Universidad Autónoma de Madrid
Facultad de Ciencias
Departamento de Matemáticas

Fractional Powers of Second Order Partial Differential Operators: Extension Problem and Regularity Theory

Memoria presentada para optar al grado de

Doctor en Matemáticas con Mención Europea

por

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Dirigido por

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Madrid, Abril de 2010

A la memoria de mi padre Raúl Stinga,
que me mira desde el Cielo.

A mi madre Eva, a mis hermanos Julieta y Sebastián.

Agradecimientos

Deseo darle las gracias a todas aquellas personas que de una u otra manera me han apoyado, animado y acompañado desde el comienzo y durante estos últimos cuatro años en mi camino por la Madre Patria.

Quisiera expresar de corazón mi agradecimiento a mi director de tesis, Prof. Dr. José Luis Torrea Hernández, sin quien este trabajo, fruto de su constante guía, ayuda, dedicación y entrega, no hubiese sido posible. Por su cálida acogida desde que puse pie en España, la formación, no sólo matemática sino también humana, el entusiasmo y la confianza que depositó en mí. De manera muy especial por la amistad, la cercanía y el cariño, sus buenos consejos de sabiduría de vida y por el buen ejemplo. Gracias también a su familia por el recibimiento.

A los miembros del Departamento de Matemáticas de la Universidad Autónoma de Madrid, por el buen ambiente de trabajo, la cordialidad y los lindos momentos compartidos que hacen que sea un placer estar en la UAM.

Al profesor István Gyöngy de la University of Edinburgh por recibirme en sendas estancias de investigación que realicé en Febrero-Marzo de 2008 y en Marzo-Abril de 2010. Por brindarme generosamente la oportunidad de aprender de sus conocimientos. Por supuesto gracias a su familia por hacerme sentir como en casa.

A los profesores Jorge Juan Betancor, Juan Carlos Fariña y Lourdes Rodríguez-Mesa de la Universidad de La Laguna, quienes me acogieron en La Laguna en Diciembre de 2009 y me brindaron su amistad.

A los profesores Hugo Aimar y Eleonor Harboure de la Universidad Nacional del Litoral, Argentina, a quienes tuve la suerte de visitar en el Instituto de Matemática Aplicada del Litoral (IMAL) de Santa Fe en dos ocasiones, Septiembre de 2009 y Febrero de 2010. A los demás miembros del IMAL por tanta amabilidad.

Al Ministerio de Educación y Ciencia y al Ministerio de Ciencia e Innovación de España por la financiación a través de la beca de Formación del Personal Investigador Ref. BES-2006-14130 asociada a los proyectos "Métodos Espectrales y Vectoriales en Análisis Armónico" MTM2005-08350-C03-01 y "Teoría Espectral de Semigrupos y su Repercusión en Análisis Armónico" MTM2008-06621-C02-01.

A quienes me ayudaron a llegar a San Luis para hacer la carrera. A los profesores de la Universidad Nacional de San Luis. En particular a Elina Mancinelli (ahora en la Universidad Nacional de Rosario) quien me puso en contacto con Martín Sombra *de la Universitat de Barcelona: a ambos les debo el haber llegado a España.*

A mis amigos de España, Argentina, Italia y Escocia.

Finalmente, aunque no menos importante, a mi familia. ¡Esto es de ustedes!

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Introducción

En los últimos cinco años se ha acrecentado el interés en el estudio de problemas no lineales en los que aparecen potencias fraccionarias de operadores diferenciales parciales de segundo orden. Esta renovada atención en operadores fraccionarios se inició con los trabajos de L. Caffarelli y L. Silvestre y sus colaboradores [67, 68, 23, 22, 26] sobre estimaciones de regularidad para soluciones de problemas no lineales que involucran al Laplaciano fraccionario.

La teoría de potencias fraccionarias de operadores en espacios de Banach es hoy en día un tema clásico del Análisis Funcional. Nombres históricos en el campo son M. Riesz, S. Bochner [9], W. Feller, E. Hille, R. S. Phillips, A. V. Balakrishnan [4], T. Kato [48], K. Yosida, J. Watanabe, M. A. Krasnosel'skii y P. E. Sobolevskii, y H. Komatsu. El artículo de H. Komatsu [51] contiene algunas notas históricas y más referencias detalladas. La teoría puede verse en el clásico libro de K. Yosida [89].

El ejemplo básico de operador fraccionario es el Laplaciano fraccionario. Denotemos por Δ al Laplaciano en $\mathbb{R}^n$, $n \geqslant 1$. Para una función f de la clase de Schwartz la transformada de Fourier nos da

$$\widehat{(-\Delta)f}(\xi) = |\xi|^2\,\widehat{f}(\xi), \qquad \xi \in \mathbb{R}^n.$$

Está claro entonces cómo definir las potencias del Laplaciano: para un número no negativo σ el Laplaciano fraccionario $(-\Delta)^\sigma$ actúa mediante

$$\widehat{(-\Delta)^\sigma f}(\xi) = |\xi|^{2\sigma}\,\widehat{f}(\xi), \qquad \xi \in \mathbb{R}^n, \tag{0.1}$$

y análogamente para las potencias negativas $(-\Delta)^{-\sigma}$, ver [72, p. 117]. Es bien conocido que el Laplaciano fraccionario juega un papel importante en conexión con los espacios de Sobolev en $\mathbb{R}^n$ [72, 73] y en los intentos de definir derivadas fraccionarias en espacios generales como son los espacios de tipo homogéneo [39]. La referencia básica para la Teoría del Potencial para el Laplaciano fraccionario es el libro de N. S. Landkof [52]. Dada la fuerte relación que existe entre $(-\Delta)^\sigma$ y los procesos de Lévy (ver Sección 0.1) la teoría se desarrolló en los últimos años usando un enfoque probabilístico [6, 7, 10, 11, 12, 14, 44, 71], véase también el libro de K. Bogdan et al. [13] que presenta un panorama completo acerca del estado actual del tema.

Respecto a potencias fraccionarias de operadores más generales, debemos decir que un logro muy importante en la última década fue la solución del problema de la raíz cuadrada de Kato por P. Auscher et al. [2]. El problema de Kato tiene sus orígenes en el trabajo de T.

Kato [48]. La conjetura establece que el dominio de la raíz cuadrada de un operador complejo uniformemente elíptico $L = -\operatorname{div}(A\nabla)$ con coeficientes medibles y acotados en $\mathbb{R}^n$, $n \geqslant 1$, es el espacio de Sobolev $H^1(\mathbb{R}^n)$ y se tiene la estimación $\|\sqrt{L}f\|_{L^2(\mathbb{R}^n)} \sim \|\nabla f\|_{L^2(\mathbb{R}^n)}$. Después de 40 años la conjetura fue demostrada con una respuesta positiva en [2]. Para darnos cuenta de su importancia citamos a Carlos Kenig en su reseña MR1933726 de [2]: "*[el artículo] provee una solución completa a cuestiones importantes que aparecen naturalmente en ecuaciones en derivadas parciales, y constituye una hermosa contribución al análisis*". En esta tesis no consideraremos el problema de caracterizar el dominio de las potencias fraccionarias de operadores.

0.1 Ejemplos de problemas que involucran operadores fraccionarios

Describimos brevemente algunos de los problemas donde surgen potencias fraccionarias de operadores diferenciales.

I. Teoría de procesos de Lévy. Sea $X = (X_t; t \geqslant 0)$ un proceso de Lévy empezando en cero con valores en $\mathbb{R}^n$, simétrico y α-estable $(0 < \alpha \leqslant 2)$. Por la fórmula de Lévy-Khintchine (un resultado profundo en la teoría de procesos estocásticos) la función característica de X toma la forma

$$\mathbb{E}(e^{i\xi\cdot X_t}) = e^{-t\kappa^\alpha|\xi|^\alpha}, \qquad \xi \in \mathbb{R}^n,\ t \geqslant 0,$$

para alguna constante positiva κ que por simplicidad tomamos igual a 1. Para $f \in \mathcal{S}$ ponemos

$$T_t f(x) := \mathbb{E}(f(X_t + x)), \qquad x \in \mathbb{R}^n,\ t \geqslant 0.$$

Entonces, por el Teorema de Fubini,

$$\begin{aligned}\widehat{T_t f}(\xi) &= \mathbb{E}\frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n} f(X_t + x)e^{-ix\cdot\xi}\,dx = \mathbb{E}\frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n} f(z)e^{-i(z-X_t)\cdot\xi}\,dz\\ &= \frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n} f(z)e^{-iz\cdot\xi}\,dz\ \mathbb{E}(e^{i\xi\cdot X_t}) = \widehat{f}(\xi)e^{-t|\xi|^\alpha}.\end{aligned}$$

Por tanto,

$$T_t f(x) = e^{-t(-\Delta)^{\alpha/2}}f(x) =: v(x,t),$$

es la solución a la ecuación de difusión fraccionaria

$$\begin{cases} \partial_t v = -(-\Delta)^{\alpha/2}v, & \text{en } \mathbb{R}^n \times (0,\infty),\\ v(x,0) = f(x), & \text{en } \mathbb{R}^n.\end{cases}$$

Notar que $0 < \alpha/2 \leqslant 1$. Véase D. Applebaum [3] o J. Bertoin [8].

