_{L^\infty(B_\rho)} + \|v - l\|_{L^\infty(B_\rho)} \leq C(\tau^\beta + \delta) + C_0 \rho^{1+\bar{\alpha}-\epsilon}.$$

First, choose ρ small enough such that $C_0 \rho^{1+\bar{\alpha}-\epsilon} \leq \rho^{1+\alpha}/3$. Then choose τ and δ such that $C\tau^\beta \leq \rho^{1+\alpha}/3$ and $C\delta \leq \rho^{1+\alpha}/3$. $\square$

Lemma 4.6.3. *Given $0 < \alpha < \bar{\alpha}$, there exist constants $C_0 > 0$, $0 < \rho < 1/2$, $0 < \delta < \rho$, depending only on n, λ, Λ, and α such that for any viscosity solution $u \in C^0(B_1)$ of*

$$\begin{cases} F^\pm(D^2 u^\pm) = f^\pm & \text{in } \Omega^\pm \\ u^+_\nu - u^-_\nu = g & \text{on } \Gamma, \end{cases}$$

with $\|u\|_{L^\infty(B_1)} \leq 1$, $\|g - 1\|_{L^\infty(\Gamma)} + C_{f^-} + C_{f^+} \leq \delta$, and $\|\psi\|_{C^{1,\alpha}(\overline{B'_1})} \leq \delta$, there exist affine functions $l^\pm(x) = A^\pm \cdot x + b$, with $|A^-| + |A^+| + |b| \leq C_0$, such that

$$\|u^\pm - l^\pm\|_{L^\infty(\Omega^\pm_\rho)} \leq \rho^{1+\alpha}.$$

Moreover, $\nabla' l^- = \nabla' l^+$ and $l^+_{x_n} - l^-_{x_n} = 1$.

Proof. Fix $0 < \varepsilon, \delta, \rho < 1/2$ to be chosen. Let v be the function given in Lemma 4.5.6. Define $l^\pm_v(x) = \nabla v^\pm(0) \cdot x + v^\pm(0)$, where $v^\pm = v|_{\overline{\Omega^\pm_{3/4}}}$. We proved that $v^\pm \in C^{1,\bar{\alpha}}(\overline{\Omega^\pm_{1/2}})$, with $\|v^\pm\|_{C^{1,\bar{\alpha}}(\overline{\Omega^\pm_{1/2}})} \leq C_0$, for some $C_0 > 0$ depending only on n, λ, Λ and α. In particular, we have that $|\nabla v^\pm(0)| + |v^\pm(0)| \leq C_0$.

We define the affine functions $l^{\pm}$ as $l_v^{\pm}$ plus a small correction, that is,

$$l^{\pm}(x) = l_v^{\pm}(x) + l_\varepsilon^{\pm}(x),$$

with $l_\varepsilon^{\pm}(x) = A_\varepsilon^{\pm} \cdot x + b_\varepsilon^{\pm}$ such that

$$\begin{aligned} b_\varepsilon^{+} &= -b_\varepsilon^{-} = \tfrac{1}{2}(v^-(0) - v^+(0)), \\ (A_\varepsilon^{+})' &= -(A_\varepsilon^{-})' = \tfrac{1}{2}(\nabla' v^-(0) - \nabla' v^+(0)), \\ (A_\varepsilon^{+})_n &= -(A_\varepsilon^{-})_n = \tfrac{1}{2}(1 - v_{x_n}^{+}(0) + v_{x_n}^{-}(0)). \end{aligned}$$

By Corollary 4.5.5, we have that $|b_\varepsilon^{\pm}| + |A_\varepsilon^{\pm}| \leq \varepsilon$. Moreover, by definition of $l^{\pm}$, it holds that

$$l^-(0) = l^+(0), \quad \nabla' l^- = \nabla' l^+, \quad \text{and} \quad l_{x_n}^{+} - l_{x_n}^{-} = 1.$$

For any $x \in \Omega_\rho^{\pm}$, by Lemma 4.5.6, and the $C^{1,\bar{\alpha}}$-estimate for $v^{\pm}$, we have that

$$\begin{aligned} |u^{\pm}(x) - l^{\pm}(x)| &= |u^{\pm}(x) - l_v^{\pm}(x) - l_\varepsilon^{\pm}(x)| \\ &\leq |u^{\pm}(x) - v^{\pm}(x)| + |v^{\pm}(x) - l_v^{\pm}(x)| + |l_\varepsilon^{\pm}(x)| \\ &\leq C\varepsilon^{1/2} + C_0\rho^{1+\bar{\alpha}} + \varepsilon\rho + \varepsilon. \end{aligned}$$

First, choose $0 < \rho < 1/2$ such that $C_0\rho^{1+\bar{\alpha}} \leq \rho^{1+\alpha}/2$. This is possible since $0 < \alpha < \bar{\alpha} < 1$. Then choose $0 < \varepsilon < \rho$ such that $C\varepsilon^{1/2} + \varepsilon\rho + \varepsilon \leq \rho^{1+\alpha}/2$. Finally, recall that $0 < \delta < \varepsilon$ is given as in Corollary 4.5.5. Therefore,

$$\|u^{\pm} - l^{\pm}\|_{L^\infty(\Omega_\rho^{\pm})} \leq \rho^{1+\alpha}.$$

$\square$

4.6.1 Proof of Theorem 4.6.1

Fix $0 < \alpha < \bar{\alpha}$. Let $C_0, \rho, \delta > 0$ be the minimum of the constants given in Lemma 4.6.2 and Lemma 4.6.3. Let $\delta_0 > 0$ to be chosen sufficiently small. First, we normalize the problem. Recall that we are assuming that $0 \in \Gamma$, that is, $\psi(0') = 0$.

(i) After a rotation, we can assume that $\nu(0) = e_n$. In particular, $\nabla'\psi(0') = 0'$. Also, we can suppose that $[\psi]_{C^{1,\alpha}(0)} \leq \delta_0$. Recall that

$$[\psi]_{C^{1,\alpha}(0)} = \sup_{x' \in B_1', \, x' \neq 0'} \frac{|\nabla'\psi(x')|}{|x'|^\alpha}.$$

Indeed, let $K^\alpha = [\psi]_{C^{1,\alpha}(0)}/\delta_0$, and consider $v(y) = u(y/K)$, for $y \in B_1$. Then v satisfies

$$\begin{cases} F_K(D^2 v^\pm) = f_K^\pm & \text{in } \tilde{\Omega}^\pm \\ v_\nu^+ - v_\nu^- = g_K & \text{on } \tilde{\Gamma}, \end{cases}$$

where $F_K(M) = K^{-2}F(K^2 M)$, for $M \in \mathcal{S}^n$, $\tilde{\Omega}^\pm = \{y \in B_1 : y/K \in \Omega^\pm\}$, $\tilde{\Gamma} = \{y \in B_1 : y/K \in \Gamma\}$, $f_K^\pm(y) = K^{-2} f^\pm(y/K)$, for $y \in \tilde{\Omega}^\pm$, and $g_K(y) = K^{-1} g(y/K)$, for $y \in \tilde{\Gamma}$. In particular, it holds that $F_K \in \mathcal{E}_{\lambda,\Lambda}$, that is, F_K is a fully nonlinear operator with the same ellipticity constants as F. Also, $f_K^\pm$ satisfy

$$\begin{aligned}
\Big(\fint_{B_r \cap \tilde{\Omega}^\pm} |f_K^\pm(y)|^n \, dy \Big)^{1/n} &= \Big(\fint_{B_r \cap \tilde{\Omega}^\pm} |K^{-2} f^\pm(y/K)|^n \, dy \Big)^{1/n} \\
&= \Big(\fint_{B_{r/K} \cap \Omega^\pm} |K^{-2} f^\pm(x)|^n \, K^n dx \Big)^{1/n} \\
&= \frac{1}{K^2} \Big(\fint_{B_{r/K} \cap \Omega^\pm} |f^\pm(x)|^n \, dx \Big)^{1/n} \\
&\leq K^{-2} C_{f^\pm} (r/K)^{\alpha-1} = K^{-(1+\alpha)} C_{f^\pm} r^{\alpha-1}.
\end{aligned}$$

Hence, $C_{f_K^\pm} = K^{-(1+\alpha)} C_{f^\pm}$. Moreover, g_K satisfies

$$
\begin{aligned}
[g_K]_{C^{0,\alpha}(0)} &= \sup_{y \in B_1,\, y \neq 0} \frac{|g_K(y) - g_K(0)|}{|y|^\alpha} = K^{-1} \sup_{y \in B_1,\, y \neq 0} \frac{|g(y/K) - g(0)|}{|y|^\alpha} \\
&\leq K^{-(1+\alpha)} [g]_{C^{1,\alpha}(0)}.
\end{aligned}
$$

If $y \in \tilde{\Gamma}$, then $y_n = \tilde{\psi}(y')$, with $\tilde{\psi}(y') = K\psi(y'/K)$. Moreover,

$$
\begin{aligned}
[\tilde{\psi}]_{C^{1,\alpha}(0)} &= \sup_{y' \in B_1',\, y' \neq 0'} \frac{|\nabla' \tilde{\psi}(y')|}{|y'|^\alpha} = \sup_{y' \in B_1',\, y' \neq 0'} \frac{|\nabla' \psi(y'/K)|}{|y'|^\alpha} \\
&\leq K^{-\alpha} [\psi]_{C^{1,\alpha}(0)} = \delta_0.
\end{aligned}
$$

If we show that there exist affine functions $l_K^\pm(y) = A_K^\pm \cdot y + b_K$ such that

$$
|v^\pm(y) - l_K^\pm(y)| \leq C_K |y|^{1+\alpha} \qquad \text{for all } y \in \tilde{\Omega}_{1/2},
$$

and there exists $C_0 > 0$ depending only on n, λ, Λ, and α, such that

$$
|A_K^-| + |A_K^+| + |b_K| + |C_K| \leq C_0 \big(|g_K(0)| + [g_K]_{C^{0,\alpha}(0)} + C_{f_K^-} + C_{f_K^+} \big)
$$

then rescaling back, we get that

$$
|u^\pm(x) - l^\pm(x)| \leq C|x|^{1+\alpha} \qquad \text{for all } x \in \Omega_{(2K)^{-1}},
$$

with $l^\pm(x) = A^\pm \cdot x + b$, $A^\pm = K A_K^\pm$, $b = b_K$, $C = K^{1+\alpha} C_K$, and

$$
\begin{aligned}
&K^{-1}|A^-| + K^{-1}|A^+| + |b| + K^{-(1+\alpha)}|C| \\
&\qquad \leq C_0 \big(K^{-1}|g(0)| + K^{-(1+\alpha)}[g]_{C^{0,\alpha}(0)} + K^{-(1+\alpha)} C_{f^-} + K^{-(1+\alpha)} C_{f^+} \big).
\end{aligned}
$$

Multiplying by $K^{1+\alpha}$, and using that $K^\alpha = \delta_0^{-1}[\psi]_{C^{1,\alpha}(0)} \geq 1$, we get

$$
|A^-| + |A^+| + |b| + |C| \leq C_0 \delta_0^{-1} [\psi]_{C^{1,\alpha}(0)} \big(|g(0)| + [g]_{C^{0,\alpha}(0)} + C_{f^-} + C_{f^+} \big).
$$

(*ii*) Assume that $\|u\|_{L^\infty(B_1)} \leq 1$, $C_{f^-} + C_{f^+} \leq \delta_0/2$, and

$$[g]_{C^{0,\alpha}(0)} = \sup_{x \in \Gamma \cap B_1,\, x \neq 0} \frac{|g(x) - g(0)|}{|x|^\alpha} \leq \delta_0/2.$$

Indeed, let $K = \|u\|_{L^\infty(B_1)} + \delta_0^{-1}([g]_{C^{0,\alpha}(0)} + C_{f^-} + C_{f^+})$, and consider $v = u/K$. Then v satisfies

$$\begin{cases} F_K(D^2 v^\pm) = f_K^\pm & \text{in } \Omega^\pm \\ v_\nu^+ - v_\nu^- = g_K & \text{on } \Gamma, \end{cases}$$

where $F_K(M) = K^{-1}F(KM)$, for $M \in \mathcal{S}^n$, $f_K^\pm = K^{-1}f^\pm$, and $g_K = K^{-1}g$. Moreover, $\|v\|_{L^\infty(B_1)} \leq 1$, and $[g_K]_{C^{0,\alpha}(0)} + C_{f_K^-} + C_{f_K^+} \leq \delta_0$.

(*iii*) If $g(0) \neq 0$, we can suppose that $g(0) = 1$. Indeed, we consider $v = u/g(0)$, and argue similarly as in (*ii*). The case $g(0) = 0$ will be addressed at the end.

For simplicity, we use the same notation as in the statement, that is, ψ, u, F, $f^\pm$ and g.

Under these assumptions, it is enough to prove the following:

Claim. *For all $k \geq 1$, there exist affine functions $l_k^\pm(x) = A_k^\pm \cdot x + b_k$ with*

$$\rho^{k-1}|A_k^- - A_{k-1}^-| + \rho^{k-1}|A_k^+ - A_{k-1}^+| + |b_k - b_{k-1}| \leq C_0 \rho^{(k-1)(1+\alpha)},$$

where $A_0^\pm = 0$, $b_0 = 0$, $C_0 > 0$ depends only on n, λ, Λ, and α, and such that

$$\|u^\pm - l_k^\pm\|_{L^\infty(\Omega_{\rho^k}^\pm)} \leq \rho^{k(1+\alpha)}.$$

Moreover, $\nabla' l_k^- = \nabla' l_k^+$, and $(l_k^+)_{x_n} - (l_k^-)_{x_n} = 1$.

We prove the claim by induction. For $k = 1$, by the normalization, we are under the assumptions of Lemma 4.6.3. Indeed, by (i), we have that

$$\begin{aligned}\|\psi\|_{C^{1,\alpha}(\overline{B_1'})} &= \|\psi - \psi(0')\|_{L^\infty(B_1')} + \|\nabla\psi - \nabla'\psi(0')\|_{L^\infty(B_1')} + [\nabla\psi]_{C^{0,\alpha}(\overline{B_1'})}\\ &\leq 3[\psi]_{C^{1,\alpha}(0)} \leq 3\delta_0 \leq \delta.\end{aligned}$$

Moreover, by (ii) and (iii), it follows that

$$\|g - 1\|_{L^\infty(\Gamma)} = \|g - g(0)\|_{L^\infty(\Gamma)} \leq [g]_{C^{0,\alpha}(0)} \leq \delta_0 \leq \delta.$$

Hence, by Lemma 4.6.3, there exist $l_1^\pm(x) = A_1^\pm \cdot x + b_1$, with $|A_1^-| + |A_1^+| + |b_1| \leq C_0$ such that

$$\|u^\pm - l_1^\pm\|_{L^\infty(\Omega_\rho^\pm)} \leq \rho^{1+\alpha}.$$

Moreover, $\nabla' l_1^- = \nabla' l_1^+$, and $(l_1^+)_{x_n} - (l_1^-)_{x_n} = 1$.

For the induction step, assume that the claim holds for some $k \geq 1$, and let $l_k^\pm$ be such affine functions. Denote by

$$\begin{aligned}\tilde{\Omega}_k^\pm &= \{x \in B_1 : \rho^k x \in \Omega^\pm\},\\ \tilde{\Gamma}_k &= \{x \in B_1 : \rho^k x \in \Gamma\}.\end{aligned}$$

Note that if ψ_k is a parametrization of $\tilde{\Gamma}_k$ in B_1', then $\psi_k(x') = \rho^{-k}\psi(\rho^k x')$. In particular, $\nabla'\psi_k(x') = \nabla'\psi(\rho^k x)$, and thus, for $x \in \tilde{\Gamma}_k$, if $\nu_k(x)$ is the normal vector on x pointing at $\tilde{\Omega}_k^+$, then $\nu_k(x) = \nu(\rho^k x)$. Define $l_k = l_k^+ \chi_{\tilde{\Omega}_k^+} + l_k^- \chi_{\tilde{\Omega}_k^-}$. Consider the rescaled function

$$v(x) = \frac{u(\rho^k x) - l_k(\rho^k x)}{\rho^{k(1+\alpha)}} \qquad \text{for } x \in \overline{B_1}.$$

Then v satisfies

$$\begin{cases} F_k^{\pm}(D^2 v^{\pm}) = f_k^{\pm} & \text{in } \tilde{\Omega}_k^{\pm} \\ v_{\nu_k}^{+} - v_{\nu_k}^{-} = g_k & \text{on } \tilde{\Gamma}_k, \end{cases} \tag{4.6.1}$$

in the viscosity sense, where

$$\begin{aligned} F_k^{\pm}(M) &= \rho^{k(1-\alpha)} F^{\pm}(\rho^{k(\alpha-1)} M), \quad \text{for } M \in \mathcal{S}^n \\ f_k^{\pm}(x) &= \rho^{k(1-\alpha)} f^{\pm}(\rho^k x), \quad \text{for } x \in \tilde{\Omega}_k^{\pm} \\ g_k(x) &= \rho^{-k\alpha}(g(\rho^k x) - \nu_n(\rho^k x)), \quad \text{for } x \in \tilde{\Gamma}_k. \end{aligned}$$

By the induction hypothesis, $\|v\|_{L^\infty(B_1)} \le 1$. Notice that

$$\begin{aligned} \Big(\fint_{B_r \cap \tilde{\Omega}_k^{\pm}} |f_k^{\pm}(y)|^n \, dy \Big)^{1/n} &= \Big(\fint_{B_r \cap \tilde{\Omega}_k^{\pm}} \rho^{nk(1-\alpha)} |f^{\pm}(\rho^k y)|^n \, dy \Big)^{1/n} \\ &= \rho^{k(1-\alpha)} \Big(\fint_{B_{r\rho^k} \cap \Omega_k^{\pm}} |f^{\pm}(x)|^n \, dx \Big)^{1/n} \\ &\le \rho^{k(1-\alpha)} C_{f^{\pm}} (r\rho^k)^{\alpha-1} = C_{f^{\pm}} r^{\alpha-1}. \end{aligned} \tag{4.6.2}$$

Hence, $C_{f_k^{\pm}} = C_{f^{\pm}}$, and $C_{f_k^+} + C_{f_k^-} \le \delta_0$. Moreover,

$$\|g_k\|_{L^\infty(\tilde{\Gamma}_k)} \le [g]_{C^{0,\alpha}(0)} + [\nu_n]_{C^{0,\alpha}(0)} \le \delta_0 + \delta_0 = 2\delta_0. \tag{4.6.3}$$

However, we cannot apply Lemma 4.6.2 to v since it has a jump discontinuity on $\tilde{\Gamma}_k$. In fact, if $v^{\pm} = v\big|_{\overline{\tilde{\Omega}_k^{\pm}}}$, then for $x \in \tilde{\Gamma}_k$, by the normalization (i), and the induction hypothesis, we have

$$\begin{aligned} |(v^- - v^+)(x)| = \frac{|l_k^-(\rho^k x) - l_k^+(\rho^k x)|}{\rho^{k(1+\alpha)}} &= \rho^{-k\alpha}|x_n| \le \rho^{-k\alpha} \sup_{x \in \tilde{\Gamma}_k} |x_n| \\ &\le \sup_{x' \in B_1'} \frac{|\psi_k(x')|}{\rho^{k\alpha}} \le [\psi]_{C^{1,\alpha}(0)} \le \delta_0. \end{aligned} \tag{4.6.4}$$

Let $w \in C^0(B_1)$, with $w^{\pm} = w|_{\overline{\tilde{\Omega}_k^{\pm}}}$, be the viscosity solutions of the following Dirichlet problems:

$$\begin{cases} F_k^{\pm}(D^2 w^{\pm}) = 0 & \text{in } \tilde{\Omega}_k^{\pm} \\ w = \frac{1}{2}(v^+ + v^-) & \text{on } \tilde{\Gamma}_k \\ w = v & \text{on } \partial B_1. \end{cases}$$

We will prove that w satisfies the assumptions of Lemma 4.5.1. By the maximum principle, $\|w\|_{L^\infty(B_1)} \le \|v\|_{L^\infty(\partial B_1)} \le 1$. Moreover, $v^{\pm} - w^{\pm} \in S(f_k^{\pm})$ in $\tilde{\Omega}_k^{\pm}$, $v^{\pm} - w^{\pm} = \pm\frac{1}{2}\rho^{-k\alpha} x_n$ on $\tilde{\Gamma}_k$, and $v^{\pm} - w^{\pm} = 0$ on $\partial\tilde{\Omega}_k^{\pm} \setminus \tilde{\Gamma}_k$. Then by the classical ABP, and (4.6.4), we see that

$$\|v^{\pm} - w^{\pm}\|_{L^\infty(\tilde{\Omega}_k^{\pm})} \le \|v^{\pm} - w^{\pm}\|_{L^\infty(\tilde{\Gamma}_k^{\pm})} + C\|f_k^{\pm}\|_{L^n(\tilde{\Omega}_k^{\pm})} \le C\delta_0. \tag{4.6.5}$$

since $\|f_k^{\pm}\|_{L^n(\tilde{\Omega}_k^{\pm})} \le |B_1|^{1/n} C_{f^{\pm}} \le C(n)\delta_0$ by (4.6.2) with $r = 1$. By boundary pointwise $C^{1,\alpha}$ estimates (see [38, Theorem 1.6]), for any $x_0 \in \tilde{\Gamma}_k \cap B_{3/4}$, we have

$$\begin{aligned} |\nabla(v^{\pm} - w^{\pm})(x_0)| &\le C\big(\|v^{\pm} - w^{\pm}\|_{L^\infty(\tilde{\Omega}_k^{\pm})} + \tfrac{1}{2}\rho^{-k\alpha}\|\psi_k\|_{C^{1,\alpha}(x_0)} + C_{f_k^{\pm}}\big) \\ &\le \tilde{C}\delta_0, \end{aligned} \tag{4.6.6}$$

where the last inequality follows from (4.6.2), (4.6.5), and the normalization (i). Indeed:

$$\begin{aligned} \rho^{-k\alpha}\|\psi_k\|_{L^\infty(B_1')} &= \sup_{x' \in B_1'} \frac{|\psi_k(x')|}{\rho^{k\alpha}} = \sup_{x' \in B_1'} \frac{|\psi(\rho^k x')|}{\rho^{k(1+\alpha)}} \le [\psi]_{C^{1,\alpha}(0)} \le \delta_0, \\ \rho^{-k\alpha}\|\nabla'\psi_k\|_{L^\infty(B_1')} &= \sup_{x' \in B_1'} \frac{|\nabla'\psi_k(x')|}{\rho^{k\alpha}} = \sup_{x' \in B_1'} \frac{|\nabla'\psi(\rho^k x')|}{\rho^{k\alpha}} \le [\psi]_{C^{1,\alpha}(0)} \le \delta_0, \\ \rho^{-k\alpha}[\nabla'\psi_k]_{C^{0,\alpha}(\overline{B_1'})} &= \sup_{x',y' \in B_1', x' \ne y'} \frac{|\nabla'\psi(\rho^k x') - \nabla'\psi(\rho^k y')|}{\rho^{k\alpha}|x'-y'|^\alpha} \le [\psi]_{C^{1,\alpha}(\overline{B_1'})} \le \delta_0. \end{aligned}$$

Hence, $\rho^{-k\alpha}\|\psi_k\|_{C^{1,\alpha}(x_0)} \leq \rho^{-k\alpha}\|\psi_k\|_{C^{1,\alpha}(\overline{B_1'})} \leq 3\delta_0$.

