Regularity for $C^{1,\alpha}$ Interface Transmission Problems

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  - Stability Result and $C^{1,\alpha}$ Regularity
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History

Mauro Picone (1885 – 1977)

- 1927: Founder of the first applied math institute called *Istituto per le Applicazioni del Calcolo*
- 1954: Pioneer of *transmission type problems* related to elasticity theory
- 1955-1957: Lions, Stampacchia and Campanato contributed to Picone's problem

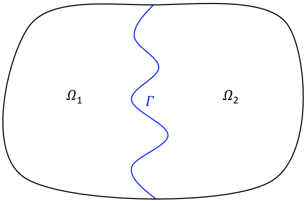
History

- M. Picone, *Sur un problème nouveau pour l'équation linéaire aux dérivées partielles de la théorie mathematique classique de l'élasticité*, Colloque sur les équations aux dérivées partielles, Bruxelles (1954).

PICONE'S TRANSMISSION PROBLEM.

$$\begin{cases} h_i \Delta \mathbf{u}_i + (h_i + k_i)\nabla(\operatorname{div} \mathbf{u}_i) + \mathbf{F}_i = 0 & \text{in } \Omega_i, \quad i \in \{1,2\} \\ \mathbf{u_1} = \mathbf{u_2} & \text{on } \Gamma \\ \mathbf{p}(\mathbf{u_1}) = -\mathbf{p}(\mathbf{u_2}) & \text{on } \Gamma \\ \mathbf{p}(\mathbf{u_i}) = \varphi_i & \text{on } \partial\Omega_i \setminus \Gamma, \quad i \in \{1,2\} \end{cases}$$

- Ω_i region occupied by material $i \in \{1,2\}$
- Γ touching surface between materials (interface)
- $\mathbf{u_i}$ displacement vector
- $\mathbf{p}(\mathbf{u_i})$ pressure field
- $\mathbf{F_i}$ force
- h_i, k_i Lamé parameters
- φ_i given function



A Generalization of the Problem of Transmission

- M. Schechter, *A Generalization of the Problem of Transmission*, Annali della Scuola Normale Superiore di Pisa, Classe di Scienze 3e série, tome 14, no 3 (1960)

PROBLEM. Find the pair (u_1, u_2), with $u_1 : \bar{\Omega}_1 \to \mathbb{R}$ and $u_2 : \bar{\Omega}_2 \to \mathbb{R}$ such that

$$\begin{array}{llllll} L^{(1)}(u_1) = 0 & \text{in} & \Omega_1, & u_1 = h_1 & \text{on} & \partial\Omega_1 \setminus \Gamma \\ L^{(2)}(u_2) = 0 & \text{in} & \Omega_2, & u_2 = h_2 & \text{on} & \partial\Omega_2 \setminus \Gamma \end{array}$$

and the conditions on the common boundary (transmission conditions)

$$\begin{array}{rcll} u_1 - u_2 & = & f & \text{on} \quad \Gamma \\ \partial_\nu u_1 - \partial_\nu u_2 & = & g & \text{on} \quad \Gamma \end{array}$$

The operators are given by the following expression

$$Lu = a_{ij}(x) D_{ij} u + b_i(x) D_i u + c(x) u$$

- Schechter assumes that the domains and the data are smooth
- Other transmission conditions: Ladyzhenskaya – Uraltseva (diffraction problems)

Our Transmission Problem

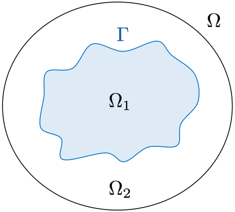
Joint work with L. A. Caffarelli and P. R. Stinga

PROBLEM. Find the pair of functions (u_1, u_2) such that

$$\begin{cases} \Delta u_1 = 0 & \text{in } \Omega_1 \\ \Delta u_2 = 0 & \text{in } \Omega_2 \\ u_2 = 0 & \text{on } \partial\Omega \\ u_1 = u_2 & \text{on } \Gamma \\ \partial_\nu u_1 - \partial_\nu u_2 = g & \text{on } \Gamma \end{cases} \tag{TP}$$

where

* Ω smooth bounded domain in $\mathbb{R}^n$
* ν is the normal vector pointing at Ω_1
* $\Gamma = \partial\Omega_1$ is an $n-1$ dimensional $C^{1,\alpha}$ manifold
* $g \in C^{0,\alpha}(\Gamma)$ and $g \geq c_0 > 0$



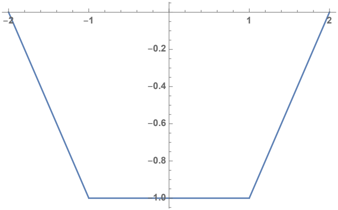
GOAL. Show that $u_i \in C^{1,\alpha}(\overline{\Omega}_i)$, for $i = 1, 2$.