Un enfoque sencillo y muy bonito considerando caminos aleatorios con saltos largos puede verse en [86].

`II. El problema de Signorini.` En "Questioni di elasticità non linearizzata e semilinearizzata (Cuestiones de elasticidad no lineal y semilineal)", *Rendiconti di Matematica e delle sue applicazioni* **18** (1959), 95–139, Antonio Signorini propuso encontrar la configuración de una membrana elástica en equilibrio que se encuentra por encima de un obstáculo *fino* dado, digamos de codimensión 1. En términos matemáticos el problema se puede formular como sigue: dada una función suave φ en $\mathbb{R}^n$ que tiende a cero en el infinito, la solución del problema de Signorini es la función $u = u(x, y)$ que satisface

$$\begin{aligned} \partial_{yy}u + \Delta_x u &= 0, && \text{en } \mathbb{R}^n \times (0,\infty), \\ u(x,0) &\geqslant \varphi(x), && \text{en } \mathbb{R}^n, \\ \partial_y u(x,0) &\leqslant 0, && \text{en } \mathbb{R}^n, \\ \partial_y u(x,0) &= 0, && \text{en } \{u(x,0) > \varphi(x)\}, \end{aligned} \tag{0.2}$$

y es cero en el infinito. Véanse [18, 38, 49] y [67, 68].

Una observación muy sencilla proporciona una descripción equivalente del problema donde aparece el Laplaciano fraccionario. La solución de (0.2) con dato de frontera $f(x) := u(x, 0)$ viene dada por convolución con el núcleo de Poisson en el semiespacio superior:

$$u(x, y) = e^{-y(-\Delta_x)^{1/2}} f(x).$$

Tomamos la derivada de u con respecto y y la evaluamos en cero para obtener

$$\partial_y u(x, y)\Big|_{y=0} = -(-\Delta_x)^{1/2} f(x).$$

Luego, el problema de Signorini se puede reescribir como

$$\begin{aligned} \partial_{yy}u + \Delta_x u &= 0, && \text{en } \mathbb{R}^n \times (0,\infty), \\ u(x,0) &\geqslant \varphi(x), && \text{en } \mathbb{R}^n, && (0.3) \\ (-\Delta_x)^{1/2}u(x,0) &\geqslant 0, && \text{en } \mathbb{R}^n, && (0.4) \\ (-\Delta_x)^{1/2}u(x,0) &= 0, && \text{en } \{u(x,0) > \varphi(x)\}, && (0.5) \end{aligned}$$

con condición de frontera cero en el infinito.

Consideremos un problema de Signorini donde el Laplaciano $-\Delta_x$ en (0.2) se reemplaza por un operador diferencial parcial de segundo orden $L = L_x$ en $\mathbb{R}^n$. El semigrupo de Poisson asociado

$$u(x, y) = e^{-y L_x^{1/2}} f(x),$$

es la solución de

$$\begin{cases} \partial_{yy}u - L_x u = 0, & \text{en } \mathbb{R}^n \times (0,\infty), \\ u(x,0) = f(x), & \text{en } \mathbb{R}^n. \end{cases}$$

Por tanto en esta situación el problema se puede formular en términos de una potencia fraccionaria de L:

$$\begin{aligned} \partial_{yy}u - L_x u &= 0, && \text{en } \mathbb{R}^n \times (0,\infty), \\ u(x,0) &\geqslant \varphi(x), && \text{en } \mathbb{R}^n, \\ L_x^{1/2}u(x,0) &\geqslant 0, && \text{en } \mathbb{R}^n, \\ L_x^{1/2}u(x,0) &= 0, && \text{en } \{u(x,0) > \varphi(x)\}. \end{aligned}$$

Se puede proponer también un caso completamente no lineal donde $(\partial_{yy} + \Delta_x)u$ se reemplaza por $F(D^2u(x,y))$, véase E. Milakis y L. Silvestre [58].

`III. Problemas de obstáculo.` Dado un proceso de Lévy X_t simétrico, α-estable, con $X_0 = 0$, consideremos el tiempo óptimo de parada τ que maximiza la función

$$v(x) = \sup_\tau \mathbb{E}\left[\varphi(X_\tau) : \tau < \infty\right].$$

Entonces se puede ver que v es una solución del problema de frontera libre

$$\begin{aligned} v(x) &\geqslant \varphi(x), && \text{en } \mathbb{R}^n, \\ (-\Delta)^\sigma v(x) &\geqslant 0, && \text{en } \mathbb{R}^n, \\ (-\Delta)^\sigma v(x) &= 0, && \text{en } \{v(x) > \varphi(x)\}, \end{aligned}$$

con condición de borde cero en el infinito y $\sigma := \alpha/2$. Este tipo de problema aparece en Matemática Financiera como un modelo para asignación de precios de opciones americanas [29, 67, 68]. Nótese que $\sigma = 1$ es el problema del obstáculo clásico en $\mathbb{R}^n$ [38, 49, 20]. Como hemos visto el problema de Signorini da lugar al conjunto de ecuaciones (0.3)-(0.4)-(0.5) que es el problema del obstáculo para el Laplaciano fraccionario con $\sigma = 1/2$ y $v(x) := u(x,0)$.

Las propiedades de regularidad de la solución y la frontera libre del problema del obstáculo para el Laplaciano fraccionario fueron estudiadas por L. Silvestre [67, 68] y por L. Caffarelli, S. Salsa y L. Silvestre [22].

`IV. Mecánica de fluidos.` La ecuación quasi-geostrófica (QG para abreviar) disipativa es de la forma

$$\theta_t + u \cdot \nabla\theta = -\kappa(-\Delta)^\sigma\theta, \qquad x \in \mathbb{R}^2,\ t > 0,$$

donde $\kappa > 0$, $0 \leqslant \sigma \leqslant 1$ y $\theta = \theta(x,t)$ es una función escalar. El campo vectorial u es un campo de velocidades 2-dimensional determinado a partir de θ mediante una función de flujo ψ a través de las relaciones

$$u = (u_1, u_2) = (-\partial_{x_2}\psi, \partial_{x_1}\psi), \qquad (-\Delta)^{1/2}\psi = \theta,$$

con lo cual

$$(u_1, u_2) = (-R_2\theta, R_1\theta),$$

siendo los operadores R_i las transformadas de Riesz clásicas $\partial_{x_i}(-\Delta)^{-1/2}$, $i = 1, 2$. Esta ecuación modela la evolución de la temperatura en la frontera 2-D de un flujo quasi-geostrófico 3-D y a veces se la conoce como ecuación QG de superficie. Véanse las notas de curso de P. Constantin [27] y el libro de A. Majda y A. Bertozzi [56].

La ecuación QG también es importante porque se puede ver como un modelo juguete para investigar las propiedades de regularidad de las ecuaciones de Navier-Stokes, véanse [56, 19] y [27] y referencias en el mismo. Bajo la hipótesis de incompresibilidad del fluido las ecuaciones de Navier-Stokes, que determinan la velocidad v y la presión p del fluido son

$$u_t + (u \cdot \nabla)\, u + \nabla p = \Delta u + f, \qquad \operatorname{div} u = 0.$$

QG es una ecuación de tipo "Navier-Stokes" en 2D. En dimensión 2, las ecuaciones de Navier-Stokes se simplifican considerablemente ya que la incompresibilidad, $\operatorname{div} u = 0$, implica que $(-u_2, u_1)$ es el gradiente de alguna función: $(-u_2, u_1) = \nabla\psi$, y $\operatorname{rot} u$ es un escalar, $\theta = \operatorname{rot} u = \Delta\psi$. Luego, aplicando el rotacional, las ecuaciones de Navier-Stokes se convierten en un sistema:

$$\theta_t + u \cdot \nabla\theta = \Delta\theta, \qquad \operatorname{rot} u = \theta.$$

En el modelo QG todavía tenemos $(-u_2, u_1) = \nabla\psi$ pero el potencial ψ está relacionado con la vorticidad mediante $\theta = (-\Delta)^{1/2}\psi$ y el sistema final es

$$\theta_t + u \cdot \nabla\theta = -(-\Delta)^{1/2}\theta, \qquad (-u_2, u_1) = (R_1\theta, R_2\theta).$$

Motivados por este problema L. Caffarelli y A. Vasseur mostraron en [26] que la ecuación de difusión-dispersión con difusión fraccionaria

$$\theta_t + v \cdot \nabla\theta = -(-\Delta)^{\sigma}\theta, \qquad \operatorname{div} v = 0, \qquad x \in \mathbb{R}^n,$$

donde $v_j = T_j\theta$ para T_j un operador de integral singular y $\sigma = 1/2$ (caso crítico) tiene soluciones clásicas para cualquier dato inicial en L^2. El método de la prueba consiste en localizar al Laplaciano fraccionario $(-\Delta)^{1/2}$ via la extensión armónica $\theta^*(x, t, y)$ de $\theta(x, t)$ al semiespacio superior añadiendo una nueva variable y y usando entonces las ideas de De Giorgi [31, 42] para el problema localizado. La técnica también fue aplicada, por ejemplo, en [28] y [65] para el caso supercrítico $\sigma < 1/2$. Usando otros métodos se muestra en [50] que cuando $\sigma = 1/2$ y el dato inicial es suave, existe una única solución de QG que también es suave.