Let $x_0 \in \tilde{\Gamma}_k \cap B_{3/4}$. Suppose there exists a test function φ touching w by above at x_0 in a small neighborhood of x_0 contained in $B_{3/4}$. In particular, $\phi = \varphi - (w - v)$ is a test function that touches v by above at x_0. Therefore,

$$\phi^+_{\nu_k}(x_0) - \phi^-_{\nu_k}(x_0) \geq g_k(x_0).$$

It follows that:

$$\varphi^+_{\nu_k}(x_0) - \varphi^-_{\nu_k}(x_0) \geq g_k(x_0) + (w^+ - v^+)_{\nu_k}(x_0) - (w^- - v^-)_{\nu_k}(x_0) \equiv \tilde{g}_k(x_0).$$

Moreover, by (4.6.3) and (4.6.6), we get

$$\|\tilde{g}_k\|_{L^\infty(\tilde{\Gamma}_k \cap B_{3/4})} \leq 2\delta_0 + 2\tilde{C}\delta_0 \leq \delta.$$

Similarly, if φ is a test function touching w from below at x_0, in a small neighborhood of x_0 contained in $B_{3/4}$, then

$$\varphi^+_{\nu_k}(x_0) - \varphi^-_{\nu_k}(x_0) \leq \tilde{g}_k(x_0).$$

Hence, $w^+_{\nu_k} - w^-_{\nu_k} = \tilde{g}_k$ on $\tilde{\Gamma}_k \cap B_{3/4}$ in the viscosity sense. Applying Lemma 4.6.2 to w, we see that there exist $C_0 > 0$ depending only on n, λ, Λ, and α, and an affine function $l(x) = A \cdot x + b$, with $|A| + |b| \leq C_0$, such that

$$\|w - l\|_{L^\infty(B_\rho)} \leq \rho^{1+\alpha}/2. \tag{4.6.7}$$

Note that we can always choose ρ sufficiently small such that the previous estimate holds (see proof of Lemma 4.6.2). Hence, by (4.6.5) and (4.6.7), we see that

$$\|v - l\|_{L^\infty(B_\rho)} \leq \|v - w\|_{L^\infty(B_\rho)} + \|w - l\|_{L^\infty(B_\rho)} \leq \tilde{C}\delta_0 + \rho^{1+\alpha}/2 \leq \rho^{1+\alpha}.$$

In particular, for any $x \in B_\rho$, we have

$$\left| \frac{u(\rho^k x) - l_k(\rho^k x)}{\rho^{k(1+\alpha)}} - l(x) \right| \le \rho^{1+\alpha},$$

or equivalently, if $y = \rho^k x$, then for any $y \in B_{\rho^{k+1}}$,

$$|u(y) - l_k(y) - \rho^{k(1+\alpha)} l(\rho^{-k} y)| \le \rho^{(k+1)(1+\alpha)}. \tag{4.6.8}$$

Define the affine approximations at the step $k+1$ as

$$l_{k+1}^\pm(y) = l_k^\pm(y) + \rho^{k(1+\alpha)} l(\rho^{-k} y).$$

If $l_{k+1}^\pm(y) = A_{k+1}^\pm \cdot y + b_{k+1}$, then

$$A_{k+1}^\pm = A_k^\pm + \rho^{k\alpha} A, \quad b_{k+1} = b_k + \rho^{k(1+\alpha)} b.$$

Using the estimate $|A| + |b| \le C_0$, we have

$$\rho^k |A_{k+1}^\pm - A_k^\pm| + |b_{k+1} - b_k| \le C_0 \rho^{k(1+\alpha)}.$$

From (4.6.8), we see that

$$\|u^\pm - l_{k+1}^\pm\|_{L^\infty(\Omega^\pm_{\rho^{k+1}})} \le \rho^{(k+1)(1+\alpha)}.$$

Moreover, by the induction hypothesis,

$$\nabla' l_{k+1}^- - \nabla' l_{k+1}^+ = \nabla' l_k^- - \nabla' l_k^+ = 0,$$
$$(l_{k+1}^+)_{x_n} - (l_{k+1}^-)_{x_n} = (l_k^+)_{x_n} - (l_k^-)_{x_n} = 1.$$

The proof of the claim is completed.

Finally, we consider the case $g(0) = 0$. As before, it is enough to prove the following:

Claim. *For all $k \geq 1$, there exist affine functions $l_k = A_k \cdot x + b_k$ such that*

$$\rho^k |A_k - A_{k-1}| + |b_k - b_{k-1}| \leq C_0 \rho^{(k-1)(1+\alpha)},$$

where $A_0 = 0$, $b_0 = 0$, $C_0 > 0$ depends only on n, λ, Λ, and α, and such that

$$\|u - l_k\|_{L^\infty(B_{\rho^k})} \leq \rho^{k(1+\alpha)}.$$

The proof is by induction. For $k = 1$, we can apply Lemma 4.6.2 to u. Indeed, $\|u\|_{L^\infty(B_1)} \leq 1$, and $\|g\|_{L^\infty(\Gamma)} + C_{f^-} + C_{f^+} \leq \delta$, given that

$$\|g\|_{L^\infty(\Gamma)} = \sup_{x \in \Gamma} |g(x) - g(0)| \leq [g]_{C^{0,\alpha}(0)} \leq \delta_0 \leq \delta.$$

Then we find an affine function $l_1(x) = A_1 \cdot x + b_1$, with $|A_1| + |b_1| \leq C_0$, such that

$$\|u - l_1\|_{L^\infty(B_\rho)} \leq \rho^{1+\alpha}.$$

Assume the claim holds for $k \geq 1$. Define

$$v(x) = \frac{u(\rho^k x) - l_k(\rho^k x)}{\rho^{k(1+\alpha)}} \qquad \text{for } x \in \overline{B_1}.$$

Then, arguing as before, we have that $v \in C^0(B_1)$ satisfies (4.6.1), with the same operator F_k, and the same right-hand sides $f_k^\pm$, but with different g_k:

$$g_k(x) = \rho^{-k\alpha} g(\rho^k x) \quad \text{for } x \in \tilde{\Gamma}_k.$$

In particular, for any $x \in \tilde{\Gamma}_k$, we have

$$|g_k(x)| = \rho^{-k\alpha}|g(\rho^k x)| \leq [g]_{C^{0,\alpha}(0)} \leq \delta_0 \leq \delta.$$

Then the claim follows for $k+1$ by applying again Lemma 4.6.2. $\square$

Remark 4.6.4. The $C^{1,\alpha}$ regularity estimate of u^+ and u^- up to the interface (Theorem 4.1.3) follows from a standard argument by patching the classical interior estimates ([15, Theorem 8.3]) and the boundary estimates (Theorem 4.6.1). For instance, see [17, Proposition 6.2].

Chapter 5

A new family of integro-differential operators related to the Monge-Ampère equation

5.1 Introduction

Integro-differential equations arise in the study of stochastic processes with jumps, such as Lévy processes. As we discussed in the introduction, a classical elliptic integro-differential operator is the fractional Laplacian,

$$\Delta^s u(x_0) = c_{n,s}\,\mathrm{PV}\int_{\mathbb{R}^n} (u(x_0+x) - u(x_0))\frac{1}{|x|^{n+2s}}dx, \qquad s \in (0,1),$$

which can be understood as an infinitesimal generator of a stable Lévy process. These types of processes are very well studied in probability, and their generators may be given by

$$L_K u(x_0) = \int_{\mathbb{R}^n} (u(x_0+x) - u(x_0) - x\cdot\nabla u(x_0))K(x)dx,$$

where the kernel K is a nonnegative function satisfying some integrability condition.

Over the last few years, there has been significant interest in studying linear and nonlinear integro-differential equations from the analytical point of view. In particular, extremal operators like

$$Fu(x_0) = \inf_{K\in\mathcal{K}} L_K u(x_0) \tag{5.1.1}$$

play a fundamental role in the regularity theory. See [10–12, 52] and the references therein. The above equation is an example of a fully nonlinear equation that appears in optimal control problems and stochastic games [32, 46]. The infimum in (5.1.1) is taken over a family of admissible kernels $\mathcal{K}$ that depends on the applications. In fact, as we discussed in Section 1.2, nonlocal Monge-Ampère equations have been developed recently in the form (5.1.1), for some choice of $\mathcal{K}$ [8, 13, 27].

For the purpose of this chapter, we recall the definition of the nonlocal Monge-Ampère operator given by L. Caffarelli and L. Silvestre [13]:

$$\mathrm{MA}^s\, u(x_0) = c_{n,s} \inf_{K \in \mathcal{K}_n^s} \int_{\mathbb{R}^n} (u(x_0 + x) - u(x_0) - x \cdot \nabla u(x_0)) K(x)\, dx,$$

where the infimum is taken over the family,

$$\mathcal{K}_n^s = \Big\{ K : \mathbb{R}^n \to \mathbb{R}_+ : |\{x \in \mathbb{R}^n : K(x) > r^{-n-2s}\}| = |B_r|,\ \forall\, r > 0 \Big\}. \quad (5.1.2)$$

In this work, we introduce a new family of operators of the form,

$$\inf_{K \in \mathcal{K}_k^s} \int_{\mathbb{R}^n} (u(x_0 + x) - u(x_0) - x \cdot \nabla u(x_0)) K(x)\, dx, \quad (5.1.3)$$

for any integer $1 \leq k < n$, which arises from imposing certain geometric conditions on the kernels. Moreover, we will see that $|y|^{-n-2s} \in \mathcal{K}_1^s \subset \mathcal{K}_k^s \subset \mathcal{K}_n^s$, for any $1 < k < n$, and thus, this family will be monotone decreasing, and bounded from above by the fractional Laplacian and by below by the Caffarelli–Silvestre nonlocal Monge-Ampère.

This chapter is organized as follows. In Section 5.2, we construct the family of admissible kernels $\mathcal{K}_k^s$, and give the precise definition of our operators for $C^{1,1}$ functions. We introduce in Section 5.3 the basic tools from the theory of rearrangements necessary for our goals. In Section 5.4, we study the infimum in (5.1.3) and obtain a representation formula, provided some condition on the level sets is satisfied (see Theorem 5.4.1). We also study the limit as $s \to 1$ and give a connection to optimal transport. The Hölder continuity of $\mathcal{F}_k^s u$ is proved in Section 5.5, following similar geometric techniques from [13]. In Section 5.6, we consider a global Poisson problem, prescribing data at infinity, and introduce a new definition of our operators for functions that are merely continuous and convex. We show existence of solutions via Perron's method and $C^{1,1}$ regularity in the full space by constructing appropriate barriers. Finally, we discuss some future directions in Section 5.7.

5.2 Construction of kernels

Let us start with the construction of the family of admissible kernels. Notice that any kernel K in $\mathcal{K}_n^s$, defined in (5.1.2), will have the same distribution function as the kernel of the fractional Laplacian, since for any $r > 0$,

$$\left\{x \in \mathbb{R}^n : |x|^{-n-2s} > r^{-n-2s}\right\} = B_r.$$

Geometrically, this means that the level sets of K are deformations in *any* direction of $\mathbb{R}^n$ of the level sets of $|x|^{-n-2s}$, preserving the n-dimensional volume.

In view of this approach, a natural way of finding an intermediate family of operators between the nonlocal Monge-Ampère and the fractional Laplacian is to consider kernels whose level sets are deformations that preserve the k-

dimensional Hausdorff measure $\mathcal{H}^k$, with $1 \leq k < n$, of the restrictions of balls in $\mathbb{R}^n$ to hyperplanes generated by $\{e_i\}_{i=1}^k$.

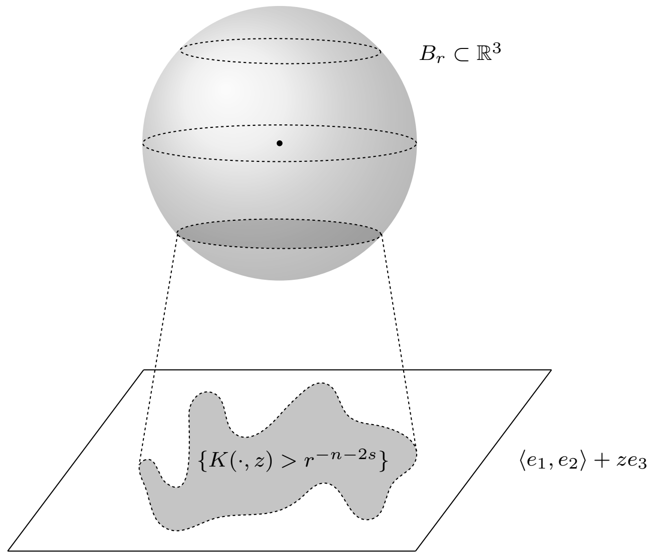

Figure 5.1: Area preserving deformation in $\mathbb{R}^3$.

We define the set of admissible kernels as follows.

Definition 5.2.1. We say that $K \in \mathcal{K}_k^s$ if for all $z \in \mathbb{R}^{n-k}$, and all $r > 0$, it holds that

$$\mathcal{H}^k\big(\{y \in \mathbb{R}^k : K(y,z) > r^{-n-2s}\}\big) = \begin{cases} \mathcal{H}^k\big(B_{(r^2-|z|^2)^{1/2}}\big) & \text{if } |z| < r \\ 0 & \text{if } |z| \geq r, \end{cases} \tag{5.2.1}$$

where $B_{(r^2-|z|^2)^{1/2}}$ is the ball in $\mathbb{R}^k$ of radius $(r^2 - |z|^2)^{1/2}$.

In Figure 5.1 we illustrate condition (5.2.1) for $k = 2$ and $n = 3$. Note that for $k = n$, we recover the definition of $\mathcal{K}_n^s$. Moreover, $|x|^{-n-2s} \in \mathcal{K}_k^s$, for all $1 \leq k \leq n$.

Proposition 5.2.2. *Let $1 \le k < n$. Then $\mathcal{K}^s_k \subset \mathcal{K}^s_{k+1} \subseteq \mathcal{K}^s_n$.*

Proof. Let $K \in \mathcal{K}^s_k$. Fix any $z \in \mathbb{R}^{n-k-1}$ and $r > 0$. Then:

$$\begin{aligned}
&\mathcal{H}^{k+1}\big(\{y \in \mathbb{R}^{k+1} : K(y,z) > r^{-n-2s}\}\big) \\
&\qquad = \int_{\mathbb{R}^{k+1}} \chi_{\{y\in\mathbb{R}^{k+1}: K(y,z)>r^{-n-2s}\}}(y)\,dy \\
&\qquad = \int_{\mathbb{R}} \Big(\int_{\mathbb{R}^k} \chi_{\{(w,t)\in\mathbb{R}^k\times\mathbb{R}\,:\,K(w,t,z)>r^{-n-2s}\}}(w,t)\,dw\Big) dt \\
&\qquad = \int_{\mathbb{R}} \mathcal{H}^k\big(\{w \in \mathbb{R}^k : K(w,t,z) > r^{-n-2s}\}\big)\,dt \equiv \mathrm{I}.
\end{aligned}$$

If $|z| \ge r$, then for any $t \in \mathbb{R}$, we have that $(t,z) \in \mathbb{R}^{n-k}$, with $|(t,z)| > r$. Therefore, by Definition 5.2.1, it follows that $\mathrm{I} = 0$. If $|z| < r$, then

$$\begin{aligned}
\mathrm{I} &= \int_{\mathbb{R}} \mathcal{H}^k\big(B_{(r^2-t^2-|z|^2)^{1/2}}\big)\,dt \\
&= \omega_k \int_{-(r^2-|z|^2)^{1/2}}^{(r^2-|z|^2)^{1/2}} (r^2-t^2-|z|^2)^{\frac{k}{2}}\,dt \\
&= \omega_k (r^2-|z|^2)^{\frac{k}{2}} \int_{-(r^2-|z|^2)^{1/2}}^{(r^2-|z|^2)^{1/2}} \Big(1 - \Big(\frac{t}{(r^2-|z|^2)^{1/2}}\Big)^2\Big)^{\frac{k}{2}} dt \\
&= \omega_k (r^2-|z|^2)^{\frac{k+1}{2}} \int_{-1}^{1} (1-\sigma^2)^{k/2}\,d\sigma \\
&= \frac{\pi^{k/2}}{\Gamma\big(\frac{k}{2}+1\big)} \frac{\pi^{1/2}\Gamma\big(\frac{k}{2}+1\big)}{\Gamma\big(\frac{k+1}{2}+1\big)} (r^2-|z|^2)^{\frac{k+1}{2}} \\
&= \omega_{k+1}(r^2-|z|^2)^{\frac{k+1}{2}} = \mathcal{H}^{k+1}\big(B_{(r^2-|z|^2)^{1/2}}\big),
\end{aligned}$$

where $\omega_l = \mathcal{H}^l(B_1) = \frac{\pi^{l/2}}{\Gamma\left(\frac{l}{2}+1\right)}$ and $B_{(r^2-|z|^2)^{1/2}}$ is the ball of radius $(r^2-|z|^2)^{1/2}$ in $\mathbb{R}^{k+1}$. □

Definition 5.2.3. A function $u : \mathbb{R}^n \to \mathbb{R}$ is said to be $C^{1,1}$ at the point x_0, and we write $u \in C^{1,1}(x_0)$, if there is a vector $p \in \mathbb{R}^n$, a radius $\rho > 0$, and a constant $C > 0$, such that

$$|u(x_0 + x) - u(x_0) - x \cdot p| \leq C|x|^2, \quad \text{for all } x \in B_\rho.$$

We denote by $[u]_{C^{1,1}(x_0)}$, the minimum constant for which this property holds, among all admissible vectors p and radii ρ.

Definition 5.2.4. Let $s \in (1/2, 1)$ and $1 \leq k < n$. For any $u \in C^0(\mathbb{R}^n) \cap C^{1,1}(x_0)$, we define

$$\mathcal{F}_k^s u(x_0) = c_{n,s} \inf_{K \in \mathcal{K}_k^s} \int_{\mathbb{R}^n} \big(u(x_0 + x) - u(x_0) - x \cdot \nabla u(x_0)\big) K(x)\, dx,$$

where $\mathcal{K}_k^s$ is the set of kernels satisfying (5.2.1) and $c_{n,s}$ is the constant in Δ^s.

As an immediate consequence of Proposition 5.2.2, we obtain that the operators are ordered.

Corollary 5.2.5. *Let $s \in (1/2, 1)$ and $1 \leq k < n$. Then for any $u \in C^0(\mathbb{R}^n) \cap C^{1,1}(x_0)$,*

$$\mathrm{MA}^s\, u(x_0) \leq \mathcal{F}_k^s u(x_0) \leq \Delta^s u(x_0).$$

Moreover, $\{\mathcal{F}_k^s\}_{k=1}^{n-1}$ is monotone decreasing.

The regularity condition on u in Definition 5.2.4 allows us to compute $\mathcal{F}_k^s u$ at the point x_0 in the classical sense. To obtain a finite number, we need to impose two extra conditions:

(P_1) An integrability condition at infinity:

$$\int_{\mathbb{R}^n} \frac{|u(x)|}{(1 + |x|)^{n+2s}}\, dx < \infty.$$

(P_2) A convexity condition at x_0:

$$\tilde{u}(x) \equiv u(x_0 + x) - u(x_0) - x \cdot \nabla u(x_0) \geq 0, \quad \text{for all } x \in \mathbb{R}^n.$$

Proposition 5.2.6. *If $u \in C^0(\mathbb{R}^n) \cap C^{1,1}(x_0)$ and satisfies (P_1) and (P_2), then*

$$0 \leq \mathcal{F}_k^s u(x_0) < \infty.$$

Proof. Let $\rho > 0$ be as in Definition 5.2.3. Then

$$\begin{aligned} 0 \leq \mathcal{F}_k^s u(x_0) &\leq \int_{\mathbb{R}^n} \big(u(x_0 + x) - u(x_0) - x \cdot \nabla u(x_0)\big) \frac{1}{|x|^{n+2s}}\, dx \\ &\leq \int_{B_\rho} \frac{[u]_{C^{1,1}(x_0)} |x|^2}{|x|^{n+2s}}\, dx + \int_{\mathbb{R}^n \setminus B_\rho(x_0)} \frac{|u(x)|}{|x - x_0|^{n+2s}}\, dx \\ &\quad + |u(x_0)| \int_{\mathbb{R}^n \setminus B_\rho} \frac{1}{|x|^{n+2s}}\, dx + |\nabla u(x_0)| \int_{\mathbb{R}^n \setminus B_\rho} \frac{|x|}{|x|^{n+2s}}\, dx \\ &\leq C(s,\rho)\big(|u(x_0)| + |\nabla u(x_0)| + [u]_{C^{1,1}(x_0)}\big) \\ &\quad + \tfrac{1+|x_0|+\rho}{\rho} \int_{\mathbb{R}^n} \frac{|u(x)|}{(1+|x|)^{n+2s}}\, dx < \infty, \quad \text{since } s \in (1/2, 1). \end{aligned}$$

□

We point out that if u is not convex at x_0, then the infimum could be $-\infty$. We show this result in the next proposition.

Proposition 5.2.7. *Let $u \in C^0(\mathbb{R}^n) \cap C^{1,1}(x_0)$. Assume that u satisfies (P_1). If there exists $\bar{x} \in \mathbb{R}^n$ with $\bar{x} = (\bar{y}, 0)$ and $\bar{y} \in \mathbb{R}^k$, such that*

$$\tilde{u}(\bar{x}) = u(x_0 + \bar{x}) - u(x_0) - \bar{x} \cdot \nabla u(x_0) < 0,$$

then $\mathcal{F}_k^s u(x_0) = -\infty$.