Example in 1D

Let $\Omega = (-2,2)$, $\Omega_1 = (-1,1)$ and $\Omega_2 = (-2,-1) \cup (1,2)$. Find (u_1, u_2) satisfying

$$\begin{cases} u_1'' = 0 \text{ in } (-1,1) \\ u_2'' = 0 \text{ in } (-2,-1) \cup (1,2) \\ u_2(-2) = 0, \quad u_2(2) = 0 \\ u_1(-1) = u_2(-1), \quad u_1(1) = u_2(1) \\ u_1'(-1) - u_2'(-1) = 1, \quad u_2'(1) - u_1'(1) = 1 \end{cases}$$

The solution is



REMARK. The primary focus is to study their behavior across the interface.

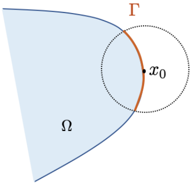
Preliminaries

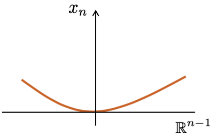
DEFINITIONS. Let Ω be a bounded domain in $\mathbb{R}^n$, and let $0 < \alpha \leq 1$.

* We say that $u \in C^{1,\alpha}(\bar{\Omega})$ if

$$\|u\|_{C^{1,\alpha}(\bar{\Omega})} := \|u\|_{C^0(\bar{\Omega})} + \|\nabla u\|_{C^0(\bar{\Omega})} + \sup_{\substack{x,y\in\bar{\Omega}\\ x\neq y}} \frac{|\nabla u(x) - \nabla u(y)|}{|x-y|^\alpha} < +\infty$$

* Let $\Gamma = \partial\Omega$. We say that Γ is a $C^{1,\alpha}$ manifold if every x_0 in Γ has a neighborhood in which Γ is the graph of a $C^{1,\alpha}$ function in $\mathbb{R}^{n-1}$.





Preliminaries

Classical Approach for Boundary Regularity

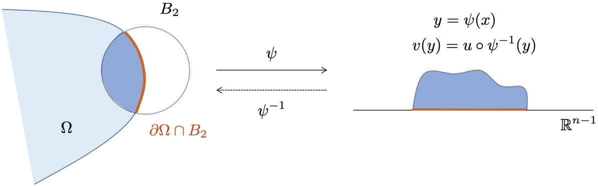
Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and consider

$$\begin{cases} \Delta u = 0 & \text{in} \quad \Omega \cap B_2 \\ u = 0 & \text{on} \quad \partial\Omega \cap B_2 \end{cases}$$

QUESTION. What can we say about the regularity of u on the boundary?

It depends on the regularity of $\partial\Omega$!

In the classical theory, a standard method is to flatten the boundary:



- v satisfies an equation with variable coefficients depending on ψ:

$$a_{ij}(x) D_{ij} v + b_i(x) D_i v = 0$$

where $a_{ij} = \nabla\psi^{(i)} \cdot \nabla\psi^{(j)}$ and $b_i = \Delta\psi^{(i)}$

- We need at least $\partial\Omega \in C^2$!

THEOREM (CAMPANATO, 1963). Let u be a measurable function defined on a bounded Lipschitz domain Ω. Then

$$u \in C^{1,\alpha}(\overline{\Omega})$$

if and only if there is $C_0 > 0$ such that for any $x \in \overline{\Omega}$, there exists a linear polynomial $P_x(z)$ such that

$$|u(z) - P_x(z)| \leq C_0 |x - z|^{1+\alpha}, \quad \forall\, z \in B_1(x) \cap \Omega$$

If C_* denotes the least constant $C_0 > 0$ for which the property above holds, then

$$\|u\|_{C^{1,\alpha}(\overline{\Omega})} \sim C_* + \sup_{x \in \overline{\Omega}} |P_x|$$

where $|P_x|$ denotes the sum of the coefficients of the polynomial $P_x(z)$.

Preliminaries

Caffarelli's Geometric Approach for $C^{2,\alpha}$ Interior Regularity

- L. A. Caffarelli, *Elliptic Second Order Equations*, Rend. Sem. Mat. Fis. Milano 58 (1988)

Let u be a bounded solution to

$$a_{ij}(x)D_{ij}u = f \quad \text{in} \quad B_1$$

where $a_{ij} \in C^\alpha(\overline{B_1})$, symmetric, uniformly elliptic, and $f \in C^\alpha(\overline{B_1})$. Assume that

$$a_{ij}(0) = I, \qquad f(0) = 0$$

GOAL. Show that $u \in C^{2,\alpha}(\overline{B_{1/2}})$ using Campanato's characterization.