Nuestra lista de problemas dada anteriormente no pretende ser exhaustiva. Sólo por mencionar algunos más, además de los modelos en Matemática Financiera [29] y en mecánica de fluidos [56, 27], hay aplicaciones en la Física Moderna, por ejemplo, cuando se consideran cinéticas fraccionarias y transporte anómalo [90, 70], cinéticas raras [64], mecánica cuántica fraccionaria [53, 54] y procesos de Lévy en mecánica cuántica [61].

0.2 Motivación

Describamos algunas de las propiedades especiales del Laplaciano fraccionario.

Lo primero a notar es que para una función $f \in \mathcal{S}$ y σ no entero, $(-\Delta)^\sigma f$ no es una función de la clase de Schwartz debido a la singularidad introducida en (las derivadas de) su transformada de Fourier (0.1). Para el resto de este trabajo restringiremos nuestra atención a $0 < \sigma < 1$, que es el caso que aparece en las aplicaciones.

Claramente la definición (0.1) no es apropiada para estimaciones de regularidad en espacios de Hölder, ya que querríamos manejar diferencias de la forma

$$(-\Delta)^\sigma f(x_1) - (-\Delta)^\sigma f(x_2), \qquad x_1, x_2 \in \mathbb{R}^n.$$

Por lo tanto es necesaria una fórmula puntual para $(-\Delta)^\sigma f(x)$. Si aplicamos la transformada de Fourier inversa en (0.1) obtenemos

$$(-\Delta)^\sigma f(x) = c_{n,\sigma}\, \mathrm{P.\,V.} \int_{\mathbb{R}^n} \frac{f(x) - f(z)}{|x - z|^{n+2\sigma}}\, dz, \qquad x \in \mathbb{R}^n, \tag{0.6}$$

donde $c_{n,\sigma}$ es una constante positiva que depende sólo de la dimensión y de σ. Puntualicemos que tal constante es importante cuando por ejemplo queremos tomar el límite $\sigma \to 1^-$ para recuperar el Laplaciano $-\Delta$ en $\mathbb{R}^n$ o $\sigma \to 0^+$ para obtener la identidad (un hecho trivial si $f \in \mathcal{S}$). Usando (0.6) se puede estudiar la interacción de $(-\Delta)^\sigma$ con espacios de Hölder C^α. En particular, si $f \in C^{0,\alpha}(\mathbb{R}^n)$ y $0 < 2\sigma < \alpha$ entonces $(-\Delta)^\sigma f \in C^{0,\alpha-2\sigma}(\mathbb{R}^n)$ y

$$|(-\Delta)^\sigma f(x_1) - (-\Delta)^\sigma f(x_2)| \leqslant C[f]_{C^\alpha} |x_1 - x_2|^{\alpha - 2\sigma}, \qquad x_1, x_2 \in \mathbb{R}^n, \tag{0.7}$$

donde C depende sólo de α, σ y n, y resultados similares valen para los espacios $C^{k,\alpha}$, véanse [67, 68]. Para más aplicaciones de las estimaciones de Schauder para el Laplaciano fraccionario ver por ejemplo [22, 25].

A partir de (0.6) observamos que $(-\Delta)^\sigma$ es un operador no local: el valor de $(-\Delta)^\sigma f(x)$ para un $x \in \mathbb{R}^n$ dado depende de los valores de f en el infinito. Esta propiedad crea complicaciones: los métodos locales clásicos en EDPs del Cálculo de Variaciones no se pueden aplicar al estudio de problemas no lineales que involucran $(-\Delta)^\sigma$. Para superar esta dificultad L. Caffarelli y L. Silvestre mostraron en [23] que cualquier potencia fraccionaria del Laplaciano se puede caracterizar como un operador que envía una condición de frontera Dirichlet a una condición de tipo Neumann a través de un problema de extensión. Expliquemos cómo se hace esto. Consideremos la función $u = u(x, y) : \mathbb{R}^n \times [0, \infty) \to \mathbb{R}$ que resuelve el problema de contorno

$$u(x, 0) = f(x), \qquad x \in \mathbb{R}^n, \tag{0.8}$$

$$\Delta_x u + \frac{1 - 2\sigma}{y}\, u_y + u_{yy} = 0, \qquad x \in \mathbb{R}^n,\ y > 0. \tag{0.9}$$

Nótese que (0.9) es una ecuación elíptica degenerada. Entonces, salvo una constante multiplicativa que depende sólo de σ,

$$-\lim_{y\to 0^+} y^{1-2\sigma} u_y(x,y) = (-\Delta)^\sigma f(x), \qquad x \in \mathbb{R}^n.$$

Podemos interpretar este resultado diciendo que la nueva variable y que agregamos a través de (0.9) para extender f al semiespacio superior codifica los valores de f en el infinito que se necesitan para calcular $(-\Delta)^\sigma f$. El problema de extensión localiza al Laplaciano fraccionario: basta conocer u en alguna (semi) bola (superior) alrededor de $(x,0)$ para obtener $(-\Delta)^\sigma f(x)$. Los problemas no lineales para el Laplaciano fraccionario (no local) se pueden localizar agregando una nueva variable, véanse [22, 26, 28, 65] para ejemplos de la técnica.

En [26] se usa el problema de extensión con $\sigma = 1/2$: (0.8)-(0.9) se convierten en el problema de la extensión armónica de f al semiespacio superior y, como vimos en la sección anterior, a través de la condición de Neumann recuperamos $(-\Delta)^{1/2} f(x)$.

Se puede aplicar el problema de extensión para probar (entre otras propiedades de regularidad) la desigualdad de Harnack para $(-\Delta)^\sigma$, véase [23]. El carácter no local del Laplaciano fraccionario hace que en la desigualdad de Harnack tengamos que suponer que $f(x) \geqslant 0$ para *todo* $x \in \mathbb{R}^n$, y no sólo para los x en alguna bola como es usual, véase [46].

Consideremos la situación en que hemos derivado un modelo (normalmente un problema de EDPs no lineal) que involucra una potencia fraccionaria de algún operador diferencial parcial de segundo orden L. Entonces tenemos que responder al menos a las siguientes cuestiones:

(I) Definición y fórmula puntual para operadores fraccionarios. Para un operador general L, el Análisis Funcional clásico nos brinda varias maneras de definir L^σ de acuerdo con sus propiedades analíticas. Sin embargo, una fórmula abstracta no es útil a la hora de tratar problemas concretos de EDPs y se necesitará una expresión puntual más o menos explícita para $L^\sigma f(x)$. Para el Laplaciano fraccionario empezamos con la definición con la transformada de Fourier (0.1) y tomando su inversa obtuvimos (0.6). ¿Qué se puede hacer en el caso general donde la transformada de Fourier no está disponible?.

(II) Teoría de regularidad para operadores fraccionarios. Avanzando hacia ejemplos concretos de operadores diferenciales de segundo orden L, podríamos preguntarnos por las estimaciones de Schauder "correctas" para L^σ o, más precisamente, el espacio de Hölder apropiado/adaptado en el que buscar propiedades de regularidad de L^σ. Otra cuestión es la validez de desigualdades de Harnack, una herramienta importante en la teoría de EDPs [45].

(III) La naturaleza no local. En general, las potencias fraccionarias de operadores diferenciales parciales de segundo orden son operadores no locales. Sería muy útil en las aplicaciones tener una caracterización de L^σ análoga a la de Caffarelli-Silvestre, como un operador de tipo Dirichlet-to-Neumann a través de un problema de extensión. De ser así, ¿cómo usar esa caracterización para obtener estimaciones de regularidad?.

En esta tesis tenemos como objetivo responder a estos interrogantes.

0.3 Descripción de los resultados

Sea L un operador diferencial parcial de segundo orden definido en algún espacio $L^2(\Omega, d\eta)$, donde Ω es un subconjunto abierto de $\mathbb{R}^n$, $n \geqslant 1$, y $d\eta$ es una medida positiva sobre Ω.

A no ser que estemos trabajando con el Laplaciano en $\mathbb{R}^n$, la transformada de Fourier no será muy útil para estudiar operadores fraccionarios. Necesitamos encontrar un lenguaje que pueda explicar los conceptos, fórmulas y propiedades con las que queremos tratar de una manera clara y unificada.

Adoptamos el **lenguaje del semigrupo del calor**, en el que el operador central es el **semigrupo de difusión del calor generado por** L que denotamos mediante

$$e^{-tL}, \qquad \text{para } t \geqslant 0.$$

Como pretendemos mostrar en esta tesis, tal escenario resulta ser muy adecuado: nos da la comprensión acerca de las fórmulas correctas (con constantes explícitas y fácilmente calculables) y también es bastante general. Aparecerá una relación interesante con funciones especiales como las funciones Gamma y de Bessel, mostrando la armonía que hay detrás.