Proof. Let $K(x) = |x - \bar{x}|^{-n-2s}$. For any $r > 0$ and $z \in \mathbb{R}^{n-k}$, if $|z| < r$, then

$$\begin{aligned}\mathcal{H}^k\big(\{y \in \mathbb{R}^k : K(y,z) > r^{-n-2s}\}\big) &= \mathcal{H}^k\big(\{y \in \mathbb{R}^k : |y - \bar{y}|^2 + |z|^2 < r^2\}\big)\\ &= \mathcal{H}^k\big(B_{(r^2-|z|^2)^{1/2}}\big).\end{aligned}$$

Also, the measure is clearly zero if $|z| \geq r$. Therefore, $K \in \mathcal{K}_k^s$. It follows that

$$\begin{aligned}\mathcal{F}_k^s u(x_0) &\leq \int_{\mathbb{R}^n} \tilde{u}(x)|x - \bar{x}|^{-n-2s}\,dx\\ &= \int_{B_\varepsilon(\bar{x})} \tilde{u}(x)|x - \bar{x}|^{-n-2s}\,dx + \int_{\mathbb{R}^n \setminus B_\varepsilon(\bar{x})} \tilde{u}(x)|x - \bar{x}|^{-n-2s}\,dx \equiv \mathrm{I} + \mathrm{II}.\end{aligned}$$

Since $u \in C^0(\mathbb{R}^n) \cap C^{1,1}(x_0)$, we have that $\tilde{u}$ is continuous. Hence, given that $\tilde{u}(\bar{x}) < 0$, then $\tilde{u}(x) < 0$, for all $x \in B_\varepsilon(\bar{x})$, for some $\varepsilon > 0$. Moreover, since $K \notin L^1(B_\varepsilon(\bar{x}))$, it follows that $\mathrm{I} = -\infty$. Arguing similarly as in the proof of Proposition 5.2.6, we see that $\mathrm{II} < \infty$. Therefore,

$$\mathcal{F}_k^s u(x_0) = -\infty.$$

$\square$

Remark 5.2.8. The operators $\mathcal{F}_k^s$ are not rotation invariant. This is because, for simplicity, in the construction of the family of admissible kernels $\mathcal{K}_k^s$ we chose the first k vectors from the canonical basis of $\mathbb{R}^n$. In general, we may take any subset of k unitary vectors, $\tau = \{\tau_i\}_{i=1}^k$, and replace the first condition on (5.2.1) by

$$\mathcal{H}^k\big(\{y \in \langle\tau\rangle^\perp : K(y + z\tau) > r^{-n-2s}\}\big) = \mathcal{H}^k\big(B_{(r^2-|z|^2)^{1/2}}\big), \tag{5.2.2}$$

for all $z \in \langle\tau\rangle$ and $r > 0$, where $\langle\tau\rangle$ denotes the span of $\{\tau_i\}_{i=1}^k$, and $\langle\tau\rangle^\perp$ the orthogonal subspace to $\langle\tau\rangle$. Let $SO(n)$ be the group of rotation matrices

$n \times n$. Since $\tau_i = Ae_i$, for some $A \in SO(n)$, it follows that any kernel K_τ satisfying (5.2.2) can be written as $K_\tau = K \circ A$, where K satisfies (5.2.1). Therefore to make the operators rotation invariant, one possibility is to take the infimum over all possible rotations. Namely,

$$\inf_{A \in SO(n)} \inf_{K \in \mathcal{K}_k^s} \int_{\mathbb{R}^n} \tilde{u}(x) K(Ax)\, dx.$$

To focus on the main ideas, we will not explore this operator in this work.

5.3 Rearrangements and measure preserving transformations

We introduce some definitions and preliminary results regarding rearrangements of nonnegative functions. For more detailed information, see for instance [2,3].

Definition 5.3.1. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a nonnegative measurable function. We define the decreasing rearrangement of f as the function f^* defined on $[0, \infty)$ given by

$$f^*(t) = \sup \big\{ \lambda > 0 : |\{x \in \mathbb{R}^n : f(x) > \lambda\}| > t \big\},$$

and the increasing rearrangement of f as the function f_* defined on $[0, \infty)$ given by

$$f_*(t) = \inf \big\{ \lambda > 0 : |\{x \in \mathbb{R}^n : f(x) \leq \lambda\}| > t \big\}.$$

We use the convention that $\inf \emptyset = \infty$.

Proposition 5.3.2. *Let $f, g : \mathbb{R}^n \to \mathbb{R}$ be nonnegative measurable functions. Then*

$$\int_0^\infty f_*(t)g^*(t)\,dt \le \int_{\mathbb{R}^n} f(x)g(x)\,dx \le \int_0^\infty f^*(t)g^*(t)\,dt.$$

The upper bound is the classical Hardy–Littlewood inequality. For the proof see [3, Theorem 2.2] or [2, Corollary 2.16]. For the sake of completeness, we give the proof of the lower bound.

Proof. For $j \ge 1$, let $f_j = f|_{B_j}$ and $g_j = g|_{B_j}$, where B_j denotes the ball of radius j centered at 0 in $\mathbb{R}^n$. By [2, Corollary 2.18], it follows that

$$\int_0^{|B_j|} (f_j)_*(t)(g_j)^*(t)\,dt \le \int_{B_j} f_j(x)g_j(x)\,dx.$$

Since $f, g \ge 0$, we get

$$\int_{B_j} f_j(x)g_j(x)\,dx \le \int_{\mathbb{R}^n} f(x)g(x)\,dx.$$

Note that for any $t \in [0, |B_j|]$, we have

$$\big\{\lambda > 0 : |\{x \in B_j : f_j(x) \le \lambda\}| > t\big\} \subset \big\{\lambda > 0 : |\{x \in \mathbb{R}^n : f(x) \le \lambda\}| > t\big\}.$$

Hence, $(f_j)_*(t) \ge f_*(t)$, and

$$\int_0^{|B_j|} (f_j)_*(t)(g_j)^*(t)\,dt \ge \int_0^{|B_j|} f_*(t)(g_j)^*(t)\,dt.$$

Moreover, $g_j \nearrow g$ pointwise on $\mathbb{R}^n$. Then by [2, Proposition 1.39], we have $(g_j)^* \nearrow g^*$ pointwise on $[0,\infty)$, as $j \to \infty$. By the monotone convergence theorem, we get

$$\lim_{j\to\infty} \int_0^{|B_j|} f_*(t)(g_j)^*(t)\,dt = \int_0^\infty f_*(t)g^*(t)\,dt.$$

Combining the previous estimates, we conclude that

$$\int_0^\infty f_*(t) g^*(t)\, dt \le \int_{\mathbb{R}^n} f(x) g(x)\, dx.$$

$\square$

Definition 5.3.3. We say that a measurable function $\psi : \mathbb{R}^l \to \mathbb{R}^m$ is a measure preserving transformation if for any measurable set E in $\mathbb{R}^m$, it holds that

$$\mathcal{H}^l(\psi^{-1}(E)) = \mathcal{H}^m(E).$$

Lemma 5.3.4. *If $\psi : \mathbb{R}^l \to \mathbb{R}^m$ is a measure preserving, then for any measurable $f : \mathbb{R}^m \to \mathbb{R}$, and any measurable set E in $\mathbb{R}^m$, it follows that*

$$\int_E f(y)\, dy = \int_{\psi^{-1}(E)} f(\psi(z))\, dz.$$

An important result by Ryff [53] provides a sufficient condition for which we can recover a function given its decreasing/increasing rearrangement, by means of a measure preserving transformation.

Theorem 5.3.5 (Ryff's theorem)**.** *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a nonnegative measurable function. If $\lim_{t\to\infty} f^*(t) = 0$, then there exists a measure preserving $\sigma : \mathrm{supp}(f) \to \mathrm{supp}(f^*)$ such that*

$$f = f^* \circ \sigma$$

almost everywhere on the support of f. Similarly, if $\lim_{t\to\infty} f_*(t) = \infty$, then $f = f_* \circ \sigma$.

We will call Ryff's map, a measure preserving σ satisfying Ryff's theorem.

Remark 5.3.6. In general, σ is not invertible. Furthermore, there may not exist a measure preserving transformation ψ such that $f^* = f \circ \psi$.

As a consequence of Ryff's theorem, we obtain a representation formula for the admissible kernels. We denote $\omega_k = \mathcal{H}^k(B_1)$.

Lemma 5.3.7. *Let $K \in \mathcal{K}_k^s$. Fix $z \in \mathbb{R}^{n-k}$ and denote by $K_z(y) = K(y,z)$. Then*

$$K_z^*(t) = \left(\left(\omega_k^{-1} t\right)^{2/k} + |z|^2\right)^{-\frac{n+2s}{2}}.$$

In particular, there exists a measure preserving $\sigma_z : \operatorname{supp}(K_z) \to (0,\infty)$, such that

$$K(y,z) = K_z^*(\sigma_z(y)), \quad \textit{for a.e. } y \in \operatorname{supp}(K_z).$$

Proof. Fix $z \in \mathbb{R}^{n-k}$. Then

$$\begin{aligned}
K_z^*(t) &= \sup\left\{\lambda > 0 : \mathcal{H}^k\big(\{y \in \mathbb{R}^k : K(y,z) > \lambda\}\big) > t\right\} \\
&= \sup\left\{\lambda < |z|^{-n-2s} : \mathcal{H}^k\left(B_{(\lambda^{-2/(n+2s)} - |z|^2)^{1/2}}\right) > t\right\} \\
&= \sup\left\{\lambda < |z|^{-n-2s} : \omega_k(\lambda^{-2/(n+2s)} - |z|^2)^{k/2} > t\right\} \\
&= \sup\left\{\lambda < |z|^{-n-2s} : \lambda^{-2/(n+2s)} > \left(\omega_k^{-1} t\right)^{2/k} + |z|^2\right\} \\
&= \left(\left(\omega_k^{-1} t\right)^{2/k} + |z|^2\right)^{-\frac{n+2s}{2}}.
\end{aligned}$$

Moreover, $\lim_{t\to\infty} K_z^*(t) = 0$. Therefore, the result follows from Theorem 5.3.5. $\square$

In view of Definition 5.3.1, we introduce the symmetric rearrangement of a function in $\mathbb{R}^n$ with respect to the first k variables as follows. Fix $k \in \mathbb{N}$ with $1 \leq k < n$. Given $x \in \mathbb{R}^n$, we denote $x = (y, z)$, with $y \in \mathbb{R}^k$ and $z \in \mathbb{R}^{n-k}$. Furthermore, for z fixed, we call f_z the restriction of f to $\mathbb{R}^k$. Namely, $f_z(y) = f(y, z)$.

Definition 5.3.8. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a nonnegative measurable function. We define the k-symmetric *decreasing* rearrangement of f as the function $f^{*,k} : \mathbb{R}^n \to [0, \infty]$ given by

$$f^{*,k}(x) = f_z^*(\omega_k |y|^k),$$

and the k-symmetric *increasing* rearrangement as the function $f_{*,k} : \mathbb{R}^n \to [0, \infty]$ given by

$$f_{*,k}(x) = (f_z)_*(\omega_k |y|^k).$$

When $k = n$, we obtain the usual symmetric rearrangement.

Remark 5.3.9. (1) Notice that $f^{*,k}$ and $f_{*,k}$ are radially symmetric and monotone decreasing/increasing, with respect to y. In the literature, this type of symmetrization is also known as the Steiner symmetrization [2, Chapter 6].

(2) By Lemma 5.3.7, we see that any kernel $K \in \mathcal{K}_k^s$ satisfies

$$K^{*,k}(x) = |x|^{-n-2s}, \qquad \text{for } x \neq 0. \tag{5.3.1}$$

5.4 Analysis of $\mathcal{F}_k^s$

Our main goal of this section is to study the infimum in the definition of the operator,

$$\mathcal{F}_k^s u(x_0) = c_{n,s} \inf_{K \in \mathcal{K}_k^s} \int_{\mathbb{R}^n} \tilde{u}(x) K(x)\, dx,$$

where $\tilde{u}(x) = u(x_0 + x) - u(x_0) - x \cdot \nabla u(x_0)$. Throughout the section, we will assume that $u \in C^0(\mathbb{R}^n) \cap C^{1,1}(x_0)$ and satisfies properties (P_1) and (P_2), so that $0 \leq \mathcal{F}_k^s u(x_0) < \infty$.

5.4.1 Analysis of the infimum

We will study the following cases:

Case 1. For all $\lambda > 0$ and $z \in \mathbb{R}^{n-k}$,

$$\mathcal{H}^k\big(\{y \in \mathbb{R}^k : \tilde{u}(y,z) \leq \lambda\}\big) < \infty.$$

Case 2. There exists some $\lambda_0 > 0$ such that for all $z \in \mathbb{R}^{n-k}$,

$$\mathcal{H}^k\big(\{y \in \mathbb{R}^k : \tilde{u}(y,z) \leq \lambda\}\big) \begin{cases} < \infty & \text{for } 0 < \lambda < \lambda_0 \\ = \infty & \text{for } \lambda \geq \lambda_0. \end{cases}$$

Case 3. For all $\lambda > 0$ and $z \in \mathbb{R}^{n-k}$,

$$\mathcal{H}^k\big(\{y \in \mathbb{R}^k : \tilde{u}(y,z) \leq \lambda\}\big) = \infty.$$

In the first case, when all of the level sets of $\tilde{u}$ have finite measure, we show that the infimum is attained at some kernel whose level sets depend on the measure preserving transformation that rearranges the level sets of $\tilde{u}$. More precisely:

Theorem 5.4.1. *Suppose that for all $\lambda > 0$ and $z \in \mathbb{R}^{n-k}$,*

$$\mathcal{H}^k\big(\{y \in \mathbb{R}^k : \tilde{u}(y,z) \leq \lambda\}\big) < \infty.$$

Then, for any $z \in \mathbb{R}^{n-k}$, there exists a measure preserving $\sigma_z : \mathbb{R}^k \to [0,\infty)$ such that

$$\mathcal{F}_k^s u(x_0) = c_{n,s} \int_{\mathbb{R}^{n-k}} \int_{\mathbb{R}^k} \frac{\tilde{u}(y,z)}{\big((\omega_k^{-1}\sigma_z(y))^{2/k} + |z|^2\big)^{\frac{n+2s}{2}}}\, dydz.$$

In particular, the infimum is attained.

Remark 5.4.2. Observe that if $\tilde{u}(\cdot,z)$ is constant in some set of positive measure, then the kernel where the infimum is attained is not unique since the integral is invariant under any measure preserving rearrangement of K within this set (see [53]).

Before we give the proof of Theorem 5.4.1, we need a lemma regarding the k-symmetric increasing rearrangement of $\tilde{u}$. By Definition 5.3.8, this is given by the following expression:

$$\tilde{u}_{*,k}(y,z) = \inf\big\{\lambda > 0 : \mathcal{H}^k\big(\{w \in \mathbb{R}^k : \tilde{u}(w,z) \leq \lambda\}\big) > \omega_k|y|^k\big\}.$$

Lemma 5.4.3. *Fix $z \in \mathbb{R}^{n-k}$. If $\mathcal{H}^k\big(\{y \in \mathbb{R}^k : \tilde{u}(y,z) \leq \lambda\}\big) < \infty$, for all $\lambda > 0$, then*

$$\lim_{|y|\to\infty} \tilde{u}_{*,k}(y,z) = \infty.$$

Proof. Assume there exists $M > 0$, independent of λ, such that

$$\mathcal{H}^k\big(\{w \in \mathbb{R}^k : \tilde{u}(w,z) \leq \lambda\}\big) \leq M, \quad \text{for all } \lambda > 0. \tag{5.4.1}$$

Then for any $y \in \mathbb{R}^k$, with $\omega_k |y|^k > M$, we have that

$$\tilde{u}_{*,k}(y, z) = \infty,$$

since $\inf \emptyset = \infty$. If (5.4.1) does not hold, then there must be an increasing sequence $\{M_\lambda\}_{\lambda > 0}$, with $M_\lambda \to \infty$, as $\lambda \to \infty$, such that

$$\mathcal{H}^k\big(\{w \in \mathbb{R}^k : \tilde{u}(w, z) \leq \lambda\}\big) = M_\lambda.$$

Then for any $M > 0$, there exists $\Lambda = \Lambda(M) > 0$ such that $M_\lambda > M$, for all $\lambda > \Lambda$. Since M_λ is monotone increasing, we can assume without loss of generality that $M_\Lambda \leq M$. Otherwise, we take Λ to be the minimum for which this property holds. Also, $\Lambda(M)$ is monotone increasing, and $\Lambda(M) \to \infty$, as $M \to \infty$. In particular, it holds that

$$\inf\{\lambda > 0 : M_\lambda > M\} \geq \Lambda(M) \to \infty \quad \text{as } M \to \infty.$$

Then for any $K > 0$, there exists $M > 0$ such that

$$\inf\{\lambda > 0 : M_\lambda > M\} \geq K.$$

Therefore, for any $y \in \mathbb{R}^k$, with $\omega_k |y|^k > M$, we have

$$\tilde{u}_{*,k}(y, z) = \inf\{\lambda > 0 : M_\lambda > \omega_k |y|^k\} \geq \inf\{\lambda > 0 : M_\lambda > M\} \geq K.$$

We conclude that

$$\lim_{|y| \to \infty} \tilde{u}_{*,k}(y, z) = \infty.$$

$\square$

Proof of Theorem 5.4.1. Since u is convex at x_0, we have that $\tilde{u}(y,z) \geq 0$. Moreover,

$$\mathcal{F}_k^s u(x_0) = c_{n,s} \inf_{K \in \mathcal{K}_k^s} \int_{\mathbb{R}^{n-k}} \int_{\mathbb{R}^k} \tilde{u}(y,z) K(y,z)\, dydz.$$

Fix $z \in \mathbb{R}^{n-k}$ and consider the functions $f(y) = \tilde{u}(y,z)$ and $g(y) = K(y,z)$. Since

$$\mathcal{H}^k\big(\{y \in \mathbb{R}^k : \tilde{u}(y,z) \leq \lambda\}\big) < \infty,$$

for any $\lambda > 0$, then by Lemma 5.4.3, we have

$$\lim_{t\to\infty} f_*(t) = \lim_{|y|\to\infty} f_{*,k}(x) = \infty,$$

with $f_{*,k}(x) = \tilde{u}_{*,k}(y,z)$ and $f_{*,k}(x) = f_*(\omega_k |y|^k)$. By Ryff's theorem (Theorem 5.3.5), there exists a measure preserving $\sigma_z : \mathbb{R}^k \to [0,\infty)$, depending on z, such that

$$\tilde{u}(y,z) = f_*(\sigma_z(y)), \tag{5.4.2}$$

for all $y \in \operatorname{supp} \tilde{u}(\cdot, z) \subseteq \mathbb{R}^k$.

Let $K(y,z) = \big((\omega_k^{-1}\sigma_z(y))^{2/k} + |z|^2\big)^{-\frac{n+2s}{2}}$. For any $r > |z|$, we have

$$\begin{aligned}
&\mathcal{H}^k\big(\{y \in \mathbb{R}^k : K(y,z) > r^{-n-2s}\}\big)\\
&\quad = \mathcal{H}^k\Big(\Big\{y \in \mathbb{R}^k : \big((\omega_k^{-1}\sigma_z(y))^{2/k} + |z|^2\big)^{-\frac{n+2s}{2}} > r^{-n-2s}\Big\}\Big)\\
&\quad = \mathcal{H}^k\big(\{y \in \mathbb{R}^k : \sigma_z(y) < \omega_k (r^2 - |z|^2)^{k/2}\}\big)\\
&\quad = \mathcal{H}^k\Big(\sigma_z^{-1}\big((0, \omega_k (r^2 - |z|^2)^{k/2})\big)\Big)\\
&\quad = \mathcal{H}^1\big((0, \omega_k (r^2 - |z|^2)^{k/2})\big)\\
&\quad = \omega_k (r^2 - |z|^2)^{k/2} = \mathcal{H}^k\big(B_{(r^2-|z|^2)^{k/2}}\big),
\end{aligned}$$

since σ_k is measure preserving (see Definition 5.3.3). Then $K \in \mathcal{K}_k^s$, and thus,

$$\mathcal{F}_k^s u(x_0) \leq c_{n,s} \int_{\mathbb{R}^{n-k}} \int_{\mathbb{R}^k} \frac{\tilde{u}(y,z)}{\left((\omega_k^{-1}\sigma_z(y))^{2/k} + |z|^2\right)^{\frac{n+2s}{2}}} \, dydz.$$

To prove the reverse inequality, let $K \in \mathcal{K}_k^s$. Applying Proposition 5.3.2, we see that

$$\begin{aligned}
\int_{\mathbb{R}^k} \tilde{u}(y,z) K(y,z) \, dy &\geq \int_0^\infty f_*(t) g^*(t) \, dt \\
&= \int_{\mathbb{R}^k} f_*(\sigma_z(y)) g^*(\sigma_z(y)) \, dy \\
&= \int_{\mathbb{R}^k} \tilde{u}(y,z) g^*(\sigma_z(y)) \, dy,
\end{aligned}$$

by Lemma 5.3.4 and (5.4.2). Moreover, by the definition of rearrangements,

$$g^*(\sigma_z(y)) = \sup\left\{\lambda > 0 : \mathcal{H}^k\left(\{w \in \mathbb{R}^k : K(w,z) > \lambda\}\right) > \sigma_z(y)\right\} = K^{*,k}(\tilde{y}, z)$$

with $\omega_k |\tilde{y}|^k = \sigma_z(y)$. By (5.3.1), we get

$$g^*(\sigma_z(y)) = \left(|\tilde{y}|^2 + |z|^2\right)^{-\frac{n+2s}{2}} = \left((\omega_k^{-1}\sigma_z(y))^{2/k} + |z|^2\right)^{-\frac{n+2s}{2}}.$$

Hence, integrating over all $z \in \mathbb{R}^{n-k}$, and taking the infimum over all kernels $K \in \mathcal{K}_k^s$, we conclude that

$$\mathcal{F}_k^s u(x) = c_{n,s} \int_{\mathbb{R}^{n-k}} \int_{\mathbb{R}^k} \frac{\tilde{u}(y,z)}{\left((\omega_k^{-1}\sigma_z(y))^{2/k} + |z|^2\right)^{\frac{n+2s}{2}}} \, dydz.$$

$\square$

Remark 5.4.4. A natural question that arises from this result is whether there exists a measure preserving $\varphi_z : \mathbb{R}^k \to \mathbb{R}^k$ such that

$$|\varphi_z(y)| = \left(\omega_k^{-1}\sigma_z(y)\right)^{1/k}.$$

In that case, we would have that the infimum is attained at a kernel K such that

$$K(y,z) = |\phi(y,z)|^{-n-2s},$$

where $\phi : \mathbb{R}^n \to \mathbb{R}^n$ is a measure preserving with $\phi(y,z) = (\varphi_z(y), z)$.

Recall that Ryff's theorem gives a representation of a function f in terms of its increasing rearrangement f_*, that is, $f = f_* \circ \sigma$, with $\sigma : \mathbb{R}^k \to \mathbb{R}$ measure preserving. If this result were also true for the *symmetric* increasing rearrangement, given by $f_\#(x) = f_*(\omega_k |x|^k)$, then there would exist a measure preserving $\varphi : \mathbb{R}^k \to \mathbb{R}^k$ such that $f = f_\# \circ \psi$. In particular,

$$f(x) = f_\#(\varphi(x)) = f_*(\omega_k |\varphi(x)|^k) = f_*(\sigma(x)).$$

Hence, it seems reasonable that $\omega_k |\varphi(x)|^k = \sigma(x)$. As far as we know, this is an open problem.