IDEA.

- If $a_{ij} \in C^\alpha(\overline{B_1}),\ a_{ij}(0) = I$, then "$a_{ij}(x) \sim I$". In particular:

$$\text{“}a_{ij}(x)D_{ij} \sim \Delta\text{”}$$

- If $f \in C^\alpha(\overline{B_1}),\ f(0) = 0$, then

$$\text{“}f(x) \sim 0\text{”}$$

- The solution to

$$\begin{cases} \Delta v = 0 & \text{in} \quad B_{1/2} \\ v = u & \text{on} \quad \partial B_{1/2} \end{cases}$$

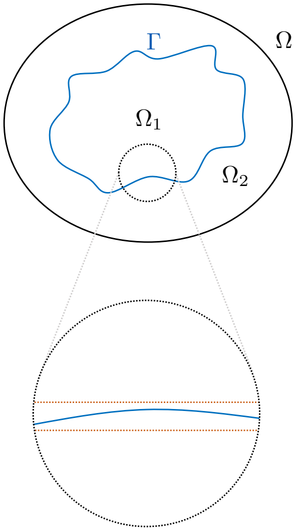
satisfies

$$\text{“}\|u - v\|_{L^\infty(B_{1/4})} \sim 0\text{”}$$

- Rescaling and iterating, we expect

"quadratic part of v approximates u at 0"

Strategy to prove $C^{1,\alpha}$ up to the boundary for our Transmission Problem



GEOMETRIC METHOD.

1. Localize and normalize the problem
2. Approximate by flat problems from above and by below
3. Find linear approximations using the flat solutions
4. Rescale and iterate

Our Transmission Problem

Recall:

$$\begin{cases} \Delta u_1 = 0 & \text{in} \quad \Omega_1 \\ \Delta u_2 = 0 & \text{in} \quad \Omega_2 \\ u_2 = 0 & \text{on} \quad \partial\Omega \\ u_1 = u_2 & \text{on} \quad \Gamma \\ \partial_\nu u_1 - \partial_\nu u_2 = g & \text{on} \quad \Gamma \end{cases} \tag{TP}$$

Consider $u = u_1\chi_{\Omega_1} + u_2\chi_{\Omega_2}$ and $\varphi \in C_c^\infty(\Omega)$. Integrating by parts, we get

$$\langle \Delta u, \varphi \rangle = \int_\Gamma g\varphi \, dS$$

DEFINITION. We say that (u_1, u_2) is a weak solution to (TP) if

$$u_1 = u\chi_{\Omega_1} \quad \text{and} \quad u_2 = u\chi_{\Omega_2},$$

where $u \in L^1(\Omega)$ is a vanishing distributional solution to

$$\Delta u = g \, dS|_\Gamma$$

Our Transmission Problem

Recall:

$$\begin{cases} \Delta u_1 = 0 & \text{in } \Omega_1 \\ \Delta u_2 = 0 & \text{in } \Omega_2 \\ u_2 = 0 & \text{on } \partial\Omega \\ u_1 = u_2 & \text{on } \Gamma \\ \partial_\nu u_1 - \partial_\nu u_2 = g & \text{on } \Gamma \end{cases} \tag{TP}$$

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$$\Delta u = g\, dS|_\Gamma$$

Our Transmission Problem

EXISTENCE AND UNIQUENESS. Let $G(x,y)$ be the Green's function in Ω. The function

$$u(x) = C_n \int_\Gamma G(x,y) g(y)\, dS$$

is well defined for all $x \in \Omega$, with

$$\|u\|_{L^\infty(\Omega)} \le C_{n,\Gamma} \|g\|_{L^\infty(\Gamma)}$$

Moreover, u is the weak solution to (TP), and $u \in H_0^1(\Omega) \cap C^{0,\gamma}(\overline{\Omega})$, for all $0 < \gamma < 1$, with

$$\|u\|_{C^{0,\gamma}(\overline{\Omega})} \le C_{n,\Gamma,\gamma} \|g\|_{L^\infty(\Gamma)}$$

GOAL. Show that $u_i \in C^{1,\alpha}(\overline{\Omega}_i)$, for $i = 1, 2$, without flattening the boundary.

$C^{1,\alpha}$ Regularity for the Flat Problem

EXISTENCE AND REGULARITY FOR THE FLAT PROBLEM.