Bajo este punto de vista repasaremos el Laplaciano fraccionario y también estudiaremos las preguntas que nos pusimos antes para un operador general L. La caracterización de L^σ como un operador de tipo Dirichlet-to-Neumann a través de un problema de extensión se obtendrá gracias al lenguaje adoptado. Las consideraciones de regularidad las analizamos en un caso: las potencias fraccionarias del oscilador armónico

$$H = -\Delta + |x|^2, \qquad \text{en } \mathbb{R}^n.$$

Este es un operador importante en mecánica cuántica, véase por ejemplo el libro de R. P. Feynmann y A. R. Hibbs [37]. Para $H^\sigma = (-\Delta + |x|^2)^\sigma$ obtendremos la desigualdad de Harnack y las estimaciones de Schauder adaptadas.

Procedamos a describir los próximos capítulos.

0.3.1 Capítulo 2: Preliminares

Como ya puntualizamos, el semigrupo de difusión del calor e^{-tL} generado por un operador de segundo orden L es el operador clave en nuestro trabajo.

En este capítulo mostramos cómo expresar varios operadores asociados a L como semigrupos de Poisson $e^{-y\sqrt{L}}$, integrales fraccionarias $L^{-\sigma}$, potencias fraccionarias L^σ y transformadas de Riesz asociadas a L en términos de e^{-tL}, véanse **fórmulas** (2.3), (2.5), (2.8), (2.9) y (2.10). Las expresiones están basadas en la definición de la función Gamma. Para el operador fraccionario tenemos la siguiente fórmula bonita y manejable con la cual comenzar:

$$L^\sigma f(x) = \frac{1}{\Gamma(-\sigma)} \int_0^\infty \left(e^{-tL} f(x) - f(x)\right) \frac{dt}{t^{1+\sigma}}. \tag{0.10}$$

Los conceptos dados anteriormente se entienden muy bien en dos ejemplos particulares (y al mismo tiempo generales), a saber, cuando L tiene espectro discreto y cuando $L = -\Delta$, el Laplaciano en $\mathbb{R}^n$, que tiene espectro continuo.

Mostramos cómo el lenguaje de semigrupos simplifica e ilumina el ejemplo básico, el Laplaciano fraccionario. En efecto, usando (0.10) con $L = -\Delta$ se deriva la fórmula puntual (0.6) con la constante exacta $c_{n,\sigma}$ sin necesidad de aplicar la transformada inversa de Fourier (**Lema 2.1**). La **Proposición 2.3** establece que

$$(-\Delta)^\sigma f(x) \to -\Delta f(x), \qquad \text{cuando } \sigma \to 1^-,$$

para todo x donde f es C^2 y la **Proposición 2.5** dice que

$$(-\Delta)^\sigma f(x) \to f(x), \qquad \text{cuando } \sigma \to 0^+,$$

para cualquier x que sea un punto de continuidad Hölder de f.

Se presentan los preliminares sobre el oscilador armónico H (espectro discreto).

Finalmente recolectamos la teoría espectral básica que necesitaremos. El contexto abstracto es muy útil ya que nos permite tratar ambos ejemplos (espectro discreto y continuo) de forma unificada.

0.3.2 Capítulo 3: Definición y problema de extensión para potencias fraccionarias de operadores diferenciales parciales de segundo orden

Este capítulo contiene **parte de los resultados del artículo [79]**.

Una vez que L^σ está dado a través del semigrupo del calor e^{-tL} mediante (0.10), exploramos una caracterización en el espíritu de [23].

Para resolver (0.8)-(0.9) Caffarelli y Silvestre se dieron cuenta que (0.9) se puede pensar como la extensión armónica de f a $2 - 2\sigma$ dimensiones. A partir de ahí establecieron la solución fundamental $\Gamma_\sigma(x, y)$ de (0.9), que tiene la propiedad de que $\Gamma_\sigma(x, 0)$ es la solución fundamental de $(-\Delta)^\sigma$, esto es, *la solución fundamental de la extensión es una extensión de la solución fundamental* $(-\Delta)^{-\sigma}$. Entonces, usando una ecuación conjugada, obtuvieron una fórmula de Poisson para u:

$$u(x, y) = \int_{\mathbb{R}^n} P_y^{\sigma,\Delta}(x - z) f(z)\, dz.$$

La función $P_y^{\sigma,\Delta}(z)$ es el núcleo de Poisson.

Nuestro enfoque de semigrupos clarifica el caso del Laplaciano y establecerá un escenario general para el problema de extensión que nos permitirá incluir otros operadores L.

Como sugieren los ejemplos, suponemos que L es autoadjunto. Entonces obtenemos en el **Teorema 3.1** la caracterización de L^σ que generaliza el resultado de Caffarelli-Silvestre.

Esto es, consideremos el siguiente problema de extensión al semiespacio superior:

$$\begin{aligned} u(x,0) &= f(x), && \text{en } \Omega; \\ -L_x u + \frac{1-2\sigma}{y}\, u_y + u_{yy} &= 0, && \text{en } \Omega \times (0,\infty). \end{aligned}$$

Mostramos que una solución u viene dada explícitamente mediante

$$u(x,y) = \frac{1}{\Gamma(\sigma)} \int_0^\infty e^{-tL}(L^\sigma f)(x) e^{-\frac{y^2}{4t}}\, \frac{dt}{t^{1-\sigma}}. \tag{0.11}$$

Esta fórmula es una de las principales novedades aquí. Entonces se demuestra que para cada $x \in \Omega$,

$$\lim_{y\to 0^+} \frac{u(x,y)-u(x,0)}{y^{2\sigma}} = \frac{\Gamma(-\sigma)}{4^\sigma \Gamma(\sigma)} L^\sigma f(x) = \frac{1}{2\sigma} \lim_{y\to 0^+} y^{1-2\sigma} u_y(x,y).$$

La expresión para u (0.11) requiere conocer la acción de L^σ en f. Para mejorar esa situación derivamos la fórmula de Poisson:

$$u(x,y) = \frac{y^{2\sigma}}{4^\sigma \Gamma(\sigma)} \int_0^\infty e^{-tL} f(x) e^{-\frac{y^2}{4t}}\, \frac{dt}{t^{1+\sigma}} = \int_\Omega P_y^{\sigma,L}(x,z) f(z)\, d\eta(z), \tag{0.12}$$

donde no aparece ninguna potencia fraccionaria de L y $P_y^{\sigma,L}(x,z)$ es el núcleo de Poisson.

Cuando $L = -\Delta$ se recuperan el resultado de extensión y el núcleo de Poisson de [23]. Vale la pena mencionar que en [23] no se establecen ni la fórmula explícita (0.11) ni la primera identidad de (0.12), sino la fórmula de Poisson de convolución con el núcleo de Poisson como explicamos más arriba.

La demostración de la caracterización mediante la extensión está siempre dada en nuestro lenguaje de semigrupos y usa de manera esencial el Teorema Espectral. Sin embargo, antes de presentarla y por propósitos de exposición, probamos el resultado para L con espectro discreto y para el Laplaciano.

En el **Teorema 3.2** se describen más propiedades que conciernen a la fórmula de Poisson, como principios del máximo y estimaciones en L^p para el problema de extensión. Mostramos que la fórmula de Poisson para u se puede derivar con la idea inteligente de [23]: usar la solución fundamental (que involucra al núcleo del semigrupo del calor generado por L) y una ecuación conjugada apropiada para inferir el núcleo de Poisson.

Se estudia la ecuación conjugada en detalle. Para ello se definen ecuaciones de Cauchy-Riemann adaptadas a la ecuación de extensión, véase (3.15).

Se proporcionan ejemplos.

El lector podría preguntarse cómo *adivinar* la fórmula explícita (0.11) para la solución u. Presentamos una prueba muy bonita del resultado de extensión en términos de expansiones ortogonales y también las derivación de soluciones de Neumann locales. Las ecuaciones resultantes son ecuaciones de Bessel cuyas soluciones en términos de funciones de Bessel nos darán la clave para (0.11).

0.3.3 Capítulo 4: Definición, propiedades básicas y desigualdad de Harnack para las potencias fraccionarias del oscilador armónico

Aquí recogemos **resultados de** [**79**].

Se aplica la fórmula (0.10) con $L = H = -\Delta + |x|^2$, el oscilador armónico, para derivar la fórmula puntual

$$H^\sigma f(x) = \int_{\mathbb{R}^n} (f(x) - f(z))F_\sigma(x, z)\ dz + f(x)B_\sigma(x), \qquad x \in \mathbb{R}^n,$$

donde el núcleo $F_\sigma(x, z)$ y la función $B_\sigma(x)$ están dadas en términos del núcleo del calor para H, véase el **Teorema 4.3**. Luego el oscilador armónico fraccionario es un operador no local. Se obtienen algunos principios del máximo y de comparación (**Teorema 4.6 y Corolario 4.7**).