As an immediate consequence of Theorem 5.4.1, we obtain the following representation of the function $\mathcal{F}_k^s u$ in terms of the k-symmetric increasing rearrangement of $\tilde{u}$.

Corollary 5.4.5. *Under the assumptions of Theorem 5.4.1, we have*

$$\mathcal{F}_k^s u(x_0) = \Delta^s \tilde{u}_{*,k}(0).$$

Proof. Note that $\tilde{u}_{*,k}(0) = 0$, since $\tilde{u}(0) = 0$. Therefore, using the same notation as in the proof of Theorem 5.4.1, we showed that

$$\begin{aligned}
\mathcal{F}_k^s u(x_0) &= c_{n,s} \int_{\mathbb{R}^{n-k}} \int_0^\infty f_*(t) g^*(t)\, dt dz \\
&= \omega_k c_{n,s} \int_{\mathbb{R}^{n-k}} \int_0^\infty f_*(\omega_k r^k) g^*(\omega_k r^k) r^{k-1}\, dr dz \\
&= c_{n,s} \int_{\mathbb{R}^{n-k}} \int_{\mathbb{R}^k} f_*(\omega_k |y|^k) g^*(\omega_k |y|^k)\, dy dz \\
&= c_{n,s} \int_{\mathbb{R}^{n-k}} \int_{\mathbb{R}^k} \tilde{u}_{*,k}(y,z) K^{*,k}(y,z)\, dy dz \\
&= c_{n,s} \int_{\mathbb{R}^{n-k}} \int_{\mathbb{R}^k} \frac{\tilde{u}_{*,k}(y,z)}{(|y|^2+|z|^2)^{\frac{n+2s}{2}}}\, dy dz = \Delta^s \tilde{u}_{*,k}(0).
\end{aligned}$$

□

From the previous result and the fact that the family of operators $\{\mathcal{F}_k\}_{k=1}^{n-1}$ is monotone decreasing, we see that the fractional Laplacian of the k-symmetric rearrangements are ordered at the origin.

Corollary 5.4.6. *Suppose we are under the assumption of Theorem 5.4.1. Then*

$$\Delta^s \tilde{u}_{*,k+1}(0) \leq \Delta^s \tilde{u}_{*,k}(0).$$

Next we treat the second case.

Theorem 5.4.7. *Suppose that there exists some $\lambda_0 > 0$ such that for all $z \in \mathbb{R}^{n-k}$,*

$$\mathcal{H}^k\big(\{y \in \mathbb{R}^k : \tilde{u}(y,z) \leq \lambda\}\big) \begin{cases} < \infty & \text{for } 0 < \lambda < \lambda_0 \\ = \infty & \text{for } \lambda \geq \lambda_0. \end{cases}$$

Then there exists a kernel $K_0 \in \mathcal{K}_k^s$, *with* $\operatorname{supp} K_0(\cdot, z) \subseteq \{y \in \mathbb{R}^k : \tilde{u}(y,z) \leq \lambda_0\}$, *such that*

$$\mathcal{F}_k^s u(x_0) = c_{n,s} \int_{\mathbb{R}^{n-k}} \int_{\mathbb{R}^k} \tilde{u}(y,z) K_0(y,z) \, dydz.$$

In particular, the infimum is attained.

Proof. Fix $z \in \mathbb{R}^{n-k}$. For $j \geq 1$, define the set

$$A_j(z) = \left\{ y \in \mathbb{R}^k : \tilde{u}(y,z) \leq \lambda_0 - \tfrac{1}{j} \right\}.$$

For simplicity, we drop the notation of z. We have that $\mathcal{H}^k(A_j) < \infty$, $A_j \subseteq A_{j+1}$, and

$$A_\infty = \bigcup_{j=1}^{\infty} A_j = \left\{ y \in \mathbb{R}^k : \tilde{u}(y,z) < \lambda_0 \right\}.$$

Observe that if $K \in \mathcal{K}_k^s$, then

$$\mathcal{H}^k\big(\{y \in \mathbb{R}^k : K(y,z) > 0\}\big) = \lim_{r \to 0} \mathcal{H}^k\big(\{y \in \mathbb{R}^k : K(y,z) > r\}\big) = \infty.$$

Hence, we need to distinguish two cases:

Case 2.1. Assume that $\mathcal{H}^k(A_\infty) = \infty$. Let $K \in \mathcal{K}_k^s$ and $v_j = \tilde{u}\chi_{A_j}$. By Proposition 5.3.2,

$$\int_{A_j} \tilde{u}(y,z) K(y,z) \, dy = \int_{\mathbb{R}^k} v_j(y,z) K(y,z) \, dy \geq \int_0^\infty (v_j)_*(t) K^*(t) \, dt.$$

By Lemma 5.3.4, for any measure preserving $\sigma : \mathbb{R}^k \to [0, \infty)$, it follows that

$$\int_0^\infty (v_j)_*(t) K^*(t) \, dt = \int_{\mathbb{R}^k} (v_j)_*(\sigma(y)) K^*(\sigma(y)) \, dy.$$

By Ryff's theorem (Theorem 5.3.5), there exists $\sigma_j : A_j \to [0, \mathcal{H}^k(A_j)]$ measure preserving such that $v_j = (v_j)_* \circ \sigma_j$ in A_j. Therefore,

$$\int_{A_j} \tilde{u}(y,z) K(y,z)\, dy \geq \int_{A_j} \tilde{u}(y,z) K^*(\sigma_j(y))\, dy. \tag{5.4.3}$$

We claim that $\sigma_{j+1}(y) \leq \sigma_j(y)$, for all $y \in A_j$. Indeed, since $A_j \subseteq A_{j+1}$, we have

$$\begin{cases} v_j(y) = v_{j+1}(y), & \text{for all } y \in A_j \\ v_j(y) \leq v_{j+1}(y), & \text{for all } y \in A_{j+1} \setminus A_j. \end{cases}$$

In particular, for all $y \in A_j$,

$$(v_{j+1})_*(\sigma_{j+1}(y)) = (v_j)_*(\sigma_j(y)) \leq (v_{j+1})_*(\sigma_j(y)).$$

Since $(v_{j+1})_*$ is monotone increasing, we must have

$$\sigma_{j+1}(y) \leq \sigma_j(y), \quad \text{for all } y \in A_j.$$

Therefore, there exists $\sigma_\infty : A_\infty \to [0, \infty)$ measure preserving such that

$$\sigma_\infty(y) = \lim_{j \to \infty} \sigma_j(y).$$

Define the kernel K_0 as

$$K_0(y,z) = \left((\omega_k^{-1} \sigma_\infty(y))^{k/2} + |z|^2 \right)^{-\frac{n+2s}{2}} \chi_{A_\infty}(y).$$

Since $\mathcal{H}^k(A_\infty) = \infty$, then $K_0 \in \mathcal{K}_k^s$. Furthermore, note that $\operatorname{supp} K_0(\cdot, z) = \overline{A_\infty} = \{y \in \mathbb{R}^k : \tilde{u}(y,z) \leq \lambda_0\}$ and $K_0(y,z) = K_0^*(\sigma_\infty(y))$, for all $y \in A_\infty$.

Then by Fatou's lemma, Lemma 5.3.7, and (5.4.3), we get

$$\begin{aligned}
\int_{\mathbb{R}^k} \tilde{u}(y,z) K_0(y,z)\,dy &= \int_{A_\infty} \tilde{u}(y,z) K_0^*(\sigma_\infty(y))\,dy \\
&\le \liminf_{j\to\infty} \int_{A_j} \tilde{u}(y,z) K_0^*(\sigma_j(y))\,dy \\
&= \liminf_{j\to\infty} \int_{A_j} \tilde{u}(y,z) K^*(\sigma_j(y))\,dy \\
&\le \int_{\mathbb{R}^k} \tilde{u}(y,z) K(y,z)\,dy,
\end{aligned}$$

for any $K \in \mathcal{K}_k^s$. Integrating over z and taking the infimum over all kernels K, we conclude the result.

Case 2.2. Assume that $\mathcal{H}^k(A_\infty) < \infty$. Set $A = \{y \in \mathbb{R}^k : \tilde{u}(y,z) = \lambda_0\}$. Then

$$\mathcal{H}^k(A) = \infty, \tag{5.4.4}$$

since $\{y \in \mathbb{R}^k : \tilde{u}(y,z) \le \lambda_0\} = A_\infty \cup A$. Fix $\varepsilon > 0$ and define

$$v_\varepsilon(y,z) = \tilde{u}(y,z)\chi_{A_\infty}(y) + \max\{\lambda_0, (\lambda_0+\varepsilon)\phi(y,z)\}\chi_A(y),$$

with $\phi(y,z) = 1 - e^{-|y|^2-|z|^2}$. Note that $0 < \phi \le 1$, $\phi(y,z) \to 1$, as $|(y,z)| \to \infty$, and $\phi(y,z) \approx |y|^2 + |z|^2$, as $|(y,z)| \to 0$. Also, $\{v_\varepsilon\}_{\varepsilon>0}$ is a monotone increasing sequence, and

$$\begin{aligned}
\lim_{\varepsilon\to 0} v_\varepsilon(y,z) &= \tilde{u}(y,z)\chi_{A_\infty}(y) + \max\big\{\lambda_0, \lim_{\varepsilon\to 0}(\lambda_0+\varepsilon)\phi(y,z)\big\}\chi_A(y) \\
&= \tilde{u}(y,z)\chi_{A_\infty}(y) + \max\{\lambda_0, \lambda_0\phi(y,z)\}\chi_A(y) = \tilde{u}(y,z)\chi_{A_\infty\cup A}(y).
\end{aligned} \tag{5.4.5}$$

For any $j \in \mathbb{N}$, with $j > 1/\varepsilon$, consider the set

$$B_j^\varepsilon(z) = \big\{y \in \mathbb{R}^k : v_\varepsilon(y,z) \le \lambda_0 + \varepsilon - \tfrac{1}{j}\big\}.$$

Then $B_j^\varepsilon \subseteq B_{j+1}^\varepsilon$ and $B_\infty^\varepsilon = \bigcup_{j>1/\varepsilon} B_j^\varepsilon = \{y \in \mathbb{R}^k : v_\varepsilon(y,z) < \lambda_0 + \varepsilon\}$. Moreover, we have

$$\mathcal{H}^k(B_j^\varepsilon) \le \mathcal{H}^k(A_\infty) + \mathcal{H}^k\big(\{y \in A : \max\{\lambda_0, (\lambda_0+\varepsilon)\phi(y,z)\} \le \lambda_0 + \varepsilon - \tfrac{1}{j}\}\big). \tag{5.4.6}$$

Choose $R > 0$ large enough (depending on ε, j, λ_0, and z) so that

$$(\lambda_0 + \varepsilon)e^{-R^2 - |z|^2} < \tfrac{1}{j}.$$

Then $(\lambda_0+\varepsilon)\phi(y,z) > \lambda_0 + \varepsilon - \frac{1}{j} > \lambda_0$, for all $y \in B_R^c$, and thus,

$$\mathcal{H}^k\big(\{y \in A \cap B_R^c : \max\{\lambda_0, (\lambda_0+\varepsilon)\phi(y,z)\} \le \lambda_0 + \varepsilon - \tfrac{1}{j}\}\big) = 0. \tag{5.4.7}$$

By (5.4.6) and (5.4.7), we see that

$$\mathcal{H}^k(B_j^\varepsilon(z)) \le \mathcal{H}^k(A_\infty) + \mathcal{H}^k(A \cap B_R) < \infty.$$

Furthermore, $A \subseteq B_\infty^\varepsilon$, and thus, by (5.4.4), we get

$$\mathcal{H}^k(B_\infty^\varepsilon) \ge \mathcal{H}^k\big(A) = \infty.$$

In particular, v_ε satisfies the assumptions of Case 1, so there exists $K_\varepsilon \in \mathcal{K}_k^s$,

$$K_\varepsilon(y,z) = \big((\omega_k^{-1}\sigma_\varepsilon(y))^{k/2} + |z|^2\big)^{-\frac{n+2s}{2}} \chi_{B_\infty^\varepsilon}(y), \tag{5.4.8}$$

with $\sigma_\varepsilon : B_\infty^\varepsilon \to [0,\infty)$ measure preserving, depending on v_ε, such that

$$\inf_{K \in \mathcal{K}_k^s} \int_{\mathbb{R}^{n-k}} \int_{\mathbb{R}^k} v_\varepsilon(y,z) K(y,z)\, dydz = \int_{\mathbb{R}^{n-k}} \int_{\mathbb{R}^k} v_\varepsilon(y,z) K_\varepsilon(y,z)\, dydz. \tag{5.4.9}$$

Finally, we need to pass to the limit. First, we prove that $\{\sigma_\varepsilon\}_{\varepsilon>0}$ is monotone decreasing. Indeed, let $V_\varepsilon = \{y \in \mathbb{R}^k : v_\varepsilon(y,z) = \tilde{u}(y,z)\}$. In particular, $A_\infty \subseteq V_\varepsilon \subseteq A_\infty \cup A$. Also, $V_{\varepsilon_2} \subseteq V_{\varepsilon_1}$, for any $\varepsilon_1 \leq \varepsilon_2$. By Ryff's theorem, recall that

$$v_{\varepsilon_1}(y,z) = (v_{\varepsilon_1})_*(\sigma_{\varepsilon_1}(y)) \quad \text{and} \quad v_{\varepsilon_2}(y,z) = (v_{\varepsilon_2})_*(\sigma_{\varepsilon_2}(y)).$$

Since $v_{\varepsilon_2}(y,z) = v_{\varepsilon_1}(y,z)$, for all $y \in V_{\varepsilon_2}$, and $v_{\varepsilon_1}(y,z) \leq v_{\varepsilon_2}(y,z)$, for all $y \in \mathbb{R}^k$, we see that

$$(v_{\varepsilon_2})_*(\sigma_{\varepsilon_2}(y)) = (v_{\varepsilon_1})_*(\sigma_{\varepsilon_1}(y)) \leq (v_{\varepsilon_2})_*(\sigma_{\varepsilon_1}(y)), \quad \text{for all } y \in V_{\varepsilon_2}.$$

Since $(v_{\varepsilon_2})_*$ is monotone increasing, we must have that $\sigma_{\varepsilon_2}(y) \leq \sigma_{\varepsilon_1}(y)$, for all $y \in V_{\varepsilon_2}$. Hence, there exists $\sigma_0 : B_\infty \to [0,\infty)$ measure preserving such that

$$\sigma_0(y) = \lim_{\varepsilon \to 0} \sigma_\varepsilon(y),$$

where $B_\infty = \bigcap_{\varepsilon>0} B_\infty^\varepsilon = \{y \in \mathbb{R}^k : \tilde{u}(y,z) \leq \lambda_0\} = A_\infty \cup A$. In particular, the sequence of kernels $\{K_\varepsilon\}_{\varepsilon>0}$ is monotone decreasing. Define

$$K_0(y,z) = \lim_{\varepsilon \to 0} K_\varepsilon(y,z). \tag{5.4.10}$$

By (5.4.8) and (5.4.10), we have

$$K_0(y,z) = \big((\omega_k^{-1}\sigma_0(y))^{k/2} + |z|^2\big)^{-\frac{n+2s}{2}} \chi_{B_\infty}(y).$$

Moreover, $K_0 \in \mathcal{K}_k^s$ since $K_\varepsilon \in \mathcal{K}_k^s$, and for any $r > 0$, it follows that

$$\mathcal{H}^k\big(D_0(r)\big) = \lim_{\varepsilon \to 0} \mathcal{H}^k(D_\varepsilon(r)),$$

where $D_\varepsilon(r) = \{y \in \mathbb{R}^k : K_\varepsilon(y,z) > r^{-(n+2s)}\}$.

Finally, using (5.4.5), (5.4.9), (5.4.10), and the monotone convergence theorem, we get

$$\begin{aligned}
\int_{\mathbb{R}^{n-k}} \int_{\mathbb{R}^k} \tilde{u}(y,z) K_0(y,z)\, dydz &= \int_{\mathbb{R}^{n-k}} \int_{\mathbb{R}^k} \lim_{\varepsilon \to 0} \big(v_\varepsilon(y,z) K_\varepsilon(y,z) \big)\, dydz \\
&= \lim_{\varepsilon \to 0} \int_{\mathbb{R}^{n-k}} \int_{\mathbb{R}^k} v_\varepsilon(y,z) K_\varepsilon(y,z)\, dydz \\
&= \lim_{\varepsilon \to 0} \inf_{K \in \mathcal{K}_k^s} \int_{\mathbb{R}^{n-k}} \int_{\mathbb{R}^k} v_\varepsilon(y,z) K(y,z)\, dydz \\
&\le \inf_{K \in \mathcal{K}_k^s} \int_{\mathbb{R}^{n-k}} \int_{\mathbb{R}^k} \Big(\lim_{\varepsilon \to 0} v_\varepsilon(y,z) \Big) K(y,z)\, dydz \\
&= \inf_{K \in \mathcal{K}_k^s} \int_{\mathbb{R}^{n-k}} \int_{\mathbb{R}^k} \tilde{u}(y,z) \big(K(y,z) \chi_{A_\infty \cup A}(y) \big)\, dydz \\
&= \inf_{K \in \mathcal{K}_k^s} \int_{\mathbb{R}^{n-k}} \int_{\mathbb{R}^k} \tilde{u}(y,z) K(y,z)\, dydz.
\end{aligned}$$

The last equality follows from the following observation: since

$$\tilde{\mathcal{K}}_k^s = \Big\{ K \in \mathcal{K}_k^s : \operatorname{supp} K(\cdot, z) \subseteq A_\infty \cup A \Big\} \subseteq \mathcal{K}_k^s,$$

then the infimum over all kernels in $\mathcal{K}_k^s$ is less than or equal to the infimum over $\tilde{\mathcal{K}}_k^s$. Moreover, the reverse inequality holds trivially. $\square$

Finally, we deal with the third case, that is, when all of the level sets of $\tilde{u}$ have infinite measure. In particular, notice that

$$\tilde{u}_{*,k}(x) = 0, \quad \text{for all } x \in \mathbb{R}^n.$$

This is the only case where the infimum is not attained. Indeed, we prove in the following theorem that the infimum is equal to zero.

Theorem 5.4.8. *Suppose that for all $\lambda > 0$ and $z \in \mathbb{R}^{n-k}$,*

$$\mathcal{H}^k\big(\{y \in \mathbb{R}^k : \tilde{u}(y,z) \le \lambda\}\big) = \infty.$$

Then $\mathcal{F}^s_k u(x_0) = 0$.

Proof. From (P_2), we have that $\mathcal{F}^s_k u(x_0) \ge 0$. To prove the reverse inequality, it is enough to find a sequence of kernels $\{K_\varepsilon\}_{\varepsilon>0} \subset \mathcal{K}^s_k$ such that

$$\liminf_{\varepsilon\to 0} \int_{\mathbb{R}^{n-k}} \int_{\mathbb{R}^k} \tilde{u}(y,z) K_\varepsilon(y,z)\, dydz = 0. \tag{5.4.11}$$

Fix $\varepsilon > 0$ and $z \in \mathbb{R}^{n-k}$. For any $j \ge 0$, we define the set

$$U_j \equiv U_j(z) = \big\{y \in \mathbb{R}^k : \tilde{u}(y,z) < \varepsilon 2^{-j(n+2s)} e^{-|z|^2}\big\}.$$

Note that $U_{j+1} \subseteq U_j$. Also, by assumption, with $\lambda = \varepsilon 2^{-j(1+2s)} e^{-|z|^2}$, we have that

$$\mathcal{H}^k(U_j) = \infty, \quad \text{for all } j \ge 0.$$

We will construct $K_\varepsilon \in \mathcal{K}^s_k$ by describing first where to locate each level set of the form:

$$A_{-1} \equiv A_{-1}(z) = \big\{y \in \mathbb{R}^k : 0 < K_\varepsilon(y,z) \le 1\big\}$$
$$A_j \equiv A_j(z) = \big\{y \in \mathbb{R}^k : 2^{j(n+2s)} < K_\varepsilon(y,z) \le 2^{(j+1)(n+2s)}\big\} \text{ for } j \ge 0.$$

Recall that $K \in \mathcal{K}^s_k$ if for all $r > 0$, we have $\mathcal{H}^k\big(\{y \in \mathbb{R}^k : K(y,z) > r^{-(n+2s)}\}\big) = \mathcal{H}^k\big(\{y \in \mathbb{R}^k : (|y|^2 + |z|^2)^{-\frac{n+2s}{2}} > r^{-(n+2s)}\}\big)$. In view of this definition, we define the sets

$$B_{-1} \equiv B_{-1}(z) = \big\{y \in \mathbb{R}^k : 0 < (|y|^2 + |z|^2)^{-\frac{n+2s}{2}} \le 1\big\}$$
$$B_j \equiv B_j(z) = \big\{y \in \mathbb{R}^k : 2^{j(n+2s)} < (|y|^2 + |z|^2)^{-\frac{n+2s}{2}} \le 2^{(j+1)(n+2s)}\big\} \text{ for } j \ge 0.$$

Note that

$$\begin{cases} \mathcal{H}^k(A_{-1}) = \mathcal{H}^k(B_{-1}) = \infty \\ \mathcal{H}^k(A_j) = \mathcal{H}^k(B_j) < \infty, \qquad \text{for all } j \geq 0. \end{cases}$$

More precisely, for $j \geq 0$, if $|z| < 2^{-(j+1)} < 2^{-j}$, then

$$\begin{aligned} \mathcal{H}^k(A_j) &= \mathcal{H}^k\big(B_{(2^{-2j}-|z|^2)^{1/2}}\big) - \mathcal{H}^k\big(B_{(2^{-2(j+1)}-|z|^2)^{1/2}}\big) \\ &= \omega_k(2^{-2j} - |z|^2)^{k/2} - \omega_k(2^{-2(j+1)} - |z|^2)^{k/2} \leq \omega_k 2^{-kj}. \end{aligned}$$

If $2^{-(j+1)} \leq |z| < 2^{-j}$, then

$$\mathcal{H}^k(A_j) = \mathcal{H}^k\big(B_{(2^{-2j}-|z|^2)^{1/2}}\big) = \omega_k(2^{-2j} - |z|^2)^{k/2} \leq \omega_k(\tfrac{3}{4})^{k/2} 2^{-kj}.$$

If $|z| \geq 2^{-j} > 2^{-(j+1)}$, then

$$\mathcal{H}^k(A_j) = 0.$$

Therefore, $\mathcal{H}^k(A_j) \leq c2^{-kj}$, where $c > 0$ only depends on k. It follows that

$$\mathcal{H}^k\Big(\bigcup_{j=0}^{\infty} A_j\Big) = \sum_{j=0}^{\infty} \mathcal{H}^k(A_j) \leq c\sum_{j=0}^{\infty} 2^{-jk} < \infty. \tag{5.4.12}$$

For any $i \geq 0$, let $\mathcal{D}_i$ be the collection of all dyadic closed cubes of the form

$$[m2^{-i}, (m+1)2^{-i}]^k = [m2^{-i}, (m+1)2^{-i}] \times \cdots \times [m2^{-i}, (m+1)2^{-i}].$$

Note that if $Q \in \mathcal{D}_i$, then $l(Q) = 2^{-i}$, where $l(Q)$ denotes the side length of the cube Q. For any $j \geq 0$, since U_j is an open set, by a standard covering argument, we have that there exists a family of dyadic cubes $\mathcal{F}_j$ such that

$$U_j = \bigcup_{Q \in \mathcal{F}_j} Q$$

satisfying the following properties:

1. For any $Q \in \mathcal{F}_j$, there exists some $i \geq 0$ such that $Q \in \mathcal{D}_i$.

2. $\mathrm{Int}(Q) \cap \mathrm{Int}(\tilde{Q}) = \emptyset$, for any $Q, \tilde{Q} \in \mathcal{F}_j$, with $Q \neq \tilde{Q}$.