Given $0 < \alpha, \gamma < 1$, let $g \in C^{0,\alpha}(T)$ and $f \in C^{0,\gamma}(\overline{B_2})$. Then there is a unique classical solution $v \in C^\infty(B_2 \setminus T) \cap C^{0,\gamma}(\overline{B_2})$ to the flat transmission problem

$$\begin{cases} \Delta v^+ = 0 & \text{in } B_2^+ \\ \Delta v^- = 0 & \text{in } B_2^- \\ v = f & \text{on } \partial B_2 \\ v^+ = v^- & \text{on } T \\ v^+_{x_n} - v^-_{x_n} = g & \text{on } T \end{cases} \tag{FTP}$$

where $T = \{y_n = 0\} \cap B_2$. Moreover, $v^\pm \in C^{1,\alpha}(\overline{B_1^\pm})$, with

$$\|v^\pm\|_{C^{1,\alpha}(\overline{B}_1^\pm)} \le C\Big(\|g\|_{C^{0,\alpha}(T)} + \|f\|_{L^\infty(\partial B_2)}\Big)$$

$C^{1,\alpha}$ Regularity for the Flat Problem

Proof.

Up to subtracting a harmonic function, we can reduce to zero boundary data on ∂B_2.

- Let v^+ be the solution to

$$\begin{cases} \Delta v^+ = 0 & \text{in } B_2^+ \\ v^+ = 0 & \text{on } \partial B_2^+ \setminus T \\ v^+_{x_n} = \frac{g}{2} & \text{on } T \end{cases}$$

- By Agmon – Douglis – Nirenberg (1959), we have $v^+ \in C^\infty(B_2^+) \cap C^{1,\alpha}(\overline{B_1^+})$, with

$$\|v^+\|_{C^{1,\alpha}(\overline{B_1^+})} \leq C\|g\|_{C^{0,\alpha}(T)}$$

- Consider $v^-(x', x_n) = v^+(x', -x_n)$. Then $v^- \in C^\infty(B_2^-) \cap C^{1,\alpha}(\overline{B_1^-})$ satisfies

$$\begin{cases} \Delta v^- = 0 & \text{in } B_2^- \\ v^- = 0 & \text{on } \partial B_2^- \setminus T \\ v^-_{x_n} = -\frac{g}{2} & \text{on } T \end{cases}$$

- Then

$$v = v^+ \chi_{B_2^+} + v^- \chi_{B_2^-}$$

is the unique solution to (FTP). Moreover, $v \in C^\infty(B_2 \setminus T) \cap \mathrm{Lip}(\overline{B_2})$, and

$$\|v^\pm\|_{C^{1,\alpha}(\overline{B_1^\pm})} \leq C\|g\|_{C^{0,\alpha}(T)}$$

□

$C^{1,\alpha}$ Pointwise Boundary Regularity for the Curved Problem

SETTING.

- $0 \in \Gamma = \{(y', \psi(y')) : y' \in B_1'\}$, and $\psi \in C^{1,\alpha}(0)$, i.e.,

$$[\psi]_{C^{1,\alpha}(0)} = \sup_{x' \in B_1'} \frac{|\psi(x') - \psi(0') - \nabla'\psi(0') \cdot x'|}{|x'|^{1+\alpha}} < +\infty$$

- $u \in C^{0,\gamma}(\overline{B_1})$ is the weak solution to

$$\Delta u = g\, dS\big|_{\Gamma}$$

  where $g \geq c_0 > 0$, $g \in C^{\alpha}(0)$

GOAL. Show that $u_i \in C^{1,\alpha}(0)$, for $i = 1, 2$

Recall the main difficulties are:

- Γ is not flat, so we CANNOT decouple the problem
- Γ is not smooth enough, so we CANNOT flatten the boundary

Recall our approach is based on:

- Campanato's characterization of $C^{1,\alpha}$ spaces
- Approximation technique inspired by Caffarelli's geometric idea

$C^{1,\alpha}$ Pointwise Boundary Regularity for the Curved Problem

Caffarelli – S.C. – Stinga

MAIN THEOREM. Under the same conditions stated before, we have that

$$u_i = u\big|_{\Omega_i} \in C^{1,\alpha}(0)$$

i.e., there are linear polynomials

$$P_i(x) = A_i \cdot x + B_i, \quad i = 1, 2$$

such that

$$|u_i(x) - P_i(x)| \le D_i |x|^{1+\alpha}, \quad \forall\, x \in \Omega_i \cap B_{1/2}$$

with

$$|A_i| + |B_i| + |D_i| \le C_0 \Big([g]_{C^\alpha(0)} + \|g\|_{L^\infty(\Gamma)} \Big)$$

and $C_0 > 0$ depending only on n and $[\psi]_{C^{1,\alpha}(0)}$.