Estudiamos en detalle el problema de extensión para H^σ (**Teorema 4.13**) ya que usamos las ideas de [23] para probar la desigualdad de Harnack para H^σ: para cada $R > 0$ y $x_0 \in \mathbb{R}^n$, existe una constante positiva $C = C_{\sigma,R,n,x_0}$ tal que

$$\sup_{B_{R/2}(x_0)} f \leqslant C \inf_{B_{R/2}(x_0)} f,$$

para todas las funciones no negativas $f : \mathbb{R}^n \to \mathbb{R}$ que son C^2 en $B_R(x_0)$ y que satisfacen $H^\sigma f(x) = 0$ para todo $x \in B_R(x_0)$, véase el **Teorema 4.10**.

Para demostrar la desigualdad de Harnack para $(-\Delta)^\sigma$ los autores de [23] aprovechan la teoría general de ecuaciones elípticas degeneradas desarrollada por E. Fabes, D. Jerison, C. Kenig y R. Serapioni en 1982-83. En nuestro caso no es necesaria la teoría general, sino la desigualdad de Harnack para operadores de Schrödinger degenerados probada por C. E. Gutiérrez en [41].

0.3.4 Capítulo 5: Interacción del oscilador armónico fraccionario con los espacios de Hölder adaptados a H y estimaciones de Schauder

Este capítulo **corresponde a** [**80**].

Para derivar estimaciones de Schauder para H^σ (del tipo (0.7)) primero necesitamos encontrar el espacio de Hölder correcto naturalmente asociado a H.

Se introduce una nueva clase de espacios de Hölder $C_H^{k,\alpha}$, $0 < \alpha \leqslant 1$, $k \in \mathbb{N}_0$, adaptada a H, que llamamos **espacios de Hermite-Hölder**. Estos espacios (más pequeños que las clases $C^{k,\alpha}(\mathbb{R}^n)$ clásicas) están definidos de forma tal que permiten a las funciones que contienen algún crecimiento en el infinito, siendo la moral detrás de esto que el Laplaciano $-\Delta$ dicta la regularidad y el potencial $|x|^2$ en el operador juega un papel relevante sólo en el infinito. Véase la **Definición 5.1** para la definición de $C_H^{0,\alpha}$.

La factorización del oscilador armónico en términos de operadores diferenciales de primer orden

$$H = \frac{1}{2}\sum_{i=1}^{n}\left[\,(\partial_{x_i} + x_i)\,(-\partial_{x_i} + x_i) + (-\partial_{x_i} + x_i)\,(\partial_{x_i} + x_i)\,\right],$$

nos proporciona la definición correcta del operador "derivada" naturalmente asociado a H, a saber

$$A_i = \partial_{x_i} + x_i, \qquad A_{-i} = -\partial_{x_i} + x_i.$$

Entonces los espacios $C_H^{k,\alpha}$ se definen de la forma usual como el conjunto de todas las funciones diferenciables cuyas $A_{\pm i}$-derivadas hasta orden k pertenecen a $C_H^{0,\alpha}$ (**Definición 5.2**).

El primer teorema principal, **Teorema A**, básicamente dice que

$$H^\sigma : C_H^{0,\alpha} \to C_H^{0,\alpha-2\sigma}, \qquad 2\sigma < \alpha,$$

y

$$H^\sigma : C_H^{1,\alpha} \to C_H^{0,\alpha-2\sigma+1}, \qquad 2\sigma \geqslant \alpha,$$

continuamente, y similarmente para espacios de orden superior. Luego H^σ actúa como una *derivada fraccionaria.*

El resultado anterior se puede interpretar diciendo que los espacios de Hölder $C_H^{k,\alpha}$ son las clases razonables para derivar estimaciones de Schauder para H^σ. En efecto, esto es lo que asegura el segundo resultado principal (**Teorema B**)

$$H^{-\sigma} : C_H^{0,\alpha} \to C_H^{0,\alpha+2\sigma}, \qquad \alpha + 2\sigma \leqslant 1,$$

$$H^{-\sigma} : C_H^{0,\alpha} \to C_H^{1,\alpha+2\sigma-1}, \qquad 1 < \alpha + 2\sigma \leqslant 2,$$

y

$$H^{-\sigma} : C_H^{0,\alpha} \to C_H^{2,\alpha+2\sigma-2}, \qquad 2 < \alpha + 2\sigma \leqslant 3,$$

continuamente. Luego $H^{-\sigma}$ es un operador *derivada fraccionaria inversa* en $C_H^{k,\alpha}$.

Una de las tareas principales en el capítulo será derivar las fórmulas puntuales para los operadores $H^\sigma u$ y $H^{-\sigma}u$ (y sus derivadas) cuando u pertenece al espacio de Hermite-Hölder.

Para demostrar nuestros resultados también tenemos que considerar las transformadas de Hermite-Riesz $\mathcal{R}_i$ y $\mathcal{R}_{ij}$, $i, j = 1, \dots, n$, cuando actúan en $C_H^{0,\alpha}$. Como $C_H^{0,\alpha}$ es el espacio correcto, se prueba el resultado esperado:

$$\mathcal{R}_i, \mathcal{R}_{ij} : C_H^{0,\alpha} \to C_H^{0,\alpha},$$

continuamente.

En algunos artículos recientes, B. Bongioanni, E. Harboure y O. Salinas estudiaron la acotación de las integrales fraccionarias [15] y de las transformadas de Riesz [16] asociadas a una cierta clase de operadores de Schrödinger $\mathcal{L} = -\Delta + V$, en espacios de tipo $BMO_{\mathcal{L}}^{\beta}$, $0 \leqslant \beta < 1$, utilizando técnicas de Análisis Armónico. En [15, Proposition 4] mostraron que los espacios $BMO_{\mathcal{L}}^{\beta}$ coinciden con un espacio de tipo Hölder $\Lambda_{\mathcal{L}}^{\beta}$, $0 < \beta < 1$, con normas equivalentes. En el caso $V = |x|^2$, nuestro espacio $C_H^{0,\beta}$ coincide con su espacio Λ_H^{β}, para $0 < \beta < 1$. Los espacios BMO asociados a $\mathcal{L}$ fueron definidos y estudiados por primera vez en [33], véase también [57]. Para la acotación de operadores relacionados con el oscilador armónico en el espacio Euclídeo BMO clásico véase [76].

Muy recientemente supimos del artículo de R. F. Bass [5] donde el autor, motivado por el Laplaciano fraccionario, considera lo que él denomina *stable-like operators* y estudia su interacción con los espacios de Hölder clásicos C^{α} para α no entero. Nuestro oscilador armónico fraccionario no es un ejemplo de ese tipo de operadores ya que el Lema 4.8 del Capítulo 4 hace que la Assumption 1.1 en [5] no valga. Véanse también [66, 24, 25, 47, 21].

Notación. A lo largo de esta tesis $\mathcal{S}$ es la clase de Schwartz de funciones $C^{\infty}(\mathbb{R}^n)$ con decaimiento rápido en el infinito, la letra C denota una constante que puede cambiar en cada aparición y dependerá de los parámetros involucrados (cuando sea necesario puntualizaremos esta dependencia con subíndices) y Γ es la función Gamma [1, 34]:

$$\Gamma(z) = \int_0^{\infty} e^{-t} t^{z-1}\, dt, \qquad \Re z > 0.$$

Sin mencionarlo, aplicaremos en repetidas ocasiones la desigualdad $r^{\nu} e^{-r} \leqslant C_{\nu} e^{-r/2}$, $\nu \geqslant 0$, $r > 0$.

Chapter 1

Introduction

During the last five years it has been an increasing interest in the study of nonlinear problems in which fractional powers of second order partial differential operators appear. This renewed attention on fractional operators started with the works by L. Caffarelli and L. Silvestre and collaborators [67, 68, 23, 22, 26] on regularity estimates for solutions of problems involving the fractional Laplacian.

The theory of fractional powers of operators on Banach spaces is nowadays a classical topic in Functional Analysis. Historic names in the field are M. Riesz, S. Bochner [9], W. Feller, E. Hille, R. S. Phillips, A. V. Balakrishnan [4], T. Kato [48], K. Yosida, J. Watanabe, M. A. Krasnosel'skii and P. E. Sobolevskii, and H. Komatsu. The paper by H. Komatsu [51] contains some historical notes and more detailed references. The subject can be found in the classical book by K. Yosida [89].