3. If $x \in Q \in \mathcal{F}_j$, then Q is the maximal dyadic cube contained in U_j that contains x.

Analogously, for the sets B_j, with $j \geq -1$, there exists a family of dyadic cubes $\tilde{\mathcal{F}}_j$ satisfying properties $(1)-(3)$ such that

$$\mathrm{Int}(B_j) = \bigcup_{Q \in \tilde{\mathcal{F}}_j} Q.$$

Note that $\tilde{\mathcal{F}}_j \cap \tilde{\mathcal{F}}_{j+1} = \emptyset$ since $B_j \cap B_{j+1} = \emptyset$.

We will construct the sets A_j by properly translating the dyadic cubes partitioning the sets B_j into U_j. In particular, we will prove that

$$\begin{cases} A_0 = T_0(B_0) \subset U_0 \\ A_j = T_j(B_j) \subset U_j \setminus \bigcup_{i=0}^{j-1} A_i, \quad \text{for all } j \geq 1 \\ A_{-1} = T_{-1}(B_{-1}) \subset U_0 \setminus \bigcup_{i=0}^{\infty} A_i, \end{cases}$$

for some *translation* mappings $T_j : \tilde{\mathcal{F}}_j \to \mathcal{F}_j$ to be determined.

We start with the case $j = 0$. For any $i \geq 0$, denote by

$$m_i = \mathcal{H}^0(\mathcal{F}_0 \cap \mathcal{D}_i) \quad \text{and} \quad n_i = \mathcal{H}^0(\tilde{\mathcal{F}}_0 \cap \mathcal{D}_i),$$

where $\mathcal{H}^0(E)$ is equal to the cardinal of the set E. Note that $m_i, n_i \in \mathbb{Z}^+ \cup \{\infty\}$.

We will recursively place B_0 into U_0. First, fix $i = 0$. If $m_0 \geq n_0$, then for any $\tilde{Q} \in \tilde{\mathcal{F}}_0 \cap \mathcal{D}_0$, there exists some $\tau \in \mathbb{R}^k$ and some $Q \in \mathcal{F}_0 \cap \mathcal{D}_0$, such that $Q = \tilde{Q} + \tau$. Then define

$$
\begin{array}{rrcl} T_0: & \tilde{\mathcal{F}}_0 \cap \mathcal{D}_0 & \to & \mathcal{F}_0 \cap \mathcal{D}_0 \\ & \tilde{Q} & \mapsto & Q. \end{array} \tag{5.4.13}
$$

Moreover, we can define T_0 one-to-one since $m_0 \geq n_0$, and we can always choose a different Q for each $\tilde{Q}$. Note that there are p_0 cubes in $\mathcal{F}_0 \cap D_0$, with $p_0 = m_0 - n_0$, that have not been used. Hence, to all of these cubes, divide each side in half, so that each cube gives rise to 2^k cubes with side length 2^{-1}. Call this collection of new cubes $\mathcal{Q} = \{Q_l\}_{l=1}^{2^{kp_0}} \subset \mathcal{D}_1$, and add them to the family $\mathcal{F}_0 \cap \mathcal{D}_1$. Namely, we replace $\mathcal{F}_0 \cap \mathcal{D}_1$ by $(\mathcal{F}_0 \cap \mathcal{D}_1) \cup \mathcal{Q}$.

If $m_0 < n_0$, then take q_0 cubes in $\tilde{\mathcal{F}}_0 \cap \mathcal{D}_0$, with $q_0 = n_0 - m_0$, and divide each side in half. Call this collection of new cubes $\tilde{\mathcal{Q}} = \{\tilde{Q}_l\}_{l=1}^{2^{kq_0}} \subset \mathcal{D}_1$. Then, we replace $\tilde{\mathcal{F}}_0$ by $\hat{\mathcal{F}}_0$, where

$$
\begin{aligned}
&\hat{\mathcal{F}}_0 \cap \mathcal{D}_0 = (\tilde{\mathcal{F}}_0 \setminus \tilde{\mathcal{Q}}) \cap \mathcal{D}_0 \\
&\hat{\mathcal{F}}_0 \cap \mathcal{D}_1 = (\tilde{\mathcal{F}}_0 \cup \tilde{\mathcal{Q}}) \cap \mathcal{D}_1 \\
&\hat{\mathcal{F}}_0 \cap \mathcal{D}_i = \tilde{\mathcal{F}}_0 \cap \mathcal{D}_i, \quad \text{for all } i \geq 2.
\end{aligned}
$$

If $\hat{n}_0 = \mathcal{H}^0(\hat{\mathcal{F}}_0 \cap \mathcal{D}_0)$, then $m_0 = \hat{n}_0$. Hence, by the same argument as in the previous case, we find T_0 as in (5.4.13). For $i \geq 1$, we can repeat the same process until we run out of cubes from $\tilde{\mathcal{F}}_0$ (or the modified family). We know the process will end since $\mathcal{H}^k(B_0) < \mathcal{H}^k(U_0)$. When this happens, we will have constructed a one-to-one mapping $T_0 : \tilde{\mathcal{F}}_0 \to \mathcal{F}_0$, since $\tilde{\mathcal{F}}_0 = \bigcup_{i=0}^{\infty} \tilde{\mathcal{F}}_0 \cap \mathcal{D}_i$ and $\mathcal{F}_0 = \bigcup_{i=0}^{\infty} \mathcal{F}_0 \cap \mathcal{D}_i$. Then define

$$
A_0 = T_0(B_0) \subset U_0.
$$

Iterating this process, we find a sequence of translation mappings $\{T_j\}_{j=0}^{\infty}$ with $T_j : \tilde{F}_j \to \mathcal{F}_j$, and a sequence of disjoint sets $\{A_j\}_{j=0}^{\infty}$ such that

$$A_j = T_j(B_j) \subset U_j \setminus \bigcup_{i=0}^{j-1} A_i.$$

The case $j = -1$ is somewhat special since $\mathcal{H}^k(A_{-1}) = \mathcal{H}^k(B_{-1}) = \infty$. We will see that

$$A_{-1} = T_{-1}(B_{-1}) \subset U_0 \setminus \bigcup_{i=0}^{\infty} A_i.$$

This is possible because $\mathcal{H}^k(U_0 \setminus \bigcup_{i=0}^{\infty} A_i) = \infty$ using (5.4.12). Indeed, we can write

$$\big\{y \in \mathbb{R}^k : 0 < K_\varepsilon(y,z) \leq 1\big\} = \bigcup_{j=0}^{\infty} \big\{2^{-(j+1)(n+2s)} < K_\varepsilon(y,z) \leq 2^{-j(n+2s)}\big\}.$$

Now call

$$C_j = \big\{2^{-(j+1)(n+2s)} < (|y|^2 + |z|^2)^{-\frac{n+2s}{2}} \leq 2^{-j(n+2s)}\big\}, \quad \text{for } j \geq 0.$$

Then $B_{-1} = \bigcup_{j=0}^{\infty} C_j$, with $\mathcal{H}^k(C_j) < \infty$, for all $j \geq 0$. Hence, instead of partitioning all of B_{-1} into dyadic cubes, we partition each of its disjoint components C_j. Arguing as before, we place them into $U_0 \setminus \bigcup_{i=0}^{\infty} A_i$ recursively, according to the following scheme:

$$\begin{cases} T_{-1}^0(C_0) \subset U_0 \setminus \bigcup\limits_{i=0}^{\infty} A_i \\ T_{-1}^j(C_j) \subset U_0 \setminus \Big(\bigcup\limits_{i=0}^{\infty} A_i \cup \bigcup\limits_{i=0}^{j-1} C_i\Big), \quad \text{for } j \geq 1, \end{cases}$$

where T_{-1}^j is defined as before. At the end of this process, we find a translation map T_{-1} with $T_{-1}(Q) = T_{-1}^j(Q)$, for $Q \in C_j$. Therefore, we define

$$A_{-1} = T_{-1}(B_{-1}).$$

Lastly, let $y \in \mathbb{R}^k = A_{-1} \cup \left(\bigcup_{j=0}^{\infty} A_j\right)$. In particular, there exists some $j \geq -1$ such that $y \in A_j$. Furthermore, recall that $A_j = T_j(B_j)$, where T_j is a one-to-one and onto translation map. Hence, there exists a unique $w \in B_j$ such that $y = T_j(w) = w + \tau$, for some $\tau \in \mathbb{R}^k$. Let $T_z : \mathbb{R}^k \to \mathbb{R}^k$ be given by $T_z(y) = w$. Note that T_z is measure preserving. Then we define the kernel K_ε as

$$K_\varepsilon(y,z) = \left(|T_z(y)|^2 + |z|^2\right)^{-\frac{n+2s}{2}}.$$

We have that

$$\begin{aligned}\int_{\mathbb{R}^k} \tilde{u}(y,z)K_\varepsilon(y,z)\,dy &= \int_{A_{-1}} \tilde{u}(y,z)K_\varepsilon(y,z)\,dy + \sum_{j=0}^{\infty}\int_{A_j} \tilde{u}(y,z)K_\varepsilon(y,z)\,dy \\ &\equiv \mathrm{I} + \mathrm{II}.\end{aligned}$$

For I, we use that $\tilde{u}(y,z) \leq \varepsilon e^{-|z|^2}$, since $A_{-1} \subset U_0$. Then by Lemma 5.3.7 and Lemma 5.3.4:

$$\begin{aligned}\mathrm{I} &\leq \varepsilon e^{-|z|^2} \int_{\{0<K_\varepsilon(y,z)\leq 1\}} K_\varepsilon(y,z)\,dy \\ &= \varepsilon e^{-|z|^2} \int_{\{0<|\sigma_z(y)|^{-n-2s}\leq 1\}} |\sigma_z(y)|^{-n-2s}\,dy \\ &= \varepsilon e^{-|z|^2} \int_{\{|y|\geq 1\}} |y|^{-n-2s}\,dy = C\varepsilon e^{-|z|^2},\end{aligned}$$

where $C > 0$ depends only on n and s. For II, we use that $\tilde{u}(y,z) \leq \varepsilon 2^{-j(n+2s)}e^{-|z|^2}$, since $A_j \subset U_j$ and $K_\varepsilon(y,z) \leq 2^{(j+1)(n+2s)}$ in A_j, by definition. Then

$$\begin{aligned}\mathrm{II} &\le \varepsilon e^{-|z|^2} \sum_{j=0}^{\infty} 2^{-j(n+2s)} 2^{(j+1)(n+2s)} \mathcal{H}^k(A_j) \\ &\le c\varepsilon e^{-|z|^2} 2^{n+2s} \sum_{j=0}^{\infty} 2^{-kj} \le C\varepsilon e^{-|z|^2},\end{aligned}$$

where $C > 0$ depends only on n, s, and k.

Integrating over z, we see that

$$\int_{\mathbb{R}^{n-k}} \int_{\mathbb{R}^k} \tilde{u}(y,z) K_\varepsilon(y,z)\, dydz \le C\varepsilon \int_{\mathbb{R}^{n-k}} e^{-|z|^2}\, dz \le \tilde{C}\varepsilon.$$

Letting $\varepsilon \to 0$, we conclude (5.4.11). □

5.4.2 Limit as $s \to 1$

Let $u \in C^2(\mathbb{R}^n)$. We define $\mathrm{MA}_k\, u$ as the Monge-Ampère operator acting on u, with respect to the first k variables, that is,

$$\mathrm{MA}_k\, u(x) = k\Big(\det\big((u_{ij}(x))_{1\le i,j\le k}\big)\Big)^{1/k},$$

with $D^2u(x) = (u_{ij}(x))_{1\le i,j\le n}$. We define $\Delta_{n-k}u$ as the Laplacian of u, with respect to the last $n-k$ variables, that is,

$$\Delta_{n-k}u(x) = \sum_{i=k+1}^{n} u_{ii}(x).$$

Then under some *special* conditions, it holds that

$$\lim_{s\to 1} \mathcal{F}_k^s u(x) = \mathrm{MA}_k\, u(x) + \Delta_{n-k}u(x). \tag{5.4.14}$$

In particular, the family $\{\mathcal{F}_k^s\}_{k=1}^{n-1}$ can be understood as nonlocal analogs of concave second order elliptic operators, which are decomposed into a Monge-Ampère operator restricted to $\mathbb{R}^k$ and a Laplacian restricted to $\mathbb{R}^{n-k}$.

Indeed, by Corollary 5.4.5, we have $\mathcal{F}_k^s u(x) = \Delta^s \tilde{u}_{*,k}(0)$. Since the k-symmetric rearrangement does not depend on s and $\Delta^s \to \Delta$, as $s \to 1$, passing to the limit we see that

$$\lim_{s\to 1} \mathcal{F}_k^s u(x) = \Delta \tilde{u}_{*,k}(0).$$

Suppose that $\tilde{u}_{*,k}(y,z) = \tilde{u}(\varphi_z^{-1}(y), z)$, where $\varphi_z : \mathbb{R}^k \to \mathbb{R}^k$ is an invertible measure preserving transformation, with $\varphi_z(0) = 0$, and

$$\omega_k |\varphi_z(y)|^{1/k} = \sigma_z(y).$$

Recall that σ_z is given in Theorem 5.4.1 (see also Remark 5.4.4). In this case,

$$\Delta \tilde{u}_{*,k}(0) = \Delta_y \tilde{u}(\varphi_z^{-1}(y), z) + \Delta_z \tilde{u}(\varphi_z^{-1}(y), z)\Big|_{(y,z)=(0,0)}. \tag{5.4.15}$$

For the first term, we use that

$$\mathrm{MA}_k\, u(x) = \inf_{\psi \in \Psi} \Delta(\tilde{u} \circ \psi)(0),$$

where $\Psi = \big\{\psi : \mathbb{R}^k \to \mathbb{R}^k \text{ measure preserving such that } \psi(0) = 0\big\}$, and the fact that the infimum is attained when $\tilde{u} \circ \psi$ is a radially symmetric increasing function [13]. Hence,

$$\Delta_y \tilde{u}(\varphi_z^{-1}(y), z)\Big|_{(y,z)=(0,0)} = \mathrm{MA}_k\, u(x). \tag{5.4.16}$$

For the second term, call $\phi(y,z) = (\varphi_z^{-1}(y), z)$ and compute:

$$\Delta_z(\tilde{u} \circ \phi)(0) = \operatorname{tr}\left(D_z\phi(0)^T D_z^2\tilde{u}(\phi(0)) D_z\phi(0)\right) + \nabla_z\tilde{u}(\phi(0))^T \cdot \Delta_z\phi(0).$$

Recall that $\phi(0) = 0$ and $\tilde{u}(y,z) = u(x+(y,z)) - u(x) - \nabla_y u(x)\cdot y - \nabla_z u(x)\cdot z$. Then

$$\nabla_z\tilde{u}(\phi(0)) = 0, \quad D_z^2\tilde{u}(\phi(0)) = D_z^2 u(x), \quad \text{and} \quad D_z\phi(0) = (0, I_{n-k}),$$

where I_{n-k} denotes the identity matrix in M_{n-k}. Therefore,

$$\Delta_z\tilde{u}(\varphi_z^{-1}(y), z)\big|_{(y,z)=(0,0)} = \Delta_z(\tilde{u}\circ\phi)(0) = \operatorname{tr}\left(D_z^2 u(x)\right) = \Delta_{n-k}u(x). \quad (5.4.17)$$

Combining (5.4.15), (5.4.16) and (5.4.17), we conclude (5.4.14).

5.4.3 Connection to optimal transport

In Corollary 5.4.5, we obtained a representation of the function $\mathcal{F}_k^s u$ in terms of the k-symmetric increasing rearrangement. Using this representation, we find an equivalent expression of $\mathcal{F}_k^s u$ that can be understood from the viewpoint of optimal transport.

Theorem 5.4.9. *Suppose we are under the assumptions of Theorem 5.4.1. Then for any $z \in \mathbb{R}^{n-k}$, $z \neq 0$, there exists an invertible map $\varphi_z : \mathbb{R}^k \to \mathbb{R}^k$ such that*

$$\mathcal{F}_k^s u(x) = c_{n,s}\int_{\mathbb{R}^{n-k}}\int_{\mathbb{R}^k} \frac{\tilde{u}(\varphi_z^{-1}(y), z)}{\left(|y|^2+|z|^2\right)^{\frac{n+2s}{2}}}\,dydz. \quad (5.4.18)$$

Moreover, if $\sigma_z : \mathbb{R}^k \to [0,\infty)$ is the Ryff's map given in Theorem 5.4.1, then φ_z is measure preserving if and only if

$$\omega_k|\varphi_z(y)|^k = \sigma_z(y), \qquad \textit{for a.e. } y \in \mathbb{R}^k. \quad (5.4.19)$$

The key tool to prove Theorem 5.4.9 is Brenier–McCann's theorem, a very well-known result in the theory of optimal transport [7, 44]. We state it here in the form that we will use it.

Theorem 5.4.10. *Let $f, g \in L^1(\mathbb{R}^k)$. Assume that*

$$\|f\|_{L^1(\mathbb{R}^k)} = \|g\|_{L^1(\mathbb{R}^k)}.$$

Then there exists a convex function $\psi : \mathbb{R}^k \to \mathbb{R}$ whose gradient $\nabla\psi$ pushes forward $f\,dy$ to $g\,dy$. Namely, for any measurable function h in $\mathbb{R}^k$,

$$\int_{\mathbb{R}^k} h(y)g(y)\,dy = \int_{\mathbb{R}^k} h(\nabla\psi(y))f(y)\,dy. \tag{5.4.20}$$

Moreover, $\nabla\psi : \mathbb{R}^k \to \mathbb{R}^k$ is invertible and unique.

In the literature, $\nabla\psi$ is known as the (optimal) transport map.

Proof of Theorem 5.4.9. Fix $z \in \mathbb{R}^{n-k}$, $z \neq 0$, and consider $f_z, g_z \in L^1(\mathbb{R}^k)$ given by

$$f_z(y) = \left(|y|^2 + |z|^2\right)^{-\frac{n+2s}{2}} \quad \text{and} \quad g_z(y) = \left((\omega_k^{-1}\sigma_z(y))^{2/k} + |z|^2\right)^{-\frac{n+2s}{2}},$$

where $\sigma_z : \mathbb{R}^k \to [0, \infty)$ is given in Theorem 5.4.1. Note that

$$\begin{aligned}
\|f\|_{L^1(\mathbb{R}^k)} &= \int_{\mathbb{R}^k} \left((\omega_k^{-1}\sigma_z(y))^{2/k} + |z|^2\right)^{-\frac{n+2s}{2}} dy \\
&= k\omega_k \int_0^\infty \left(r^2 + |z|^2\right)^{-\frac{n+2s}{2}} r^{k-1}\,dr \\
&= \int_{\mathbb{R}^k} \left(|y|^2 + |z|^2\right)^{-\frac{n+2s}{2}} dy = \|g\|_{L^1(\mathbb{R}^k)},
\end{aligned}$$

since σ_z is measure preserving. By Theorem 5.4.10, there exists a convex function $\psi_z : \mathbb{R}^k \to \mathbb{R}$ (depending on z) whose gradient $\nabla\psi_z$ pushes forward $f_z\,dy$ to $g_z\,dy$. Moreover, $\nabla\psi_z$ is invertible and unique. Call $\varphi_z = (\nabla\psi_z)^{-1}$. Using (5.4.20), with $h(y) = \tilde{u}(y,z)$, we see that

$$\int_{\mathbb{R}^k} \frac{\tilde{u}(y,z)}{\big((\omega_k^{-1}\sigma_z(y))^{2/k} + |z|^2\big)^{\frac{n+2s}{2}}}\,dy = \int_{\mathbb{R}^k} \frac{\tilde{u}(\varphi_z^{-1}(y),z)}{\big(|y|^2+|z|^2\big)^{\frac{n+2s}{2}}}\,dy. \tag{5.4.21}$$

Integrating over $z \in \mathbb{R}^{n-k}$, we obtain (5.4.18).

It remains to show that φ_z is measure preserving if and only if (5.4.19) holds. Indeed, for any measurable set $E \subset \mathbb{R}^k$, we have

$$\begin{aligned}
\mathcal{H}^k\big(\varphi_z^{-1}(E)\big) &= \int_{\varphi_z^{-1}(E)} dy = \int_{\varphi_z^{-1}(E)} \frac{\big(|y|^2+|z|^2\big)^{\frac{n+2s}{2}}}{\big(|y|^2+|z|^2\big)^{\frac{n+2s}{2}}}\,dy \\
&= \int_{\varphi_z^{-1}(E)} \frac{\big(|\varphi_z(\varphi_z^{-1}(y))|^2+|z|^2\big)^{\frac{n+2s}{2}}}{\big(|y|^2+|z|^2\big)^{\frac{n+2s}{2}}}\,dy \\
&= \int_E \frac{\big(|\varphi_z(y)|^2+|z|^2\big)^{\frac{n+2s}{2}}}{\big(\big(\omega_k^{-1}\sigma_z(y)\big)^{2/k}+|z|^2\big)^{\frac{n+2s}{2}}}\,dy,
\end{aligned}$$

where the last equality follows from (5.4.21) with

$$h(y) = \big(|\varphi_z(y)|^2+|z|^2\big)^{\frac{n+2s}{2}}\chi_E(y).$$

Therefore,

$$\mathcal{H}^k\big(\varphi_z^{-1}(E)\big) = \mathcal{H}^k(E)$$

if and only if $\omega_k|\varphi_z(y)|^k = \sigma_z(y)$, for *a.e.* $y \in \mathbb{R}^k$.