$C^{1,\alpha}$ Pointwise Boundary Regularity for the Curved Problem

REMARK. The estimate

$$|u(x) - P(x)| \leq D|x|^{1+\alpha}, \quad \forall\, x \in B_1 \tag{1}$$

is very rigid, since P is exactly the first order Taylor polynomial of u at 0.

* Idea. Find P and show (1) by approximation. Given $0 < \lambda < 1$, it is equivalent to show that there is a sequence of linear polynomials $\{P_k\}_{k\geq 0}$ such that

  $$|u(x) - P_k(x)| \leq \lambda^{k(1+\alpha)}, \quad \forall\, x \in B_{\lambda^k}$$

  and also,

  $$|P_{k+1}(x) - P_k(x)| \leq C\lambda^{k(1+\alpha)}, \quad \forall\, x \in B_1$$

* In our problem, we need $\{P_k^1\}_{k\geq 0}$ and $\{P_k^2\}_{k\geq 0}$ such that

  $$|u_i(x) - P_k^i(x)| \leq \lambda^{k(1+\alpha)}, \quad \forall\, x \in \Omega_i \cap B_{\lambda^k}, \quad i = 1, 2$$

BREAK!

Questions or Comments?

Recapitulating...

We are studying the transmission problem

$$\begin{cases} \Delta u_1 = 0 & \text{in} \quad \Omega_1 \\ \Delta u_2 = 0 & \text{in} \quad \Omega_2 \\ u_2 = 0 & \text{on} \quad \partial\Omega \\ u_1 = u_2 & \text{on} \quad \Gamma \\ \partial_\nu u_1 - \partial_\nu u_2 = g & \text{on} \quad \Gamma \end{cases} \qquad \text{(TP)}$$

that can be written in the weak form as

$$\begin{cases} \Delta u = g\, dS|_\Gamma & \text{on} \quad \Omega \\ u = 0 & \text{on} \quad \partial\Omega \end{cases}$$

- ⋆ We assume Γ is $C^{1,\alpha}$ and g is C^α, $g \geq c_0 > 0$
- ⋆ We expect solutions u_1 and u_2 to be $C^{1,\alpha}$ up to the boundary
- ⋆ We assume $0 \in \Gamma$, and we would like to show that

$$u_1, u_2 \in C^{1,\alpha}(0)$$

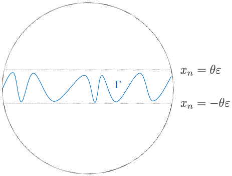
- ⋆ To prove that, we will find linear approximations $\{P_k^i\}_{k \geq 1}$ such that

$$|u_i(x) - P_k^i(x)| \leq \lambda^{k(1+\alpha)}, \quad \forall\, x \in \Omega_i \cap B_{\lambda^k}, \quad i = 1, 2$$

Geometry of the interface

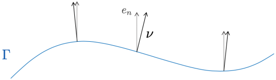
DEFINITIONS. Let $\Gamma = \{(y', \psi(y')) : y' \in B_1'\}$ for a function ψ. Fix $0 < \varepsilon, \theta < 1$.

* We say that Γ is $\theta\varepsilon$-flat in B_1 if $\Gamma \subset \{x \in B_1 : |x_n| < \theta\varepsilon\}$



* We say that Γ is ε-horizontal in B_1 if

$$1 - \varepsilon \leq \nu(x) \cdot (0', 1) = \left(1 + |\nabla' \psi(x')|^2\right)^{-1/2} \leq 1, \quad \forall\, x \in \Gamma$$



Stability result

Caffarelli–S.-C.–Stinga

APPROXIMATION WITH FLAT PROBLEM BY BELOW.

Let $0 < \varepsilon, \theta, \delta, \gamma < 1$ be given. Assume that Γ is $\theta\varepsilon$-flat and ε-horizontal in B_2.
Let $u \in C^{0,\gamma}(\overline{B_1})$ be a weak solution to the transmission problem

$$\Delta u = g\, dS\big|_{\Gamma} \quad \text{in} \quad B_1$$

where $g \in L^\infty(\Gamma)$, and

$$\|g - 1\|_{L^\infty(\Gamma)} \leq \delta$$

Then there is a classical solution $v \in C^{0,\gamma}(\overline{B_1})$ to the flat transmission problem

$$\begin{cases} \Delta v = (1 + \eta)\, dS\big|_{T} & \text{in} \quad B_1 \\ v = u & \text{on} \quad \partial B_1 \end{cases}$$

where $T = \{y_n = -\theta\varepsilon\} \cap B_1$, and $\eta > 0$ depending only on θ, δ and ε, such that

$$\|u - v\|_{L^\infty(B_{1/2})} \leq C(\theta + \delta + \varepsilon^\gamma)$$

where $C > 0$ depends only on n and Γ.