The basic example of fractional operator is the fractional Laplacian. Let Δ denote the Laplacian in $\mathbb{R}^n$, $n \geqslant 1$. For a Schwartz's class function f the Fourier transform gives

$$\widehat{(-\Delta)f}(\xi) = |\xi|^2 \, \widehat{f}(\xi), \qquad \xi \in \mathbb{R}^n.$$

Then it is clear how to define the powers of the Laplacian: for a nonnegative number σ the fractional Laplacian $(-\Delta)^\sigma$ acts as

$$\widehat{(-\Delta)^\sigma f}(\xi) = |\xi|^{2\sigma} \, \widehat{f}(\xi), \qquad \xi \in \mathbb{R}^n, \tag{1.1}$$

and analogously for the negative powers $(-\Delta)^{-\sigma}$, see [72, p. 117]. It is well known that the fractional Laplacian plays an important role in connection with Sobolev spaces in $\mathbb{R}^n$ [72, 73] and in the attempts to define fractional derivatives in general spaces like spaces of homogeneous type [39]. The basic reference for the Potential Theory for the fractional Laplacian is the book by N. S. Landkof [52]. Because of the strong relation between $(-\Delta)^\sigma$ and Lévy processes (see Section 1.1) the theory was developed in the last years using the probabilistic approach [6, 7, 10, 11, 12, 14, 44, 71], see also the book by K. Bogdan et al. [13] that presents a complete overview on the actual state of the field.

Regarding fractional powers of more general operators, we must point out that a very important achievement of the last decade was the solution of the Kato square root problem by P. Auscher et al. [2]. The Kato problem originates from the work by T. Kato [48]. The conjecture states that the domain of the square root of a uniformly complex elliptic operator $L = -\operatorname{div}(A\nabla)$ with bounded measurable coefficients in $\mathbb{R}^n$, $n \geqslant 1$, is the Sobolev space $H^1(\mathbb{R}^n)$ with the estimate $\|\sqrt{L}f\|_{L^2(\mathbb{R}^n)} \sim \|\nabla f\|_{L^2(\mathbb{R}^n)}$. After 40 years the conjecture was proved with positive answer in [2]. To notice its importance we quote Carlos Kenig in his review MR1933726 of [2]: "*[the paper] provides a complete solution to important questions that arise naturally in partial differential equations, and constitute a beautiful contribution to analysis*". In this dissertation we will not consider the problem of characterizing the domain of fractional powers of operators.

1.1 Examples of problems involving fractional operators

We briefly describe some of the problems where fractional powers of differential operators arise.

I. Theory of Lévy processes. Let $X = (X_t; t \geqslant 0)$ be a symmetric α-stable $(0 < \alpha \leqslant 2)$ $\mathbb{R}^n$-valued Lévy process starting at 0. By the Lévy-Khintchine formula (a deep result in the theory of stochastic processes) the characteristic function of X takes the form

$$\mathbb{E}(e^{i\xi\cdot X_t}) = e^{-t\kappa^\alpha|\xi|^\alpha}, \qquad \xi \in \mathbb{R}^n,\ t \geqslant 0,$$

for some positive constant κ that for simplicity we take equal to 1. For $f \in \mathcal{S}$ set

$$T_t f(x) := \mathbb{E}(f(X_t + x)), \qquad x \in \mathbb{R}^n,\ t \geqslant 0.$$

Then, by Fubini's Theorem,

$$\begin{aligned}\widehat{T_t f}(\xi) &= \mathbb{E}\frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n} f(X_t + x)e^{-ix\cdot\xi}\,dx = \mathbb{E}\frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n} f(z)e^{-i(z-X_t)\cdot\xi}\,dz \\ &= \frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n} f(z)e^{-iz\cdot\xi}\,dz\ \mathbb{E}(e^{i\xi\cdot X_t}) = \widehat{f}(\xi)e^{-t|\xi|^\alpha}.\end{aligned}$$

Therefore,

$$T_t f(x) = e^{-t(-\Delta)^{\alpha/2}}f(x) =: v(x,t),$$

is the solution to the fractional diffusion equation

$$\begin{cases} \partial_t v = -(-\Delta)^{\alpha/2}v, & \text{in } \mathbb{R}^n \times (0,\infty), \\ v(x,0) = f(x), & \text{on } \mathbb{R}^n. \end{cases}$$

Note that $0 < \alpha/2 \leqslant 1$. See D. Applebaum [3] or J. Bertoin [8].

A nice and simple approach by considering long jump random walks is given in [86].

II. The Signorini problem. In “Questioni di elasticità non linearizzata e semilinearizzata (Issues in non linear and semilinear elasticity)”, *Rendiconti di Matematica e delle sue applicazioni* **18** (1959), 95-139, Antonio Signorini posed the question of finding the configuration of an elastic membrane in equilibrium that stays above some given *thin* obstacle, say of codimension 1. In mathematical terms the problem can be formulated as follows: given a smooth function φ in $\mathbb{R}^n$ that goes to zero at infinity, the solution of the Signorini problem is the function $u = u(x, y)$ that satisfies

$$\begin{aligned} \partial_{yy}u + \Delta_x u &= 0, && \text{in } \mathbb{R}^n \times (0,\infty), \\ u(x,0) &\geqslant \varphi(x), && \text{on } \mathbb{R}^n, \\ \partial_y u(x,0) &\leqslant 0, && \text{on } \mathbb{R}^n, \\ \partial_y u(x,0) &= 0, && \text{in } \{u(x,0) > \varphi(x)\}, \end{aligned} \tag{1.2}$$

and is zero at infinity. See [18, 38, 49] and [67, 68].

A very simple observation gives an equivalent description of the problem where the fractional Laplacian appears. The solution to (1.2) with boundary data $f(x) := u(x, 0)$ is given by convolution with the Poisson kernel in the upper half space:

$$u(x,y) = e^{-y(-\Delta_x)^{1/2}} f(x).$$

Take the derivative of u with respect to y and evaluate it at zero to get

$$\partial_y u(x,y)\Big|_{y=0} = -(-\Delta_x)^{1/2} f(x).$$

Hence, the Signorini problem can be rewritten as

$$\begin{aligned} \partial_{yy}u + \Delta_x u &= 0, && \text{in } \mathbb{R}^n \times (0,\infty), \\ u(x,0) &\geqslant \varphi(x), && \text{on } \mathbb{R}^n, && (1.3) \\ (-\Delta_x)^{1/2} u(x,0) &\geqslant 0, && \text{on } \mathbb{R}^n, && (1.4) \\ (-\Delta_x)^{1/2} u(x,0) &= 0, && \text{in } \{u(x,0) > \varphi(x)\}, && (1.5) \end{aligned}$$

with zero boundary condition at infinity.

Let us consider a Signorini problem where the Laplacian $-\Delta_x$ in (1.2) is replaced by a second order partial differential operator $L = L_x$ in $\mathbb{R}^n$. The associated Poisson semigroup

$$u(x,y) = e^{-yL_x^{1/2}} f(x),$$

is the solution to

$$\begin{cases} \partial_{yy}u - L_x u = 0, & \text{in } \mathbb{R}^n \times (0,\infty), \\ u(x,0) = f(x), & \text{on } \mathbb{R}^n. \end{cases}$$

Then the problem in this situation can be formulated in terms of a fractional power of L:

$$
\begin{aligned}
\partial_{yy}u - L_x u &= 0, && \text{in } \mathbb{R}^n \times (0,\infty),\\
u(x,0) &\geqslant \varphi(x), && \text{on } \mathbb{R}^n,\\
L_x^{1/2} u(x,0) &\geqslant 0, && \text{on } \mathbb{R}^n,\\
L_x^{1/2} u(x,0) &= 0, && \text{in } \{u(x,0) > \varphi(x)\}.
\end{aligned}
$$

A fully nonlinear case where $(\partial_{yy}+\Delta_x)u$ is replaced by $F(D^2u(x,y))$ can also be proposed, see E. Milakis and L. Silvestre [58].

`III. Obstacle problems.` For a symmetric α-stable Lévy process X_t with $X_0 = x$ consider the optimal stopping time τ to maximize the function

$$v(x) = \sup_{\tau} \mathbb{E}\left[\varphi(X_\tau) : \tau < \infty\right].$$

Then it can be seen that v is a solution of the free boundary problem

$$
\begin{aligned}
v(x) &\geqslant \varphi(x), && \text{on } \mathbb{R}^n,\\
(-\Delta)^\sigma v(x) &\geqslant 0, && \text{on } \mathbb{R}^n,\\
(-\Delta)^\sigma v(x) &= 0, && \text{in } \{v(x) > \varphi(x)\},
\end{aligned}
$$

with zero boundary condition at infinity and $\sigma := \alpha/2$. This type of problem appears in Financial Mathematics as a pricing model for American options [29, 67, 68]. Note that $\sigma = 1$ is the classical obstacle problem in $\mathbb{R}^n$ [38, 49, 20]. As we have seen above the Signorini problem gives rise to the set of equations (1.3)-(1.4)-(1.5) which is the obstacle problem for the fractional Laplacian with $\sigma = 1/2$ and $v(x) := u(x,0)$.

The regularity properties of the solution and the free boundary for the obstacle problem for the fractional Laplacian were studied by L. Silvestre [67, 68] and by L. Caffarelli, S. Salsa and L. Silvestre [22].