$\square$

5.5 Regularity of $\mathcal{F}_k^s u$

Given $x_0 \in \mathbb{R}^n$, we define the sections

$$D_{x_0}u(t) = \{x \in \mathbb{R}^n : u(x) - u(x_0) - (x - x_0) \cdot \nabla u(x_0) \leq t\}, \quad \text{for } t > 0.$$

Our main regularity result is the following.

Theorem 5.5.1. *Let $s \in (1/2, 1)$ and $1 \leq k < n$. Let $u \in C^{1,1}(\mathbb{R}^n)$ be convex. Fix $x_0 \in \mathbb{R}^n$ and $r_0, \varepsilon > 0$. Suppose that $\Lambda = \sup_{x \in B_{r_0}(x_0)} \operatorname{diam}(D_x u(\varepsilon)) < \infty$ and $M = \sup_{x \in B_{r_0}(x_0)} \mathcal{F}_k^s u(x) < \infty$. Then $\mathcal{F}_k^s u \in C^{0,1-s}(\overline{B_r(x_0)})$ with $r < \min\{r_0/4, \Lambda, \varepsilon/(8\Lambda)\}$, and*

$$[\mathcal{F}_k^s]_{C^{0,1-s}(\overline{B_r(x_0)})} \leq C_0 [u]_{C^{1,1}(\mathbb{R}^n)}$$

for some constant $C_0 > 0$ depending only on n, k, s, ε, Λ, and M.

This theorem will be a consequence of the next proposition.

Proposition 5.5.2. *Fix $x_0 \in \mathbb{R}^n$ and $\varepsilon > 0$. Suppose that*

$$\Lambda = \operatorname{diam}(D_{x_0}u(\varepsilon)) < \infty \quad \text{and} \quad [u]_{C^{1,1}(\mathbb{R}^n)} \leq 1.$$

Then for any $x_1 \in B_r(x_0)$, with $r \leq \varepsilon/(4\Lambda)$, it holds that

$$\mathcal{F}_k^s u(x_1) - \mathcal{F}_k^s u(x_0) \leq C\Lambda^{1-s}|x_1 - x_0|^{1-s} + \tfrac{4\Lambda}{\varepsilon}|x_1 - x_0| \mathcal{F}_k^s u(x_0)$$

for some $C > 0$ depending only on n, k, and s.

First, we prove Theorem 5.5.1.

Proof of Theorem 5.5.1. Without loss of generality, assume that $[u]_{C^{1,1}(\mathbb{R}^n)} \leq 1$. Otherwise, we consider $u/[u]_{C^{1,1}(\mathbb{R}^n)}$. Let $r < \min\{r_0/4, \Lambda, \varepsilon/(8\Lambda)\}$. It is enough to show that

$$[\mathcal{F}_k^s]_{C^{0,1-s}(\overline{B_r(x_0)})} \leq C_0, \tag{5.5.1}$$

for some constant $C_0 > 0$ depending only on n, k, s, ε, Λ, and M.

Let $x_1, x_2 \in \overline{B_r(x_0)}$. Then $x_2 \in \overline{B_{2r}(x_1)} \subset B_{r_0}(x_0)$, since $4r < r_0$. Moreover, $\operatorname{diam}(D_{x_1}u(\varepsilon)) \leq \Lambda < \infty$. Hence, applying Proposition 5.5.2 to u, replacing $B_r(x_0)$ by $B_{2r}(x_1)$, we get

$$\begin{aligned}\mathcal{F}_k^s u(x_2) - \mathcal{F}_k^s u(x_1) &\leq C\Lambda^{1-s}|x_2 - x_1|^{1-s} + \tfrac{4\Lambda}{\varepsilon}|x_2 - x_1|\mathcal{F}_k^s u(x_1)\\ &\leq C_0|x_2 - x_1|^{1-s},\end{aligned}$$

where $C_0 = C\Lambda^{1-s} + 4\Lambda^{1+s}M/(\varepsilon 2^s)$. Since x_1 and x_2 are arbitrary, we conclude (5.5.1). $\square$

Before we prove Proposition 5.5.2, we need several preliminary results.

Lemma 5.5.3. *If f is monotone increasing, then*

$$\int_0^\infty f(r)\omega(r)\,dr = \int_0^\infty \int_{\mu_f(t)}^\infty \omega(r)\,drdt,$$

with $\mu_f(t) = |\{r > 0 : f(r) \leq t\}|$.

Proof. By Fubini's theorem, we have

$$\int_0^\infty \int_{\mu_f(t)}^\infty \omega(r)\,drdt = \int_0^\infty \omega(r) \int_{\{r > \mu_f(t)\}} dtdr.$$

Since f is monotone increasing, then $r > \mu_f(t)$ if and only if $t < f(r)$. Therefore,

$$\int_{\{r>\mu_f(t)\}} dt = \int_0^{f(r)} dt = f(r).$$

$\square$

Proposition 5.5.4. *Let $x \in \mathbb{R}^n$. Under the assumptions of Corollary 5.4.5 it holds that*

$$\mathcal{F}_k^s u(x) = c_{n,s} \int_0^\infty \int_{\mathbb{R}^{n-k}} \frac{1}{|z|^{n-k+2s}} W\Big(\frac{\mu_x u(t,z)^{1/k}}{|z|}\Big)\, dz dt,$$

where $\mu_x u(t,z) = \omega_k^{-1}\mathcal{H}^k\big(\{y \in \mathbb{R}^k : \tilde{u}_x(y,z) \le t\}\big)$ and

$$W(\rho) = k\omega_k \int_\rho^\infty \frac{r^{k-1}}{(1+r^2)^{\frac{n+2s}{2}}}\, dr. \tag{5.5.2}$$

Proof. By Corollary 5.4.5, we have that

$$\begin{aligned}\mathcal{F}_k^s u(x) = \Delta^s \tilde{u}_{*,k}(0) &= c_{n,s} \int_{\mathbb{R}^{n-k}} \frac{1}{|z|^{n+2s}} \Big(\int_{\mathbb{R}^k} \frac{\tilde{u}_{*,k}(y,z)}{\big(||z|^{-1}y|^2+1\big)^{\frac{n+2s}{2}}}\, dy\Big)\, dz \\ &= c_{n,s} \int_{\mathbb{R}^{n-k}} \frac{1}{|z|^{n-k+2s}} \Big(k\omega_k \int_0^\infty v(|z|r,z) \frac{r^{k-1}}{(r^2+1)^{\frac{n+2s}{2}}}\, dr\Big)\, dz,\end{aligned}$$

where $v(r,z) = \tilde{u}_{*,k}(y,z)$ for $|y| = r$.

Next we apply Lemma 5.5.3 to

$$f(r) = v(|z|r,z) \quad \text{and} \quad \omega(r) = k\omega_k r^{k-1}(r^2+1)^{-\frac{n+2s}{2}}.$$

Note that since v is the k-symmetric increasing rearrangement of $\tilde{u}$, we have

$$\begin{aligned}\mu_f(t) &= \tfrac{1}{|z|}|\{r>0 : v(r,z) < t\}| \\ &= \tfrac{\omega_k^{-1/k}}{|z|}\mathcal{H}^k\big(\{y \in \mathbb{R}^k : \tilde{u}(y,z) < t\}\big)^{1/k} \\ &= \tfrac{1}{|z|}\mu_x u(t,z)^{1/k}.\end{aligned}$$

Therefore,

$$k\omega_k \int_0^\infty v(|z|r,z) \frac{r^{k-1}}{(r^2+1)^{\frac{n+2s}{2}}}\,dr = \int_0^\infty \Big(k\omega_k \int_{\mu_x u(t,z)^{1/k}/|z|}^\infty \frac{r^{k-1}}{(r^2+1)^{\frac{n+2s}{2}}}\,dr \Big) dt$$
$$= \int_0^\infty W\Big(\frac{\mu_x u(t,z)^{1/k}}{|z|}\Big)\,dt,$$

where W is given in (5.5.2). By Fubini's theorem, we conclude that

$$\mathcal{F}_k^s u(x) = c_{n,s} \int_0^\infty \int_{\mathbb{R}^{n-k}} \frac{1}{|z|^{n-k+2s}} W\Big(\frac{\mu_x u(t,z)^{1/k}}{|z|}\Big)\,dzdt.$$

□

Lemma 5.5.5. *Suppose we are under the assumptions of Proposition 5.5.2. Let $x_1 \in \overline{B_r(x_0)}$ and $d = |x_1 - x_0|$. The following holds:*

(a) If $t \in (2\Lambda d, \varepsilon]$, then $D_{x_0}u(t - 2\Lambda d) \subset D_{x_1}u(t)$.

(b) If $t \in (\varepsilon, \infty)$, then $D_{x_0}u\big(t - 2\Lambda dt/\varepsilon\big) \subset D_{x_1}u(t)$.

Proof. First we prove (a). Fix $t \in (2\Lambda d, \varepsilon]$ and let $x \in D_{x_0}u(t - 2\Lambda d)$. Then

$$u(x) - u(x_0) - (x - x_0)\cdot \nabla u(x_0) \le t - 2\Lambda d. \tag{5.5.3}$$

Using (5.5.3), convexity, and $[u]_{C^{1,1}(\mathbb{R}^n)} \le 1$, we see that

$$\begin{aligned} u(x) - u(x_1) - (x - x_1)\cdot\nabla u(x_1) &= u(x) - u(x_0) - (x - x_0)\cdot\nabla u(x_0) \\ &\quad - \big(u(x_1) - u(x_0) - (x_1 - x_0)\cdot\nabla u(x_0)\big) \\ &\quad + (x - x_1)\cdot(\nabla u(x_0) - \nabla u(x_1)) \\ &\le t - 2\Lambda d + |x - x_1| d. \end{aligned}$$

Moreover, $x \in D_{x_0}u(\varepsilon)$, since $t \leq \varepsilon$, and thus,

$$|x - x_1| \leq |x - x_0| + |x_0 - x_1| \leq \Lambda + d \leq 2\Lambda.$$

Therefore, $x \in D_{x_1}u(t)$.

Next we prove (b). Fix $t \in (\varepsilon, \infty)$ and let $x \in D_{x_0}u\big(t - 2\Lambda dt/\varepsilon\big)$. By the previous computation, we have that

$$u(x) - u(x_1) - (x - x_1) \cdot \nabla u(x_1) \leq t - 2\Lambda dt/\varepsilon + (|x - x_0| + \Lambda)d. \tag{5.5.4}$$

To control the distance from x to x_0, we need to estimate the diameter of $D_{x_0}u(t)$. Take $y \in D_{x_0}u(t) \setminus D_{x_0}u(\varepsilon)$ and let z be in the intersection between $\partial D_{x_0}u(\varepsilon)$ and the line segment joining x_0 and y. Then there is some $\lambda > 1$ such that $y - x_0 = \lambda(z - x_0)$. By convexity of u,

$$u(z) \leq \tfrac{\lambda-1}{\lambda}u(x_0) + \tfrac{1}{\lambda}u(y).$$

Therefore,

$$\begin{aligned}\lambda\varepsilon &= \lambda\big(u(z) - u(x_0) - (z - x_0) \cdot \nabla u(x_0)\big) \\ &\leq (\lambda - 1)u(x_0) + u(y) - \lambda u(x_0) - (y - x_0) \cdot \nabla u(x_0) \\ &= u(y) - u(x_0) - (y - x_0) \cdot \nabla u(x_0) \leq t,\end{aligned}$$

so $\lambda \leq t/\varepsilon$. By convexity, we have that $D_{x_0}u(t) \subset x_0 + \frac{t}{\varepsilon}(D_{x_0}u(\varepsilon) - x_0)$. It follows that

$$\operatorname{diam} D_{x_0}u(t) \leq t/\varepsilon \operatorname{diam} D_{x_0}u(\varepsilon) = \Lambda t/\varepsilon.$$

Hence, $|x - x_0| \leq \Lambda t/\varepsilon$, and by (5.5.4), we get

$$u(x) - u(x_1) - (x - x_1) \cdot \nabla u(x_1) \leq t - 2\Lambda dt/\varepsilon + (\Lambda t/\varepsilon + \Lambda)d \leq t,$$

which means that $x \in D_{x_1}u(t)$. $\square$

We are ready to give the proof of Proposition 5.5.2.

Proof of Proposition 5.5.2. Let $x_1 \in B_r(x_0)$, with $r \leq \varepsilon/(4\Lambda)$, and call $d = |x_0 - x_1|$. We will estimate $\mathcal{F}_k^s u(x_1)$ using Proposition 5.5.4:

$$\mathcal{F}_k^s u(x_1) = c_{n,s} \int_0^\infty \int_{\mathbb{R}^{n-k}} \frac{1}{|z|^{n-k+2s}} W\Big(\frac{\mu_{x_1} u(t,z)^{1/k}}{|z|}\Big)\, dz dt.$$

In view of Lemma 5.5.5, we divide the integral with respect to t in three parts:

$$\text{I. } t \in (0, 2\Lambda d], \quad \text{II. } t \in (2\Lambda d, \varepsilon], \quad \text{III. } t \in (\varepsilon, \infty).$$

Let us start with I. Since $u \in C^{1,1}(\mathbb{R}^n)$ with $[u]_{C^{1,1}(\mathbb{R}^n)} \leq 1$, then

$$\mu_{x_1} u(t,z) \geq (t - |z|^2)_+^{k/2}.$$

Hence, using that $W(\rho)$ is monotone decreasing, we get

$$W\Big(\frac{\mu_{x_1} u(t,z)^{1/k}}{|z|}\Big) \leq W\Big(\big(\tfrac{t}{|z|^2} - 1\big)_+^{\frac{1}{2}}\Big).$$

Therefore,

$$\begin{aligned}
&\int_{\mathbb{R}^{n-k}} \frac{1}{|z|^{n-k+2s}} W\Big(\frac{\mu_{x_1} u(t,z)^{1/k}}{|z|}\Big)\, dz \\
&\quad \leq \int_{\{|z|<t^{1/2}\}} \frac{1}{|z|^{n-k+2s}} W\Big(\big(\tfrac{t}{|z|^2} - 1\big)^{\frac{1}{2}}\Big)\, dz + W(0) \int_{\{|z|>t^{1/2}\}} \frac{1}{|z|^{n-k+2s}}\, dz \\
&\quad \equiv I_1 + I_2.
\end{aligned}$$

Note that $W(0) = C(n,k,s) < \infty$. Then

$$I_2 \lesssim \int_{t^{1/2}}^\infty \frac{1}{\rho^{n-k+2s}} \rho^{n-k-1}\, d\rho \approx t^{-s}.$$

For I_1, we make the change of variables, $w = z/t^{1/2}$. We see that

$$\begin{aligned}
I_1 &= \int_{\{|w|<1\}} \frac{1}{t^{\frac{n-k+2s}{2}}|w|^{n-k+2s}} W\big(\big(\tfrac{1}{|w|^2}-1\big)^{\frac{1}{2}}\big) t^{\frac{n-k}{2}}\, dw \\
&\approx \frac{1}{t^s}\int_0^1 \frac{1}{\rho^{1+2s}} W\big(\big(\tfrac{1}{\rho^2}-1\big)^{\frac{1}{2}}\big)\, d\rho.
\end{aligned}$$

Note that if $0 < \rho \le 1/2$, then $\big(\frac{1}{\rho^2}-1\big)^{\frac{1}{2}} \ge \frac{1}{\sqrt{2}\rho}$. Hence,

$$W\big(\big(\tfrac{1}{\rho^2}-1\big)^{\frac{1}{2}}\big) \le W\big(\tfrac{1}{\sqrt{2}\rho}\big) = \int_{\frac{1}{\sqrt{2}\rho}}^{\infty} \frac{r^{k-1}}{(1+r^2)^{\frac{n+2s}{2}}}\, dr \lesssim \rho^{n-k+2s}.$$

Therefore,

$$I_1 \lesssim t^{-s}\int_0^{1/2} \frac{1}{\rho^{1+2s}}\rho^{n-k+2s}\, d\rho + t^{-s}W(0)\int_{1/2}^1 \frac{1}{\rho^{1+2s}}\, d\rho \approx t^{-s},$$

since $n-k>0$. We conclude that

$$\begin{aligned}
\mathrm{I} &= c_{n,s}\int_0^{2\Lambda d}\int_{\mathbb{R}^{n-k}} \frac{1}{|z|^{n-k+2s}} W\Big(\frac{\mu_{x_1}u(t,z)^{1/k}}{|z|}\Big)\, dz dt \\
&\lesssim \int_0^{2\Lambda d} t^{-s}\, dt \approx (2\Lambda d)^{1-s} = (2\Lambda)^{1-s}|x_1-x_0|^{1-s}.
\end{aligned}$$

Next we estimate the integral for $t \in (2\Lambda d, \varepsilon]$. To this end, we use Lemma 5.5.5, part (a):

$$D_{x_0}u(t-2\Lambda d) \subset D_{x_1}u(t).$$

In particular, for any $z \in \mathbb{R}^{n-k}$ fixed, we have

$$\{y \in \mathbb{R}^k : \tilde{u}_{x_0}(y,z) \le t - 2\Lambda d\} \subset \{y \in \mathbb{R}^k : \tilde{u}_{x_1}(y,z) \le t\}.$$

Hence, $\mu_{x_0}(t - 2\Lambda d, z) \leq \mu_{x_1}(t, z)$, which yields

$$\begin{aligned}
\mathrm{II} &= c_{n,s} \int_{2\Lambda d}^{\varepsilon} \int_{\mathbb{R}^{n-k}} \frac{1}{|z|^{n-k+2s}} W\Big(\frac{\mu_{x_1} u(t,z)^{1/k}}{|z|}\Big)\, dzdt \\
&\leq c_{n,s} \int_{0}^{\varepsilon - 2\Lambda d} \int_{\mathbb{R}^{n-k}} \frac{1}{|z|^{n-k+2s}} W\Big(\frac{\mu_{x_0} u(t,z)^{1/k}}{|z|}\Big)\, dzdt.
\end{aligned}$$

Finally, we estimate the integral for $t \in [\varepsilon, \infty)$. By Lemma 5.5.5, part (b):

$$D_{x_0} u\big(t - 2\Lambda dt/\varepsilon\big) \subset D_{x_1} u(t).$$

Hence, $\mu_{x_0} u(t - 2\Lambda dt/\varepsilon, z) \leq \mu_{x_1} u(t, z)$, and

$$\begin{aligned}
\mathrm{III} &= c_{n,s} \int_{\varepsilon}^{\infty} \int_{\mathbb{R}^{n-k}} \frac{1}{|z|^{n-k+2s}} W\Big(\frac{\mu_{x_1} u(t,z)^{1/k}}{|z|}\Big)\, dzdt \\
&\lesssim \int_{\varepsilon}^{\infty} \int_{\mathbb{R}^{n-k}} \frac{1}{|z|^{n-k+2s}} W\Big(\frac{\mu_{x_0} u(t - 2\Lambda dt/\varepsilon, z)^{1/k}}{|z|}\Big)\, dzdt \\
&= \frac{1}{1 - 2\Lambda d/\varepsilon} \int_{\varepsilon - 2\Lambda d}^{\infty} \int_{\mathbb{R}^{n-k}} \frac{1}{|z|^{n-k+2s}} W\Big(\frac{\mu_{x_0} u(t,z)^{1/k}}{|z|}\Big)\, dzdt.
\end{aligned}$$

Note that

$$\begin{aligned}
\mathrm{II} + \mathrm{III} &\leq \frac{c_{n,s}}{1 - 2\Lambda d/\varepsilon} \int_{0}^{\infty} \int_{\mathbb{R}^{n-k}} \frac{1}{|z|^{n-k+2s}} W\Big(\frac{\mu_{x_0} u(t,z)^{1/k}}{|z|}\Big)\, dzdt \\
&= \frac{\varepsilon}{\varepsilon - 2\Lambda d} \mathcal{F}_k^s u(x_0).
\end{aligned}$$

Therefore, we conclude that

$$\begin{aligned}
\mathcal{F}_k^s u(x_1) - \mathcal{F}_k^s u(x_0) &\leq C\Lambda^{1-s} |x_1 - x_0|^{1-s} + \big(\tfrac{\varepsilon}{\varepsilon - 2\Lambda d} - 1\big) \mathcal{F}_k^s u(x_0) \\
&\leq C\Lambda^{1-s} |x_1 - x_0|^{1-s} + \tfrac{4\Lambda}{\varepsilon} |x_1 - x_0| \mathcal{F}_k^s u(x_0)
\end{aligned}$$

since $d < r \leq \varepsilon/(4\Lambda)$, and thus, $\varepsilon - 2\Lambda d \geq \varepsilon/2$. $\square$

5.6 A Global Poisson Problem

We consider the following Poisson problem in the full space:

$$\begin{cases} \mathcal{F}_k^s u = u - \varphi & \text{in } \mathbb{R}^n \\ (u-\varphi)(x) \to 0 & \text{as } |x| \to \infty, \end{cases} \tag{5.6.1}$$

where $\varphi : \mathbb{R}^n \to \mathbb{R}$ is nonnegative, smooth, and strictly convex. Furthermore, we ask that φ behaves asymptotically at infinity as a cone ϕ, that is,

$$\lim_{|x|\to\infty} (\varphi - \phi)(x) = 0. \tag{5.6.2}$$

Similar problems have been studied for nonlocal Monge-Ampère operators in [8, 13].

We will prove the following theorem.

Theorem 5.6.1. *There exists a unique solution u to* (5.6.1) *such that $u \in C^{1,1}(\mathbb{R}^n)$ with*

$$[u]_{C^{1,1}(\mathbb{R}^n)} \leq [\varphi]_{C^{1,1}(\mathbb{R}^n)}.$$

To define the notion of solution, we introduce a natural pointwise definition of $\mathcal{F}_k^s u$ for functions u that are merely continuous.

Definition 5.6.2. Let $u \in C^0(\mathbb{R}^n)$.

(*a*) We say that a linear function $l(y) = y \cdot p + b$, with $p \in \mathbb{R}^n$, and $b \in \mathbb{R}$, is a supporting plane of u at a point x if $l(x) = u(x)$ and $l(y) \leq u(y)$, for all $y \in \mathbb{R}^n$.

(b) We define the subdifferential of u at a point x as the set $\partial u(x)$ of all vectors $p \in \mathbb{R}^n$ such that $l(y) = y \cdot p + b$ is a supporting plane of u at x, for some $b \in \mathbb{R}$.

Definition 5.6.3. Let $u \in C^0(\mathbb{R}^n)$ be a convex function. For $x_0 \in \mathbb{R}^n$, we define

$$\mathcal{F}_k^s u(x_0) = c_{n,s} \sup_{p \in \partial u(x_0)} \inf_{K \in \mathcal{K}_k^s} \int_{\mathbb{R}^n} (u(x_0 + x) - u(x_0) - x \cdot p) K(x) \, dx.$$

Remark 5.6.4. Note that if $u \in C^{1,1}(x_0)$, then $\partial u(x_0) = \{\nabla u(x_0)\}$, and the previous definition coincides with Definition 5.2.4.