Stability result

Sketch of the proof

$$\boxed{\|u-v\|_{L^\infty(B_{1/2})} \le C(\theta+\delta+\varepsilon^\gamma)}$$

STEP 1. CONSTRUCTION OF v

Let v be the classical solution to the flat transmission problem

$$\begin{cases} \Delta v & = & (1+\eta)\, dS\big|_T \\ v & = & u \quad \text{on } \partial B_1 \end{cases}$$

where $T = \{y_n = -\theta\varepsilon\} \cap B_1$ and $0 < \eta < 1$, to be determined.

We are going to show that

$$v(x) \le u(x) + C(\theta+\delta+\varepsilon^\gamma), \qquad \forall\, x \in B_{1/2}$$

REMARK. We cannot compare Δu and Δv, since the distributions

$$\Delta u = g\, dS|_\Gamma \quad \text{and} \quad \Delta v = (1+\eta)\, dS\big|_T$$

have disjoint support.

Step 2. Averaging to increase support

Let $M = 1 + 2\theta$ and consider the average functions

$$u_\varepsilon(x) = \frac{1}{|B_\varepsilon|} \int_{B_\varepsilon(x)} u(y)\, dy, \quad x \in B_{1-\varepsilon}$$

$$v_{M\varepsilon}(x) = \frac{1}{|B_{M\varepsilon}|} \int_{B_{M\varepsilon}(x)} v(y)\, dy, \quad x \in B_{1-M\varepsilon}$$

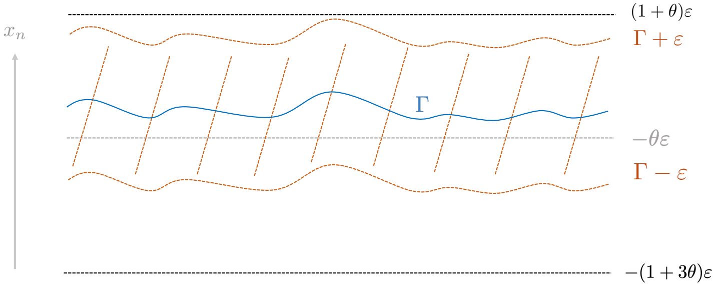
- If $B_\varepsilon(x) \cap \Gamma = \varnothing$, then by the Mean Value Theorem, $\Delta u_\varepsilon(x) = 0$
- If $B_\varepsilon(x) \cap \Gamma \neq \varnothing$, then

$$\Delta u_\varepsilon(x) = \frac{1}{|B_\varepsilon|} \int_{\Gamma \cap B_\varepsilon(x)} g(y) dS_y$$

- If $B_{M\varepsilon}(x) \cap \{y_n = 0\} = \varnothing$, then $\Delta v_{M\varepsilon}(x) = 0$
- If $B_{M\varepsilon}(x) \cap \{y_n = 0\} \neq \varnothing$,

$$\Delta v_{M\varepsilon}(x) = \frac{1}{|B_{M\varepsilon}|} \int_{T \cap B_{M\varepsilon}(x)} (1 + \eta)\, dy'$$

Stability result
Sketch of the proof



Notice that

$$\operatorname{supp}(\Delta u_\varepsilon) \subset \{x \in \mathbb{R}^n : \text{dist}\,(x,\Gamma) < \varepsilon\}, \quad \operatorname{supp}(\Delta v_{M\varepsilon}) \subset \{x \in \mathbb{R}^n : |x_n + \theta\varepsilon| < M\varepsilon\}$$

Since Γ is $\theta\varepsilon$-flat and $M = 1 + 2\theta$ it follows that

$$\operatorname{supp}(\Delta u_\varepsilon) \subset \operatorname{supp}(\Delta v_{M\varepsilon})$$

We will show that $\Delta u_\varepsilon \leq \Delta v_{M\varepsilon}$.