`IV. Fluid mechanics.` The dissipative quasi-geostrophic (QG for short) equation has the form

$$\theta_t + u\cdot\nabla\theta = -\kappa(-\Delta)^\sigma\theta, \qquad x\in\mathbb{R}^2,\ t>0,$$

where $\kappa > 0$, $0 \leqslant \sigma \leqslant 1$ and $\theta = \theta(x,t)$ is a scalar function. The vector field u is a 2-dimensional velocity field determined from θ by a stream function ψ through the relations

$$u = (u_1,u_2) = (-\partial_{x_2}\psi, \partial_{x_1}\psi), \qquad (-\Delta)^{1/2}\psi = \theta,$$

so that

$$(u_1,u_2) = (-R_2\theta, R_1\theta),$$

the operators R_i being the classical Riesz transforms $\partial_{x_i}(-\Delta)^{-1/2}$, $i = 1,2$. This equation models the temperature evolution on the 2-D boundary of a 3-D quasi-geostrophic flow and is sometimes referred to as the surface QG equation. See the lecture notes by P. Constantin [27] and the book by A. Majda and A. Bertozzi [56].

The QG equation is important also because it can be seen as a toy model to investigate regularity properties of Navier-Stokes equations, see [56, 19] and [27] and references therein. Under the assumption of incompressibility of the fluid the Navier-Stokes equations, determining the fluid velocity u and the fluid pressure p, read

$$u_t + (u \cdot \nabla)\, u + \nabla p = \Delta u + f, \qquad \operatorname{div} u = 0.$$

QG is a 2D "Navier-Stokes" type equation. In dimension 2, Navier-Stokes equations simplify considerably since the incompressibility, $\operatorname{div} u = 0$, implies that $(-u_2, u_1)$ is the gradient of some function: $(-u_2, u_1) = \nabla\psi$, and $\operatorname{curl} u$ is a scalar, $\theta = \operatorname{curl} u = \Delta\psi$. Hence, by taking the curl Navier-Stokes equations become a system:

$$\theta_t + u \cdot \nabla\theta = \Delta\theta, \qquad \operatorname{curl} u = \theta.$$

In the QG model we still have $(-u_2, u_1) = \nabla\psi$ but the potential ψ is related to the vorticity by $\theta = (-\Delta)^{1/2}\psi$ and the final system is

$$\theta_t + u \cdot \nabla\theta = -(-\Delta)^{1/2}\theta, \qquad (-u_2, u_1) = (R_1\theta, R_2\theta).$$

Motivated by this problem L. Caffarelli and A. Vasseur showed in [26] that the drift diffusion equation with fractional diffusion

$$\theta_t + v \cdot \nabla\theta = -(-\Delta)^{\sigma}\theta, \qquad \operatorname{div} v = 0, \qquad x \in \mathbb{R}^n,$$

where $v_j = T_j\theta$, T_j being a singular integral operator, and $\sigma = 1/2$ (critical case) has classical solutions for any L^2 initial data. The method of the proof is by localizing the fractional Laplacian $(-\Delta)^{1/2}$ via the harmonic extension $\theta^*(x, t, y)$ of $\theta(x, t)$ to the upper half space by adding a new variable y and then using the ideas of De Giorgi [31, 42] for the localized problem. The technique was also applied for instance in [28] and [65] for the supercritical case $\sigma < 1/2$. By using other methods it is shown in [50] that when $\sigma = 1/2$ and the initial data is smooth, there exists a unique solution to QG that is also smooth.

Our list of problems above does not pretend to be exhaustive. Just to mention a few more, besides modeling in Financial Mathematics [29] and fluid mechanics [56, 27], there are applications in Modern Physics, for instance, when considering fractional kinetics and anomalous transport [90, 70], strange kinetics [64], fractional quantum mechanics [53, 54] and Lévy processes in quantum mechanics [61].

1.2 Motivation

Let us describe some of the special features the fractional Laplacian has.

The first thing to note is that for a function $f \in \mathcal{S}$ and σ not an integer, $(-\Delta)^\sigma f$ is not a Schwartz's class function because of the singularity at the origin introduced in (the derivatives of) its Fourier transform (1.1). For the rest of this work we will restrict our attention to $0 < \sigma < 1$, that is the case that appears in applications.

Clearly definition (1.1) is not well suited for regularity estimates in Hölder spaces, since we would like to handle differences of the form

$$(-\Delta)^\sigma f(x_1) - (-\Delta)^\sigma f(x_2), \qquad x_1, x_2 \in \mathbb{R}^n.$$

Therefore a pointwise formula for $(-\Delta)^\sigma f(x)$ is needed. If we apply the inverse Fourier transform in (1.1) we get

$$(-\Delta)^\sigma f(x) = c_{n,\sigma} \,\mathrm{P.V.} \int_{\mathbb{R}^n} \frac{f(x) - f(z)}{|x-z|^{n+2\sigma}}\, dz, \qquad x \in \mathbb{R}^n, \tag{1.6}$$

where $c_{n,\sigma}$ is a positive constant depending only on dimension and σ. Let us point out that such a constant is important for instance when we want to take the limit $\sigma \to 1^-$ to recover the Laplacian $-\Delta$ in $\mathbb{R}^n$ or $\sigma \to 0^+$ to get the identity Id (a trivial fact if $f \in \mathcal{S}$). Using (1.6) the interaction of $(-\Delta)^\sigma$ with Hölder spaces C^α can be studied. In particular, if $f \in C^{0,\alpha}(\mathbb{R}^n)$ and $0 < 2\sigma < \alpha$ then $(-\Delta)^\sigma f \in C^{0,\alpha-2\sigma}(\mathbb{R}^n)$ and

$$|(-\Delta)^\sigma f(x_1) - (-\Delta)^\sigma f(x_2)| \leqslant C[f]_{C^\alpha} |x_1 - x_2|^{\alpha - 2\sigma}, \qquad x_1, x_2 \in \mathbb{R}^n, \tag{1.7}$$

where C depends only on α, σ and n, and similar results hold for $C^{k,\alpha}$ spaces, see [67, 68]. For more applications of Schauder estimates for the fractional Laplacian see for instance [22] and [25].

From (1.6) we observe that $(-\Delta)^\sigma$ is a nonlocal operator: the value of $(-\Delta)^\sigma f(x)$ for a given $x \in \mathbb{R}^n$ depends on the values of f at infinity. This property creates complications: the classical local PDE methods from the Calculus of Variations cannot be applied to the study of nonlinear problems involving $(-\Delta)^\sigma$. To overcome this difficulty L. Caffarelli and L. Silvestre showed in [23] that any fractional power of the Laplacian can be characterized as an operator that maps a Dirichlet boundary condition to a Neumann-type condition via an extension problem. Let us explain how it is done. Consider the function $u = u(x,y) : \mathbb{R}^n \times [0,\infty) \to \mathbb{R}$ that solves the boundary value problem

$$u(x,0) = f(x), \qquad x \in \mathbb{R}^n, \tag{1.8}$$

$$\Delta_x u + \frac{1-2\sigma}{y}\, u_y + u_{yy} = 0, \qquad x \in \mathbb{R}^n,\ y > 0. \tag{1.9}$$

Note that (1.9) is a degenerate elliptic equation. Then, up to a multiplicative constant depending only on σ,

$$-\lim_{y \to 0^+} y^{1-2\sigma} u_y(x,y) = (-\Delta)^\sigma f(x), \qquad x \in \mathbb{R}^n.$$

We can interpret this result as saying that the new variable y added to extend f to the upper half space via (1.9) encodes the values of f at infinity needed to compute $(-\Delta)^\sigma f$. The extension problem localizes the fractional Laplacian: it is enough to know u in some (upper half) ball around $(x, 0)$ to already obtain $(-\Delta)^\sigma f(x)$. The nonlinear problems for the (nonlocal) fractional Laplacian can be then localized adding a new variable, see [22, 26, 28, 65] for examples of the technique.

In [26] the extension problem with $\sigma = 1/2$ is used: (1.8)-(1.9) become the problem of the harmonic extension of f to the upper half space and, as we saw in the previous section, via the Neumann condition we recover $(-\Delta)^{1/2} f(x)$.

The extension problem can be applied to prove (among other regularity properties) the Harnack's inequality for $(-\Delta)^\sigma$, see [23]. The nonlocal character of the fractional Laplacian makes us to assume in the Harnack's inequality that $f(x) \geqslant 0$ for *all* $x \in \mathbb{R}^n$, and not just for x in some ball as usual, see [46].

Consider the situation where we have derived a model (normally a nonlinear PDE problem) that involves a fractional power of some second order partial differential operator L. Then we have to answer at least the following questions:

(I) Definition and pointwise formula for fractional operators. For a general operator L, classical Functional Analysis gives us several ways to define L^σ according to its analytical properties. Nevertheless, an abstract formula is not useful to treat concrete PDE problems and a more or less explicit pointwise expression for $L^\sigma f(x)$ will be needed. For the fractional Laplacian we started with the Fourier transform definition (1.1) and by taking its inverse we got (1.6). What can be done in the general case where the Fourier transform is not available?