The following properties of $\mathcal{F}_k^s u$ will be useful for our purposes. The proof is analogous to the one in [13], so we omit it here.

Lemma 5.6.5. *Let $u, v \in C^0(\mathbb{R}^n)$ be convex functions.The following holds:*

(a) *(Homogeneity). For any $\lambda > 0$,*

$$\mathcal{F}_k^s(\lambda u) = \lambda \mathcal{F}_k^s u.$$

(b) *(Monotonicity). Assume that $u(x_0) = v(x_0)$ and $u(x) \geq v(x)$ for all $x \in \mathbb{R}^n$. Then*

$$\mathcal{F}_k^s u(x_0) \geq \mathcal{F}_k^s v(x_0).$$

(c) *(Concavity). For any $x \in \mathbb{R}^n$,*

$$\mathcal{F}_k^s\Big(\frac{u+v}{2}\Big)(x) \geq \frac{\mathcal{F}_k^s u(x) + \mathcal{F}_k^s v(x)}{2}.$$

(*d*) (*Lower semicontinuity*). *Assume that* $u \in C^{1,1}(\mathbb{R}^n)$. *Then*

$$\mathcal{F}_k^s u(x_0) \leq \liminf_{x \to x_0} \mathcal{F}_k^s u(x).$$

Definition 5.6.6. Let $u \in C^0(\mathbb{R}^n)$ be a convex function. We say that u is a subsolution to $\mathcal{F}_k^s u = u - \varphi$ in $\mathbb{R}^n$ if

$$\mathcal{F}_k^s u(x_0) \geq u(x_0) - \varphi(x_0), \quad \text{for all } x_0 \in \mathbb{R}^n.$$

Similarly, u is a supersolution if

$$\mathcal{F}_k^s u(x_0) \leq u(x_0) - \varphi(x_0), \quad \text{for all } x_0 \in \mathbb{R}^n.$$

We say that u is a solution if it is both a subsolution and a supersolution.

Lemma 5.6.7. *If u and v are subsolutions, then* $\max\{u, v\}$ *is a subsolution.*

Proof. Let $w = \max\{u, v\}$. Then w is continuous and convex. Fix $x_0 \in \mathbb{R}^n$. Without loss of generality, we may assume that $u(x_0) \geq v(x_0)$. Then $w(x_0) = u(x_0)$ and $w(x) \geq u(x)$, for any $x \in \mathbb{R}^n$. By monotonicity (see Lemma 5.6.5), we have

$$\mathcal{F}_k^s w(x_0) \geq \mathcal{F}_k^s u(x_0) \geq u(x_0) - \varphi(x_0) = w(x_0) - \varphi(x_0).$$

Hence, w is a subsolution. □

We will show existence and uniqueness of solutions to (5.6.1) using Perron's method. The key ingredients are the comparison principle, and the existence of a subsolution (lower barrier) and a supersolution (upper barrier). We state this in the following proposition. We omit the proof since it is similar to that in [13].

Proposition 5.6.8. *Consider the equation $\mathcal{F}_k^s u = u - \varphi$ in $\mathbb{R}^n$. The following holds:*

(a) *(Comparison principle). Let u and v be a subsolution and supersolution, respectively. Assume that $u \leq v$ in $\mathbb{R}^n \setminus \Omega$, for some bounded domain $\Omega \subset \mathbb{R}^n$. Then $u \leq v$ in $\mathbb{R}^n$.*

(b) *(Lower-barrier). The function φ is a subsolution.*

(c) *(Upper-barrier). The function $\varphi + w$ is a supersolution, where $w = (I - \Delta^s)^{-1}\Delta^s\varphi$. In particular, $w(x) \leq C(1+|x|)^{1-2s}$, for some $C > 0$.*

An immediate consequence of the comparison principle is the uniqueness of solutions.

Lemma 5.6.9 (Uniqueness)**.** *There exists at most one solution to* (5.6.1).

Proof. Suppose by means of contradiction that there exist two functions $u, v \in C^0(\mathbb{R}^n)$ with $u \neq v$, satisfying (5.6.1). Then $|u(x) - v(x)| \to 0$, as $|x| \to \infty$. Hence, for any $\varepsilon > 0$, there exists a compact set $\Omega_\varepsilon \in \mathbb{R}^n$, depending on ε, such that

$$v(x) - \varepsilon \leq u(x) \leq v(x) + \varepsilon \quad \text{for all } x \in \mathbb{R}^n \setminus \Omega_\varepsilon.$$

Moreover, for any $x_0 \in \mathbb{R}^n$, the function $v + \varepsilon$ satisfies

$$\mathcal{F}_k^s(v+\varepsilon)(x_0) = v(x_0) - \varphi(x_0) < (v(x_0) + \varepsilon) - \varphi(x_0).$$

Therefore, v is a supersolution and by the comparison principle, it follows that $u \leq v + \varepsilon$ in $\mathbb{R}^n$. Similarly, we see that $v - \varepsilon$ is a subsolution and $u \geq v - \varepsilon$ in $\mathbb{R}^n$. Hence,

$$\|u - v\|_{L^\infty(\mathbb{R}^n)} \leq \varepsilon,$$

and letting $\varepsilon \to 0$, we get $u = v$ in $\mathbb{R}^n$, which is a contradiction. $\square$

To prove existence of a solution, we define

$$u(x) = \sup_{v \in \mathcal{S}} v(x), \tag{5.6.3}$$

where $\mathcal{S}$ is the set of admissible subsolutions given by

$$\mathcal{S} = \big\{ v \in C^{0,1}(\mathbb{R}^n) : v \text{ subsolution, } \varphi \leq v \leq \varphi + w, \\ \text{and } [v]_{C^{0,1}(\mathbb{R}^n)} \leq [\varphi]_{C^{0,1}(\mathbb{R}^n)} \big\}.$$

Note that $\mathcal{S} \neq \emptyset$ since $\varphi \in \mathcal{S}$, and the supremum is finite since $v \leq \varphi + w$, for any $v \in \mathcal{S}$. Moreover, u is convex and Lipschitz, with

$$[u]_{C^{0,1}(\mathbb{R}^n)} \leq [\varphi]_{C^{0,1}(\mathbb{R}^n)}.$$

From $\varphi \leq u \leq \varphi + w$, and the upper bound for w in Proposition 5.6.8, it follows that

$$0 \leq (u - \varphi)(x) \leq w(x) \leq C(1 + |x|)^{1-2s} \to 0,$$

as $|x| \to \infty$ since $1 - 2s < 0$.

Proposition 5.6.10. *The function u given in* (5.6.3) *is $C^{1,1}(\mathbb{R}^n)$ with*

$$[u]_{C^{1,1}(\mathbb{R}^n)} \leq [\varphi]_{C^{1,1}(\mathbb{R}^n)}.$$

Proof. We will show that for any $x_0, x_1 \in \mathbb{R}^n$,

$$0 \leq u(x_0 + x_1) - u(x_0 - x_1) - 2u(x_0) \leq [\varphi]_{C^{1,1}(\mathbb{R}^n)}|x_1|^2.$$

Indeed, the lower bound follows from convexity of u. Hence, we only need to prove the upper bound. Call $M = [\varphi]_{C^{1,1}(\mathbb{R}^n)}$. Then

$$\varphi(x_0 + x_1) - \varphi(x_0 - x_1) - M|x_1|^2 \leq 2\varphi(x_0). \tag{5.6.4}$$

Take any $v \in \mathcal{S}$ and fix $x_1 \in \mathbb{R}^n$. Define

$$\hat{v}(x_0) = \tfrac{1}{2}\big(v(x_0 + x_1) + v(x_0 - x_1) - M|x_1|^2\big), \quad \text{for } x_0 \in \mathbb{R}^n.$$

We claim that $\hat{v}$ is a subsolution to $\mathcal{F}_k^s u = u - \varphi$ in $\mathbb{R}^n$. Indeed, since $\mathcal{F}_k^s$ is homogeneous of degree 1, concave, and translation invariant (see Lemma 5.6.5), we have

$$\begin{aligned}
\mathcal{F}_k^s \hat{v}(x_0) &= \mathcal{F}_k^s\Big(\tfrac{1}{2}v(x_0 + x_1) + \tfrac{1}{2}v(x_0 - x_1)\Big) \\
&\geq \tfrac{1}{2}\mathcal{F}_k^s v(x_0 + x_1) + \tfrac{1}{2}\mathcal{F}_k^s v(x_0 - x_1) \\
&\geq \tfrac{1}{2}\big(v(x_0 + x_1) - \varphi(x_0 + x_1) + v(x_0 - x_1) - \varphi(x_0 - x_1)\big) \\
&= \tfrac{1}{2}\big(v(x_0 + x_1) - v(x_0 - x_1) - M|x_1|^2\big) \\
&\qquad - \tfrac{1}{2}\big(\varphi(x_0 + x_1) + \varphi(x_0 - x_1) - M|x_1|^2\big) \\
&\geq \hat{v}(x_0) - \varphi(x_0).
\end{aligned}$$

Moreover, using that $v \leq \varphi + w$, we get

$$\hat{v}(x_0) \leq \tfrac{1}{2}\big(\varphi(x_0 + x_1) + \varphi(x_0 - x_1) - M|x_1|^2\big) + \tfrac{1}{2}\big(w(x_0 + x_1) + w(x_0 - x_1)\big).$$

By (5.6.4) and the upper bound of w in Proposition 5.6.8, part (c), we see that

$$\hat{v}(x_0) - \varphi(x_0) \leq \tfrac{C}{2}(1 + |x_0 + x_1|^{1-2s}) + \tfrac{C}{2}(1 + |x_0 - x_1|)^{1-2s}) \to 0,$$

as $|x_0| \to \infty$ and x_1 fixed, since $1 - 2s < 0$. Then for all $\varepsilon > 0$, there is some compact set Ω_ε, depending on ε and x_1, such that

$$\hat{v}(x_0) - \varepsilon \leq \varphi(x_0), \quad \text{for all } x_0 \in \mathbb{R}^n \setminus \Omega_\varepsilon.$$

Consider $\hat{v}_\varepsilon = \max\{\hat{v} - \varepsilon, \varphi\}$. Then $\hat{v}_\varepsilon$ is a subsolution, since the maximum of subsolutions is a subsolution (see Lemma 5.6.7). Also, $\hat{v}_\varepsilon = \varphi \leq \varphi + w$ in $\mathbb{R}^n \setminus \Omega_\varepsilon$, and $\varphi + w$ is a supersolution by Proposition 5.6.8, part (c). Applying the comparison principle, we get $\varphi \leq \hat{v}_\varepsilon \leq \varphi + w$. Moreover, $[\hat{v}_\varepsilon]_{C^{0,1}(\mathbb{R}^n)} \leq [\varphi]_{C^{0,1}(\mathbb{R}^n)}$. Therefore, $\hat{v}_\varepsilon \in \mathcal{S}$.

Since $u(x_0) = \sup_{v \in \mathcal{S}} v(x_0)$, it follows that $u(x_0) \geq \hat{v}_\varepsilon(x_0) \geq \hat{v}(x_0) - \varepsilon$. Letting $\varepsilon \to 0$, we conclude that for any $v \in \mathcal{S}$ and $x_0, x_1 \in \mathbb{R}^n$,

$$u(x_0) \geq \tfrac{1}{2}\big(v(x_0 + x_1) + v(x_0 - x_1) - M|x_1|^2\big). \tag{5.6.5}$$

Finally, by definition of supremum, for any $\delta > 0$, and $x_0, x_1 \in \mathbb{R}^n$, there exist $v_1, v_2 \in \mathcal{S}$ such that $u(x_0+x_1) - \delta < v_1(x_0+x_1)$ and $u(x_0-x_1) - \delta < v_2(x_0-x_1)$. Let $v = \max\{v_1, v_2\}$. Then using (5.6.5) for this v, we get

$$u(x_0) \geq \tfrac{1}{2}\big(u(x_0 + x_1) - \delta + u(x_0 - x_1) - \delta - M|x_1|^2\big).$$

Letting $\delta \to 0$, we conclude that

$$u(x_0 + x_1) - u(x_0 - x_1) - 2u(x_0) \leq [\varphi]_{C^{1,1}(\mathbb{R}^n)}|x_1|^2.$$

$\square$

To complete the proof of Theorem 5.6.1, it remains to see that u is a solution. Hence, we need to show that u is both a subsolution and a supersolution. We will prove these results in the next two propositions.

Lemma 5.6.11. *For any $x_0 \in \mathbb{R}^n$ and $\varepsilon > 0$, the set*

$$D_{x_0}u(\varepsilon) = \big\{x \in \mathbb{R}^n : u(x) - u(x_0) - (x - x_0) \cdot \nabla u(x_0) \leq \varepsilon\big\}$$

is compact.

Proof. Let $x_0 \in \mathbb{R}^n$ and $\varepsilon > 0$. Without loss of generality, we may assume that $x_0 = 0$. Let l be the supporting plane of u at 0, that is, $l(x) = u(0) + x \cdot \nabla u(0)$. Clearly, $D_{x_0}u(\varepsilon)$ is closed. Hence, we only need to show that it is bounded. Recall that

$$\phi(x) < \varphi(x) \leq u(x), \quad \text{for all } x \in \mathbb{R}^n, \tag{5.6.6}$$

where ϕ is a cone. Note that the strict inequality in (5.6.6) follows from the strict convexity of φ. Moreover, by (5.6.1) and (5.6.2), we have

$$\lim_{|x|\to\infty} (u - \phi)(x) = 0.$$

Therefore, $D_{x_0}u(\varepsilon) \subset \{\phi < l + \varepsilon\}$. We claim that

$$\lim_{|x|\to\infty} (\phi - l)(x) = \infty. \tag{5.6.7}$$

If this condition holds, then for all $M > 0$, there exists $R > 0$, such that

$$\phi(x) - l(x) > M, \quad \text{for all } |x| > R.$$

Choosing $M = \varepsilon$, we see that $\{\phi < l + \varepsilon\} \subset B_R$, for some R depending on ε. Hence, the set $D_{x_0}u(\varepsilon)$ is bounded.

To prove the claim, we distinguish two cases. If $u(0) = 0$, then u attains an absolute minimum at 0, so $\nabla u(0) = 0$. In particular, $l(x) = 0$, for all $x \in \mathbb{R}^n$, and thus, (5.6.7) is clearly satisfied. Hence, it remains to show the claim when

$$u(0) > 0.$$

We will prove it by contradiction. If (5.6.7) is not true, then there exists a sequence of points $\{x_j\}_{j=1}^{\infty} \subset \mathbb{R}^n$ such that $|x_j| \to \infty$, as $j \to \infty$, and

$$\lim_{j\to\infty} (\phi - l)(x_j) < \infty.$$

Using that ϕ is continuous and homogeneous of degree 1, and letting $j \to \infty$, we get

$$\tfrac{\phi(x_j)}{|x_j|} - \tfrac{l(x_j)}{|x_j|} = \phi\big(\tfrac{x_j}{|x_j|}\big) - \tfrac{u(0)}{|x_j|} - \tfrac{x_j}{|x_j|} \cdot \nabla u(0) \to \phi(e) - D_e u(0) = 0,$$

where $x_j/|x_j| \to e$, up to a subsequence. Therefore, $\phi(e) = D_e u(0)$. For any $\lambda > 0$, we have

$$l(\lambda e) = u(0) + \lambda e \cdot \nabla u(0) = u(0) + \lambda\phi(e) = u(0) + \phi(\lambda e).$$

Since l is a supporting plane of u, we know that $u(x) \geq l(x)$, for all $x \in \mathbb{R}^n$, and thus,

$$u(\lambda e) \geq l(\lambda e) = \phi(\lambda e) + u(0).$$

Letting $\lambda \to \infty$, we see that

$$0 = \lim_{\lambda \to \infty} (u - \phi)(\lambda e) \geq u(0) > 0,$$

which is a contradiction.

$\square$

Proposition 5.6.12 (u is a subsolution)**.** *The function* u *given in* (5.6.3) *satisfies*

$$\mathcal{F}_k^s u(x_0) \geq u(x_0) - \varphi(x_0), \quad \textit{for all } x_0 \in \mathbb{R}^n.$$

Proof. By Proposition 5.6.10, we know that $u \in C^{1,1}(\mathbb{R}^n)$. Without loss of generality, we may assume that $[u]_{C^{1,1}(\mathbb{R}^n)} = 1$. Otherwise, consider $u/[u]_{C^{1,1}(\mathbb{R}^n)}$.

Let $x_0 \in \mathbb{R}^n$. Then the quadratic polynomial

$$P(x) = u(x_0) + \nabla u(x_0) \cdot (x - x_0) + |x - x_0|^2$$

touches u from above at x_0. Moreover, we may assume that P touches u strictly from above at x_0. If not, we replace P by $P + \varepsilon |x - x_0|^2$ with $\varepsilon > 0$ small.

Fix $\delta > 0$. Then there exists $h > 0$, with $h \to 0$ as $\delta \to 0$, such that

$$P(x) - u(x) \geq h > 0, \quad \text{for all } x \in \mathbb{R}^n \setminus B_\delta(x_0).$$

Since $u(x) = \sup_{v \in \mathcal{S}} v(x)$ and $v \in \mathcal{S}$ is uniformly continuous, there is a monotone sequence $\{v_j\}_{j=1}^\infty \subset \mathcal{S}$ such that $v_j \to u$ uniformly in compact subsets of $\mathbb{R}^n$. In particular, there exists $j_0 \geq 1$, depending on h, such that for all $j > j_0$,

$$u(x) - h < v_j(x), \quad \text{for all } x \in \overline{B_\delta(x_0)}. \tag{5.6.8}$$

Call $v = v_j$ for some $j > j_0$. It follows that

$$\begin{cases} P - v \geq h & \text{in } \mathbb{R}^n \setminus B_\delta(x_0) \\ P - v < P - u + h & \text{in } B_\delta(x_0). \end{cases}$$

Let $d = \inf_{\mathbb{R}^n}(P - v)$. Then $d = P(x_1) - v(x_1)$, for some $x_1 \in \overline{B_h(x_0)}$, with $0 \leq d < h$, and $P(x) - d \geq v(x)$, for all $x \in \mathbb{R}^n$. Hence, $P - d$ is a quadratic polynomial that touches v from above at x_1. In particular, since v is convex, then v has a unique supporting plane l at x_1, so $\partial v(x_1) = \{\nabla l\}$.

Let $\tau \geq 0$ be such that $l + \tau$ is the supporting plane of u at some point x_2. Note that x_2 approaches x_0 as h goes to 0, and thus, there exists some $r = r(h) > 0$ such that $r \to 0$, as $h \to 0$, and $x_2 \in B_r(x_0)$. Furthermore, since $l(x_1) + d = v(x_1) + d = P(x_1) \geq u(x_1)$, then $\tau \leq d < h$ (see Figure 5.2).

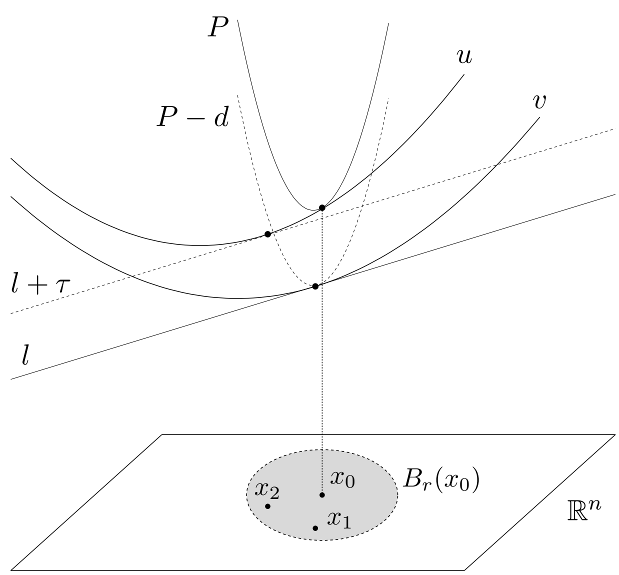

Figure 5.2: Geometry involved in the proof of Proposition 5.6.12.

Fix $\varepsilon > 0$. By Lemma 5.6.11, we have that $D_{x_0}u(\varepsilon)$ is bounded, so

$\Lambda = \operatorname{diam} D_{x_0}u(\varepsilon) < \infty$. Choose δ sufficiently small, so that $r < \varepsilon/(4\Lambda)$. Then by Proposition 5.5.2, it holds that

$$\begin{aligned}\mathcal{F}_k^s u(x_2) &\le \mathcal{F}_k^s u(x_0) + C\Lambda^{1-s}|x_2 - x_0|^{1-s} + \tfrac{4\Lambda}{\varepsilon}\mathcal{F}_k^s u(x_0)|x_2 - x_0| \\ &\le \mathcal{F}_k^s u(x_0) + C(r), \end{aligned} \tag{5.6.9}$$

where $C(r) \to 0$, as $r \to 0$. Next we will show that

$$\mathcal{F}_k^s v(x_1) - C\tau^{1-s} \le \mathcal{F}_k^s u(x_2) \tag{5.6.10}$$

for some constant $C > 0$ depending only on n, k, and s. Since $\partial v(x_1) = \{\nabla l\}$, then $v \in C^{1,1}(x_1)$, and using Proposition 5.5.4, we get

$$\mathcal{F}_k^s v(x_1) = c_{n,s}\int_0^\infty \int_{\mathbb{R}^{n-k}} \frac{1}{|z|^{n-k+2s}} W\Big(\frac{\mu_{x_1}v(t,z)^{1/k}}{|z|}\Big)\, dzdt,$$

where $\mu_x v(t,z) = \omega_k^{-1}\mathcal{H}^k\big(\{y \in \mathbb{R}^k : \tilde{v}_x(y,z) \le t\}\big)$, and W is the monotone decreasing function given in (5.5.2). Observe that since $v \le u$, l is the supporting plane of v at x_1, and $l + \tau$ is the supporting plane of u at x_2, then for any $t > 0$, it follows that

$$D_{x_2}u(t) = \{u - (l+\tau) \le t\} \subseteq \{v - l \le t + \tau\} = D_{x_1}v(t+\tau).$$

In particular, $\mu_{x_2}u(t,z) \le \mu_{x_1}v(t+\tau,z)$, for any $z \in \mathbb{R}^{n-k}$. Therefore,

$$W(\mu_{x_2}u(t,z)) \ge W(\mu_{x_1}v(t+\tau,z)),$$

which yields

$$\begin{aligned}\mathcal{F}_k^s u(x_2) &\ge c_{n,s}\int_\tau^\infty \int_{\mathbb{R}^{n-k}} \frac{1}{|z|^{n-k+2s}} W\Big(\frac{\mu_{x_1}v(t,z)^{1/k}}{|z|}\Big)\, dzdt \\ &= \mathcal{F}_k^s v(x_1) - c_{n,s}\int_0^\tau \int_{\mathbb{R}^{n-k}} \frac{1}{|z|^{n-k+2s}} W\Big(\frac{\mu_{x_1}v(t,z)^{1/k}}{|z|}\Big)\, dzdt \\ &\ge \mathcal{F}_k^s v(x_1) - C\tau^{1-s},\end{aligned}$$

where the last inequality follows from the fact that $\mu_{x_1} v(t,z) \geq C(t-|z|^2)_+^{k/2}$ and W is monotone decreasing.