Stability result

Sketch of the proof

STEP 3. COMPARISON. Take $x \in \operatorname{supp}(\Delta u_\varepsilon)$, and choose $1+\eta$ such that:

$$\begin{aligned}
\Delta u_\varepsilon(x) &= \frac{1}{|B_\varepsilon|} \int_{\Gamma \cap B_\varepsilon(x)} g \, dS \\
&= \frac{1}{|B_\varepsilon|} \int_{\{y' : (y', \varphi(y')) \in B_\varepsilon(x)\}} g(y', \psi(y')) \sqrt{1 + |\nabla' \psi(y')|^2} \, dy' \\
&\le \frac{1}{|B_\varepsilon|} \int_{T \cap B_{M\varepsilon}(x)} g(y', \psi(y')) \sqrt{1 + |\nabla' \psi(y')|^2} \, dy' \\
&\le \frac{1}{M^n |B_\varepsilon|} \int_{T \cap B_{M\varepsilon}(x)} M^n (1+\delta)(1-\varepsilon)^{-1} \, dy' \\
&= \frac{1}{|B_{M\varepsilon}|} \int_{T \cap B_{M\varepsilon}(x)} (1+\eta) \, dy' \\
&= \Delta v_{M\varepsilon}(x)
\end{aligned}$$

Now since $u = v$ on ∂B_1, we have

$$v_{M\varepsilon}(x) - u_\varepsilon(x) \le C\varepsilon^\gamma, \quad \forall\, x \in \partial B_{1-M\varepsilon}$$

By the Maximum Principle,

$$v_{M\varepsilon}(x) \le u_\varepsilon(x) + C\varepsilon^\gamma, \quad \forall\, x \in B_{1-M\varepsilon}$$

STEP 4. FINAL ESTIMATE. We have proved that

$$v_{M\varepsilon}(x) \le u_\varepsilon(x) + C\varepsilon^\gamma, \quad \forall\, x \in B_{1-M\varepsilon}$$

and we want to show a similar result for v and u. Indeed, by Hölder continuity,

$$\begin{aligned} v(x) - u(x) &\le [v(x) - v_{M\varepsilon}(x)] + [v_{M\varepsilon}(x) - u_\varepsilon(x)] + [u_\varepsilon(x) - u(x)] \\ &\le \frac{1}{|B_{M\varepsilon}|}\int_{B_{M\varepsilon}(x)} |v(x) - v(y)|\, dy + \varepsilon + \frac{1}{|B_\varepsilon|}\int_{B_\varepsilon(x)} |u(x) - u(y)|\, dy \\ &\le [v]_{C^{0,\gamma}(B_1)} M^\gamma \varepsilon^\gamma + \varepsilon + [u]_{C^{0,\gamma}(B_1)}\varepsilon^\gamma \\ &\le C\varepsilon^\gamma \end{aligned}$$

where $C > 0$ depends only on n, θ, $\|g\|_{L^\infty(\Gamma)}$ and Γ.

Arguing similarly, we can prove that

$$\|u - v\|_{L^\infty(B_{1/2})} \le C(\theta + \delta + \varepsilon^\gamma)$$

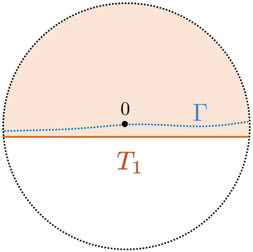
$\square$

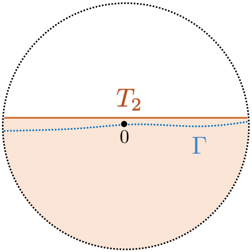
Construction of linear approximations

By induction, we will find $\{P_k^i\}_{k\geq 1}$ such that

$$|u_i(x) - P_k^i(x)| \leq \lambda^{k(1+\alpha)}, \quad \forall\, x \in \Omega_i \cap B_{\lambda^k}, \quad i = 1, 2$$

$\boxed{k = 1}$





$$\Delta v_1 = (1+\eta)\, dS\big|_{T_1} \qquad \Delta v_2 = (1+\eta)\, dS\big|_{T_2}$$

$$P_1^1(x) = v_1(0) + \nabla v_1(0) \cdot x \qquad P_1^2(x) = v_2(0) + \nabla v_2(0) \cdot x$$

Construction of linear approximations

KEY COMPUTATION. If $x \in \Omega_1 \cap B_{1/2}$, then

$$\begin{aligned}|u_1(x) - P_1^1(x)| &\leq |u_1(x) - v_1(x)| + |v_1(x) - P_1^1(x)| \\ &\leq C(\theta + \delta + \varepsilon^\gamma) + \|D^2 v_1\|_{L^\infty(\Omega_1 \cap B_{1/2})}|x|^2 \\ &\leq C(\theta + \delta + \varepsilon^\gamma) + C|x|^2\end{aligned}$$

- Choose $0 < \lambda < 1/2$, such that

$$C|x|^2 \leq \frac{\lambda^{1+\alpha}}{2}, \qquad \forall \; x \in \Omega_1 \cap B_\lambda$$

- Choose $0 < \theta, \delta, \varepsilon < \lambda$ such that

$$C(\theta + \delta + \varepsilon^\gamma) \leq \frac{\lambda^{1+\alpha}}{2}, \qquad \forall \; x \in \Omega_1 \cap B_\lambda$$

Therefore,

$$|u_i(x) - P_1^i(x)| \leq \lambda^{1+\alpha}, \quad \forall \, x \in \Omega_i \cap B_\lambda, \quad i = 1, 2$$

Rescaling and iteration

For $k \geq 1$, consider the rescaled function:

$$w(x) = \frac{u(\lambda^k x) - P_k(\lambda^k x)}{\lambda^{k(1+\alpha)}}, \quad x \in B_1$$

- By induction, $\|w\|_{L^\infty(B_1)} \leq 1$
- There is a harmonic function h *close* to w in B_1
- By the key computation, if $P_0(x) = h(0) + \nabla h(0) \cdot x$, then

$$|w(x) - P_0(x)| \leq \lambda^{1+\alpha}, \quad \forall\, x \in B_\lambda \tag{2}$$

- Define P^i_{k+1} as

$$P^i_{k+1}(x) = P^i_k(x) + \lambda^{k(1+\alpha)} P_0(\lambda^{-k} x)$$

  Then by (2) we get

$$|u_i(x) - P^i_{k+1}(x)| \leq \lambda^{(k+1)(1+\alpha)}, \quad \forall\, x \in \Omega_i \cap B_{\lambda^{k+1}}, \quad i = 1, 2$$

  Therefore $u_1, u_2 \in C^{1,\alpha}(0)$.

Possible Extension to Uniformly Elliptic Operators in Divergence Form

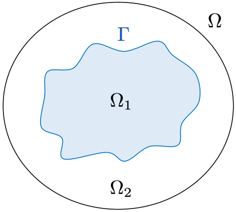
We think that a similar argument could work for symmetric uniformly elliptic operators

$$Lu = \operatorname{div}(A(x)\nabla u), \quad \lambda I \le A \le \Lambda I, \quad 0 < \lambda \le \Lambda < +\infty$$

PROBLEM. Find the pair of functions (u_1, u_2) such that

$$\begin{cases} \operatorname{div}(A_1(x)\nabla u_1) = 0 & \text{in } \Omega_1 \\ \operatorname{div}(A_2(x)\nabla u_2) = 0 & \text{in } \Omega_2 \\ u_2 = 0 & \text{on } \partial\Omega \\ u_1 = u_2 & \text{on } \Gamma \\ \nabla u_1 A_1 \nu - \nabla u_2 A_2 \nu = g & \text{on } \Gamma \end{cases} \tag{TP}$$

* Ω smooth bounded domain in $\mathbb{R}^n$
* A_1, A_2 bounded symmetric and uniformly elliptic
* ν is the normal vector pointing at Ω_1
* $\Gamma = \partial\Omega_1$ is an $n-1$ dimensional $C^{1,\alpha}$ manifold
* $g \in C^{0,\alpha}(\Gamma)$ and $g \ge c_0 > 0$



Possible Extension to Uniformly Elliptic Operators in Divergence Form

We can write the previous problem in the distributional sense as

$$Lu \equiv \operatorname{div}(A\nabla u) = g\, dS\big|_{\Gamma}$$

where $A(x) = A_1(x)\chi_{\Omega_1} + A_2(x)\chi_{\Omega_2}$.

Existence and Uniqueness.

- Littman, W., Stampacchia, G., Weinberger, H. F., *Regular Points for Elliptic Equations with Discontinuous Coefficients*, Annali della Scuola Normale Superiore di Pisa (1963).

THEOREM. For every measure μ of bounded variation the integral

$$u(x) = \int G(x,y)\, d\mu(y)$$

exists and is finite a.e., and is the weak solution vanishing on ∂B_1 of the equation

$$Lu = \mu$$

Possible Extension to Uniformly Elliptic Operators in Divergence Form

Stability result. Based on Maximum Principle and Mean Value Theorem:

- I. Blank and Z. Hao, *The Mean Value Theorem and Basic Properties of the Obstacle Problem for Divergence Form Elliptic Operators*, Communications in Analysis and Geometry (2013).

THEOREM. Let u be a solution to $Lu = 0$ in Ω, and fix any $x_0 \in \Omega$. Then there is an increasing family of sets $\{D_R(x_0)\}_{R>0}$ such that

$$B_{\underline{c}R}(x_0) \subseteq D_R(x_0) \subseteq B_{\overline{c}R}(x_0)$$

for some $\underline{c}, \overline{c} > 0$ depending only on n, λ and Λ, and such that

$$u(x_0) = \frac{1}{|D_R(x_0)|} \int_{D_R(x_0)} u(y)\, dy$$

Thank you for your attention!