Combining (5.6.9) and (5.6.10), using that v is a subsolution, and (5.6.8), we get

$$\begin{aligned}\mathcal{F}_k^s u(x_0) + C(r) \geq \mathcal{F}_k^s v(x_1) - C\tau^{1-s} &\geq v(x_1) - \varphi(x_1) - C\tau^{1-s}\\ &> u(x_1) - h - \varphi(x_1) - C\tau^{1-s}.\end{aligned}$$

Letting $\delta \to 0$, it follows that $h \to 0$, $C(r) \to 0$, $\tau \to 0$, and $x_1 \to x_0$. By continuity of u and φ, we conclude the result. $\square$

Proposition 5.6.13 (u is a supersolution)**.** *The function u given in* (5.6.3) *satisfies*

$$\mathcal{F}_k^s u(x_0) \leq u(x_0) - \varphi(x_0), \quad \textit{for all } x_0 \in \mathbb{R}^n.$$

Proof. Assume the statement is false. Then there exists some $x_0 \in \mathbb{R}^n$ such that

$$\mathcal{F}_k^s u(x_0) > u(x_0) - \varphi(x_0).$$

Without loss of generality, we may assume that $u(x_0) = 0$ and $\nabla u(x_0) = 0$. Otherwise, consider $v(x) = u(x) - u(x_0) - (x - x_0) \cdot \nabla u(x_0)$. Then there exists some $\delta > 0$ such that

$$\mathcal{F}_k^s u(x_0) \geq -\varphi(x_0) + \delta. \tag{5.6.11}$$

Fix $\varepsilon > 0$ and let $u^\varepsilon(x) = \max\{u(x), \varepsilon\}$. We will show that for ε sufficiently small, u^ε is an admissible subsolution, and thus, reaching a contradiction with u being the largest subsolution. Indeed, u^ε is convex and

$u^\varepsilon \in C^{0,1}(\mathbb{R}^n)$ with $[u^\varepsilon]_{C^{0,1}(\mathbb{R}^n)} \leq [\varphi]_{C^{0,1}(\mathbb{R}^n)}$. Moreover, note that $u^\varepsilon(x) = u(x)$, for x large. Hence, once we show that u^ε is a subsolution, it will follow from the comparison principle that $\varphi \leq u^\varepsilon \leq \varphi + w$.

If $x \in \{u_\varepsilon = u\}$, then $u_\varepsilon(x) = u(x)$ and $u_\varepsilon \geq u$ in $\mathbb{R}^n$. By monotonicity (Lemma 5.6.5),

$$\mathcal{F}_k^s u^\varepsilon(x) \geq \mathcal{F}_k^s u(x) \geq u(x) - \varphi(x) = u^\varepsilon(x) - \varphi(x),$$

since u is a subsolution, by Proposition 5.6.12.

If $x \in \{u^\varepsilon > u\}$, then $u^\varepsilon(x) = \varepsilon$ and $\partial u^\varepsilon(x) = \{0\}$. In particular,

$$\mathcal{F}_k^s u^\varepsilon(x) = \mathcal{F}_k^s u^\varepsilon(x_0). \tag{5.6.12}$$

Moreover, for any $t > 0$, we have $D_{x_0}u^\varepsilon(t) = \{u^\varepsilon - \varepsilon \leq t\} = \{u \leq t + \varepsilon\} = D_{x_0}u(t+\varepsilon)$. Therefore, in view of Proposition 5.5.4, we get

$$\begin{aligned}\mathcal{F}_k^s u^\varepsilon(x_0) &= \mathcal{F}_k^s u(x_0) - \int_0^\varepsilon \int_{\mathbb{R}^{n-k}} \frac{1}{|z|^{n-k+2s}} W\Big(\frac{\mu_{x_0}u(t,z)^{1/k}}{|z|}\Big)\, dzdt \\ &\geq \mathcal{F}_k^s u(x_0) - C\varepsilon^{1-s} \end{aligned} \tag{5.6.13}$$

since $u \in C^{1,1}(\mathbb{R}^n)$ and $\mu_{x_0}u(t,z) \geq (t - |z|^2)_+^{k/2}$.

Combining (5.6.11), (5.6.12), and (5.6.13), we see that

$$\begin{aligned}\mathcal{F}_k^s u^\varepsilon(x) = \mathcal{F}_k^s u^\varepsilon(x_0) &\geq \mathcal{F}_k^s u(x_0) - C\varepsilon^{1-s} \geq -\varphi(x_0) + \delta - C\varepsilon^{1-s} \\ &= u^\varepsilon(x) - \varphi(x) + \big(\varphi(x) - \varphi(x_0) + \delta - C\varepsilon^{1-s} - \varepsilon\big),\end{aligned}$$

since $u^\varepsilon(x) = \varepsilon$. We need the term inside the parenthesis to be nonnegative. Hence, it remains to control $\varphi(x) - \varphi(x_0)$. Since φ is smooth,

$$|\varphi(x) - \varphi(x_0)| \leq [\varphi]_{C^{0,1}(\mathbb{R}^n)}|x - x_0|.$$

We distinguish two cases. If $\{u = 0\} = \{x_0\}$, then $|x - x_0| \leq d_\varepsilon \to 0$, as $\varepsilon \to 0$. Hence, choosing ε sufficiently small, we see that

$$\varphi(x) - \varphi(x_0) + \delta - C\varepsilon^{1-s} - \varepsilon \geq \delta - [\varphi]_{C^{0,1}(\mathbb{R}^n)} d_\varepsilon - C\varepsilon^{1-s} - \varepsilon \geq 0.$$

Therefore, $u^\varepsilon \in \mathcal{S}$, which contradicts $u^\varepsilon(x_0) > u(x_0) = \sup_{v\in\mathcal{S}} v(x_0) \geq u^\varepsilon(x_0)$.

Suppose now that $\{u = 0\}$ contains more than one point. By compactness of $\{u = 0\}$ and continuity of φ, there exists some $x_1 \in \{u = 0\}$ where φ attains its maximum. Then

$$\mathcal{F}_k^s u(x_1) = \mathcal{F}_k^s u(x_0) \geq u(x_0) - \varphi(x_0) + \delta \geq u(x_1) - \varphi(x_1) + \delta.$$

Moreover, by convexity of $\{u = 0\}$ (since $u \geq \varphi \geq 0$) and φ, we must have that $x_1 \in \partial\{u = 0\}$. Hence, there exists $\{x_j\}_{j=2}^\infty \subset \{u > 0\}$ such that $x_j \to x_1$ and u is strictly convex at x_j. Namely, there is a supporting plane that touches u only at x_j.

By continuity of u, there exists some $j_0 \geq 2$ such that

$$u(x_1) > u(x_j) - \delta/4, \quad \text{for all } j > j_0.$$

By continuity of φ, there exists some $j_1 \geq 2$ such that

$$\varphi(x_1) < \varphi(x_j) + \delta/4, \quad \text{for all } j > j_1.$$

By lower semicontinuity of $\mathcal{F}_k^s u$, up to a subsequence, there exists some $j_2 \geq 2$ such that

$$\mathcal{F}_k^s u(x_j) > \mathcal{F}_k^s u(x_1) - \delta/4, \quad \text{for all } j > j_2.$$

Let $J > \max\{j_0, j_1, j_2\}$. Then

$$\mathcal{F}_k^s u(x_J) > \mathcal{F}_k^s u(x_1) - \delta/4 \geq u(x_1) - \varphi(x_1) + 3\delta/4 > u(x_J) - \varphi(x_J) + \delta/4,$$

and we can repeat the previous argument, replacing x_0 by x_J. We conclude that

$$\mathcal{F}_k^s u(x_0) \leq u(x_0) - \varphi(x_0), \quad \text{for all } x_0 \in \mathbb{R}^n.$$

$\square$

5.7 Future directions

As mentioned in the introduction, the main idea to define a nonlocal analog to the Monge-Ampère operator is to write it as a concave envelope of linear operators. More precisely,

$$n \det(D^2 u(x))^{1/n} = \inf_{M \in \mathcal{M}} \operatorname{tr}(M D^2 u(x)),$$

where $\mathcal{M} = \{M \in \mathcal{S}^n : M > 0,\ \det(M) = 1\}$ and $\mathcal{S}^n$ is the set of $n \times n$ symmetric matrices. In fact, this extremal property does not only hold for $n \det(B)^{1/n}$ with $B \in \mathcal{S}^n$ and $B > 0$. Observe that if $\lambda = (\lambda_1, \ldots, \lambda_n)$, where λ_i are the eigenvalues of B, then the function f defined in $\Gamma = \{\lambda \in \mathbb{R}^n : \lambda_i > 0,\ \text{for all } i = 1, \ldots, n\}$, given by

$$f(\lambda) = n\Big(\prod_{i=1}^n \lambda_i\Big)^{1/n} = n \det(B)^{1/n}$$

is differentiable, concave, and homogeneous of degree 1. In general, if f satisfies these conditions in an open convex set Γ in $\mathbb{R}^n$, then

$$f(\lambda) = \inf_{\mu \in \Gamma} \big\{ f(\mu) + \nabla f(\mu) \cdot (\lambda - \mu) \big\} = \inf_{\mu \in \Gamma} \nabla f(\mu) \cdot \lambda,$$

where the second identity follows by Euler's theorem. Therefore,

$$f(\lambda) = \inf_{M \in \mathcal{M}_f} \operatorname{tr}(MB),$$

where $\mathcal{M}_f = \{M \in \mathcal{S}^n : \lambda(M) \in \nabla f(\Gamma)\}$, $\nabla f(\Gamma) = \{\nabla f(\mu) : \mu \in \Gamma\}$, and $\lambda(M)$ are the eigenvalues of M.

For instance, the k-Hessian functions introduced by Caffarelli, Nirenberg, and Spruck in [9] satisfy these conditions and, in fact, fractional analogs have been recently studied by Wu [62]. It would be interesting to explore fractional analogs to a wider class of fully nonlinear concave operators, as the ones mentioned above.

We remark that the 1-Hessian is equal to the Laplacian, and the n-Hessian is equal to the Monge-Ampère operator. Moreover, for $1 < k < n$, we obtain an intermediate *discrete* family between these operators. In view of this observation, a natural question of finding a *continuous* family connecting the Laplacian with the Monge-Ampère operator arises. Here we suggest possible families that connect smoothly these two operators, passing through the k-Hessians, in some sense. Indeed, let $\alpha \in (0,1]^n$ and denote $|\alpha| = \alpha_1 + \cdots + \alpha_n$. For $\lambda \in \mathbb{R}^n_+$, we consider the functions,

$$f_\alpha(\lambda) = \Big(\sum_{\sigma \in S} \lambda_{\sigma(1)}^{\alpha_1} \cdots \lambda_{\sigma(n)}^{\alpha_n} \Big)^{\frac{1}{|\alpha|}},$$

where S is the set of all cyclic permutations of $\{1, \ldots, n\}$. Observe that for any $1 \leq k \leq n$, if $\alpha = \sum_{i \in \mathcal{I}} e_i$, with $|\mathcal{I}| = k$, then f_α is precisely the k-Hessian function. Consider any smooth simple curve $\gamma : [0,1] \to (0,1]^n$ such that

1. $\gamma(0) = e_i$, for some $1 \leq i \leq n$,

2. $\gamma(t_k) = \sum_{i \in \mathcal{I}_k} e_i$, with $|\mathcal{I}_k| = k$, and $0 < t_k < t_{k+1} < 1$, for all $1 < k < n$, and

3. $\gamma(1) = (1, \ldots, 1)$.

Then the family $\{f_\alpha\}_{\alpha \in \mathrm{Im}(\gamma)}$ is as we described. In particular, fractional analogs of these functions would give a continuous family from the fractional Laplacian to the nonlocal Monge-Ampère. We will study this problem in a forthcoming paper.

Bibliography

[1] M. Amaral and E. Teixeira. Free transmission problems. *Comm. Math. Phys.*, 337:1465–1489, 2015.

[2] A. Baernstein. *Symmetrization in Analysis.* Cambridge University Press, 2019.

[3] C. Bennett and M. Sharpley. *Interpolation of Operators.* Pure and Applied Mathematics, Academic Press, 1988.

[4] I. Blank and Z. Hao. The mean value theorem and basic properties of the obstacle problem for divergence form elliptic operators. *Comm. Anal. Geom.*, 23:129–158, 2015.

[5] M. Borsuk. *Transmission problems for elliptic second-order equations in non-smooth domains.* Frontiers in Mathematics, Birkhäuser/Springer Basel AG, Basel, 2010.

[6] M. V. Borsuk. A priori estimates and solvability of second order quasi-linear elliptic equations in a composite domain with nonlinear boundary conditions and conjunction condition. *Proc. Steklov Inst. of Math.*, 103:13–51, 1970.

[7] Y. Brenier. Polar factorization and monotone rearrangement of vector-valued functions. *Comm. Pure Appl. Math.*, 44:375–417, 1991.

[8] L. Caffarelli and F. Charro. On a fractional Monge-Ampère operator. *Ann. PDE*, 1, 2015.

[9] L. Caffarelli, L. Nirenberg, and J. Spruck. The Dirichlet problem for nonlinear second-order elliptic equations. III. Functions of the eigenvalues of the Hessian. *Acta Math.*, 155:261–301, 1985.

[10] L. Caffarelli and L. Silvestre. Regularity theory for fully nonlinear integro-differential equations. *Comm. Pure Appl. Math.*, 62:597–638, 2009.

[11] L. Caffarelli and L. Silvestre. The Evans-Krylov theorem for nonlocal fully nonlinear equations. *Ann. of Math.*, 174:1163–1187, 2011.

[12] L. Caffarelli and L. Silvestre. Regularity results for nonlocal equations by approximation. *Arch. Ration. Mech. Anal.*, 200:59–88, 2011.

[13] L. Caffarelli and L. Silvestre. A nonlocal Monge-Ampère equation. *Comm. Anal. Geom.*, 24:307–335, 2016.

[14] L. A. Caffarelli. Elliptic second order equations. *Rend. Sem. Mat. Fis. Milano*, 58:253–284, 1988.

[15] L. A. Caffarelli and X. Cabré. *Fully Nonlinear Elliptic Equations*. American Mathematical Society, Providence, 2005.

[16] L. A. Caffarelli and M. Soria-Carro. On a family of fully nonlinear integro-differential operators: From fractional laplacian to nonlocal monge-ampère. *arXiv:2111.12781*, 2021.

[17] L. A. Caffarelli, M. Soria-Carro, and P. R. Stinga. Regularity for $C^{1,\alpha}$ interface transmission problems. *Arch. Ration. Mech. Anal.*, 1:265–294, 2021.

[18] S. Campanato. Sul problema di M. Picone relativo all'equilibrio di un corpo elastico incastrato. *Ricerche Mat.*, 6:125–149, 1957.

[19] S. Campanato. Proprietà di holderianità di alcune classi di funzioni. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (3)*, 17:175–188, 1963.

[20] G. Citti and F. Ferrari. A sharp regularity result of solutions of a transmission problem. *Proc. Amer. Math. Soc.*, 140:615–620, 2012.

[21] M. G. Crandall, H. Ishii, and P. L. Lions. User's guide to viscosity solutions of second order partial differential equations. *Bull. Amer. Math. Soc.*, 27:1–67, 1992.

[22] M. G. Crandall and P. L. Lions. Viscosity solutions of hamilton-jacobi equations. *Trans. Amer. Math. Soc.*, 277:1–42, 1983.

[23] L. C. Evans and R. F. Gariepi. *Measure Theory and Fine Properties of Functions*. Studies in Advanced Mathematics, CRC Press, Boca Raton, FL, 1992.

[24] A. Figalli. *The Monge-Ampère Equation and Its Applications*. Zurich Lectures in Advanced Mathematics, European Mathematical Society, 2017.

[25] C. De Filippis. Regularity for solutions of fully nonlinear elliptic equations with nonhomogeneous degeneracy. *Proc. Roy. Soc. Edinburgh Sect. A*, 151:110–132, 2021.

[26] D. Gilbarg and N. S. Trudinger. *Elliptic partial differential equations of second order*. Classics in Mathematics, Springer-Verlag, Berlin, 2001.

[27] N. Guillen and R. W. Schwab. Aleksandrov-bakelman-pucci type estimates for integro-differential equations. *Arch. Ration. Mech. Anal.*, 206:111–117, 2012.

[28] C. E. Gutiérrez. *The Monge-Ampère Equation.* Progress in Nonlinear Differential Equations and Their Applications, 2001.

[29] G. Huaroto, E. A. Pimentel, G. C. Rampasso, and A. Święch. A fully nonlinear degenerate free transmission problem. *arXiv:2008.06917*, 2020.

[30] H. Ishii. Fully nonlinear oblique derivative problems for nonlinear second-order elliptic pdes. *Duke Math. J.*, 62:633–661, 1991.

[31] D. Kriventsov. Regularity for a local-nonlocal transmission problem. *Arch. Ration. Mech. Anal.*, 2017:1103–1195., 2015.

[32] N. V. Krylov. *Controlled diffusion processes.* Springer-Verlag, 1980.

[33] O. A. Ladyzhenskaya and N. N. Ural'tseva. *Linear and Quasilinear Elliptic Equations.* Academic Press, New York-London, 1968.

[34] D. Li and K. Zhang. Regularity for fully nonlinear elliptic equations with oblique boundary conditions. *Arch. Rational Mech. Anal.*, 228:923–967, 2018.

[35] Y. Li and L. Nirenberg. Estimates for elliptic systems from composite material. dedicated to the memory of Jurgen K. Moser. *Comm. Pure Appl. Math.*, 56:892–925, 2003.

[36] Y. Li and M. Vogelius. Gradient estimates for solutions to divergence form elliptic equations with discontinuous coefficients. *Arch. Ration. Mech. Anal.*, 153:91–151, 2000.

[37] Y. Lian, W. Xu, and K. Zhang. Boundary lipschitz regularity and the hopf lemma for fully nonlinear elliptic equations. *arXiv:1812.11357*, 2020.

[38] Y. Lian and K. Zhang. Boundary pointwise $c^{1,\alpha}$ and $c^{2,\alpha}$ regularity for fully nonlinear elliptic equations. *J. Differential Equations*, 269:1172–1191, 2020.

[39] J. L. Lions. Contributions à un problème de M. M. Picone. *Ann. Mat. Pura Appl. (4)*, 41:201–219, 1956.

[40] W. Littman, G. Stampacchia, and H. F. Weinberger. Regular points for elliptic equations with discontinuous coefficients. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (3)*, 17:43–77, 1963.

[41] F. Maggi. *Sets of Finite Perimeter and Geometric Variational Problems. An Introduction to Geometric Measure Theory.* Cambridge University Press, Cambridge, 2012.

[42] D. Maldonado and P. R. Stinga. Harnack inequality for the fractional nonlocal linearized Monge-Ampère equation. *Calc. Var. Partial Differential Equations*, 56, 2017.

[43] J. Mateu, J. Orobitg, and J. Verdera. Extra cancellation of even Calderón-Zygmund operators and quasiconformal mappings. *J. Math. Pures Appl. (9)*, 91:402–431, 2009.

[44] R. J. McCann. Existence and uniqueness of monotone measure-preserving maps. *Duke Math. J.*, 80:309–323, 1995.

[45] E. Milakis and L. E. Silvestre. Regularity for fully nonlinear elliptic equations with Neumann boundary data. *Comm. Partial Differential Equations*, 31:1227–1252, 2006.

[46] M. Nisio. Stochastic differential games and viscosity solutions of Isaacs equations. *Nagoya Math. J.*, 110:163–184, 1988.

[47] O. A. Oleinik. Boundary value problems for linear elliptic and parabolic equations with discontinuous coefficients. *Amer. Math. Soc. Transl.*, 42:175–194, 1964.

[48] G. De Philippis and A. Figalli. The Monge-Ampère equation and its link to optimal transportation. *Bull. Amer. Math. Soc.*, 51:527–580, 2014.

[49] M. Picone. Sur un problème nouveau pour l'équation linéaire aux dérivées partielles de la théorie mathematique classique de l'élasticité. *Colloque sur les équations aux dérivées partielles*, 1954.

[50] E. Pimentel and M. Santos. A fully nonlinear free transmission problem. *arXiv:2010.15910*, 2021.

[51] E. Pimentel and A. Święch. Existence of solutions to a fully nonlinear free transmission problem. *J. Differential Equations*, 320:49–63, 2022.

[52] X. Ros-Oton and J. Serra. Boundary regularity for fully nonlinear integro-differential equations. *Duke Math. J.*, 165:2079–2154, 2016.

[53] J. V. Ryff. Measure preserving transformations and rearrangements. *J. Math. Anal. Appl.*, 31:449–458, 1970.

[54] M. Schechter. A generalization of the problem of transmission. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (3)*, 14:207–236, 1960.

[55] D. De Silva, F. Ferrari, and S. Salsa. Two-phase problems with distributed source: regularity of the free boundary. *Anal. PDE*, 7:267–310, 2014.

[56] D. De Silva, F. Ferrari, and S. Salsa. Free boundary regularity for fully nonlinear non-homogeneous two-phase problems. *J. Math. Pures Appl.*, 103:658–694, 2015.

[57] D. De Silva, F. Ferrari, and S. Salsa. Perron's solutions for two-phase free boundary problems with distributed sources. *Nonlinear Anal.*, 121:382–402, 2015.

[58] D. De Silva, F. Ferrari, and S. Salsa. Regularity of transmission problems for uniformly elliptic fully nonlinear equations. *Electron. J. Differ. Equ. Conf.*, 25:55–63, 2018.

[59] L. Silvestre and B. Sirakov. Boundary regularity for viscosity solutions of fully nonlinear elliptic equations. *Comm. Partial Differential Equations*, 39:1694–1717, 2014.

[60] M. Soria-Carro and P. R. Stinga. Regularity of solutions to fully nonlinear transmission problems with flat and nonflat interfaces *(preprint)*.

[61] G. Stampacchia. Su un problema relativo alle equazioni di tipo ellittico del secondo ordine. *Ricerche Mat.*, 5:3–24, 1956.

[62] Y. Wu. Regularity of fractional analogue of k-Hessian operators and a non-local one-phase free boundary problem. *Ph.D. Thesis, The University of Texas at Austin*, 2019.