(II) Regularity theory for fractional operators. Going to concrete examples of second order differential operators L we may ask for the "right" Schauder estimates for L^σ or, more precisely, the proper/adapted Hölder space to look for regularity properties of L^σ. Another question is the validity of Harnack's inequalities, an important tool in the theory of PDEs [45].

(III) The nonlocal nature. In general, fractional powers of linear second order partial differential operators are nonlocal operators. It would be very useful in applications to have an analogous Caffarelli-Silvestre characterization of L^σ as a Dirichlet-to-Neumann map via an extension problem. If so, how to use such a characterization to get regularity estimates?

In this dissertation we aim to answer these questions.

1.3 Description of the results

Let L be a second order partial differential operator defined in some $L^2(\Omega, d\eta)$ space, where Ω is an open subset of $\mathbb{R}^n$, $n \geqslant 1$, and $d\eta$ is a positive measure on Ω.

Unless we are dealing with the Laplacian in $\mathbb{R}^n$, the Fourier transform will not be very useful to study fractional operators. We need to find a language that can explain in a clear and unified way the concepts, formulas and properties we want to deal with.

We adopt the **heat semigroup language**, in which the central operator is the **heat-diffusion semigroup generated by** L that we denote by

$$e^{-tL}, \qquad \text{for } t \geqslant 0.$$

As we pretend to show in this dissertation, such a setting turns out to be very well suited: it gives the insight in the correct formulas (with explicitly and easily computable constants) and it is also very general. An interesting relationship with special functions such as Gamma and Bessel functions will appear, showing the underlying harmony.

Under this point of view we will revisit the fractional Laplacian and we will also analyze the questions posed above for a general operator L. The characterization of L^σ as a Dirichlet-to-Neumann type operator via an extension problem will be obtained thanks to the language adopted. To go to regularity considerations we move to a case study: fractional powers of the harmonic oscillator

$$H = -\Delta + |x|^2, \qquad \text{in } \mathbb{R}^n.$$

This is an important operator in quantum mechanics, see for instance the book by R. P. Feynmann and A. R. Hibbs [37]. For $H^\sigma = (-\Delta + |x|^2)^\sigma$ Harnack's inequality and adapted Schauder estimates will be obtained.

Let us proceed to describe the next chapters.

1.3.1 Chapter 2: Preliminaries

As we already pointed out, the heat-diffusion semigroup e^{-tL} generated by a second order operator L is the key operator in our work.

In this chapter we show how to express several operators associated to L like Poisson semigroups $e^{-y\sqrt{L}}$, fractional integrals $L^{-\sigma}$, fractional powers L^σ and Riesz transforms associated to L in terms of e^{-tL}, see **formulas** (2.3), (2.5), (2.8), (2.9) **and** (2.10). The expressions are based on the definition of the Gamma function. For the fractional operator we have the following nice and handle formula to start with:

$$L^\sigma f(x) = \frac{1}{\Gamma(-\sigma)} \int_0^\infty \left(e^{-tL} f(x) - f(x)\right) \frac{dt}{t^{1+\sigma}}. \tag{1.10}$$

The concepts given above are well understood in two particular (and at the same time general) examples, namely when L has a discrete spectrum and when $L = -\Delta$, the Laplacian in $\mathbb{R}^n$, that has a continuous spectrum.

We show how the semigroup language simplifies and clarifies the basic example, the fractional Laplacian. Indeed, using (1.10) for $L = -\Delta$ the pointwise formula (1.6) with the exact constant $c_{n,\sigma}$ is derived without need of applying the inverse Fourier transform (**Lemma 2.1**). **Proposition 2.3** establishes that

$$(-\Delta)^\sigma f(x) \to -\Delta f(x), \qquad \text{as } \sigma \to 1^-,$$

for all x where f is C^2 and **Proposition 2.5** says that

$$(-\Delta)^\sigma f(x) \to f(x), \qquad \text{as } \sigma \to 0^+,$$

for x being any point of Hölder continuity of f.

The preliminaries about the harmonic oscillator H (discrete spectrum) are presented.

Finally we collect the basics of the spectral theory we will need. The abstract context is very useful since it allows us to treat both examples (discrete and continuous spectrum) in an unified way.

1.3.2 Chapter 3: Definition and extension problem for fractional powers of second order partial differential operators

This chapter contains **part of the results of paper** [**79**].

Once L^σ is given via the heat semigroup e^{-tL} by (1.10), we explore a characterization in the spirit of [23].

To solve (1.8)-(1.9) Caffarelli and Silvestre noted that (1.9) can be though as the harmonic extension of f in $2-2\sigma$ dimensions more. From there they established the fundamental solution $\Gamma_\sigma(x,y)$ for (1.9), that has the property that $\Gamma_\sigma(x,0)$ is the fundamental solution of $(-\Delta)^\sigma$, that is, *the fundamental solution of the extension is an extension of the fundamental solution* $(-\Delta)^{-\sigma}$. Then, by using a conjugate equation, a Poisson formula for u was obtained:

$$u(x,y) = \int_{\mathbb{R}^n} P_y^{\sigma,\Delta}(x-z)f(z)\,dz.$$

The function $P_y^{\sigma,\Delta}(z)$ is the Poisson kernel.

Our semigroup approach gives light to the Laplacian case and will establish a general setting for the extension problem that will allow us to include other operators L.

As examples suggest, we assume that L is self-adjoint. Then we get in **Theorem 3.1** the characterization of L^σ that generalizes the Caffarelli-Silvestre result. That is, consider the following extension problem to the upper half space:

$$\begin{aligned} u(x,0) &= f(x), && \text{on } \Omega; \\ -L_x u + \frac{1-2\sigma}{y}\,u_y + u_{yy} &= 0, && \text{in } \Omega\times(0,\infty). \end{aligned}$$

We show that a solution u is given explicitly by

$$u(x,y) = \frac{1}{\Gamma(\sigma)}\int_0^\infty e^{-tL}(L^\sigma f)(x)e^{-\frac{y^2}{4t}}\,\frac{dt}{t^{1-\sigma}}. \tag{1.11}$$

This formula is one of the main novelties here. Then it is proved that for each $x \in \Omega$,

$$\lim_{y\to 0^+}\frac{u(x,y)-u(x,0)}{y^{2\sigma}} = \frac{\Gamma(-\sigma)}{4^\sigma\Gamma(\sigma)}L^\sigma f(x) = \frac{1}{2\sigma}\lim_{y\to 0^+} y^{1-2\sigma}u_y(x,y).$$

The expression for u (1.11) requires to know the action of L^σ on f. To improve that situation the Poisson formula is derived:

$$u(x,y) = \frac{y^{2\sigma}}{4^\sigma\Gamma(\sigma)} \int_0^\infty e^{-tL} f(x) e^{-\frac{y^2}{4t}} \frac{dt}{t^{1+\sigma}} = \int_\Omega P_y^{\sigma,L}(x,z) f(z)\, d\eta(z), \tag{1.12}$$

where no fractional power of L is involved and $P_y^{\sigma,L}(x,z)$ is the Poisson kernel.

When $L = -\Delta$ the extension result and the Poisson kernel of [23] is recovered. It is worth mentioning that in [23] the explicit formula (1.11) and the first identity of (1.12) are not established, but the Poisson convolution formula with the Poisson kernel as explained above.

The proof of the extension characterization is given always in our semigroup language and uses the Spectral Theorem in an essential way. Nevertheless, before presenting it and for exposition purposes, we prove the result for L having a discrete spectrum and for the Laplacian.

In **Theorem 3.2** more properties concerning the Poisson formula, as maximum principles and L^p estimates for the extension problem, are described. We show that the Poisson formula for u can be derived with the clever idea of [23]: use the fundamental solution (that involves the kernel of the heat semigroup generated by L) and an appropriate conjugate equation to infer the Poisson kernel.

The conjugate equation is studied in detail. To that end Cauchy-Riemann equations adapted to the extension equation are defined, see (3.15).

Examples are also provided.

The reader could ask how the explicit formula (1.11) for the solution u can be figured out. We present a very nice proof of the extension result in terms of orthogonal expansions and also the derivation of local Neumann solutions. The resulting equations are Bessel equations whose solutions in terms of Bessel functions will give us the clue for (1.11).

1.3.3 Chapter 4: Definition, basic properties and Harnack's inequality for the fractional powers of the harmonic oscillator

Here we collect **results from** [**79**].

Formula (1.10) is applied with $L = H = -\Delta + |x|^2$, the harmonic oscillator, to derive the pointwise formula

$$H^\sigma f(x) = \int_{\mathbb{R}^n} (f(x) - f(z)) F_\sigma(x,z)\, dz + f(x) B_\sigma(x), \qquad x \in \mathbb{R}^n,$$

where the kernel $F_\sigma(x,z)$ and the function $B_\sigma(x)$ are given in terms of the heat kernel for H, see **Theorem 4.3**. Hence the fractional harmonic oscillator is a nonlocal operator. Some maximum and comparison principles are obtained (**Theorem 4.6 and Corollary 4.7**).

The extension problem for H^σ is studied in detail (**Theorem 4.13**) because we use the ideas from [23] to prove the Harnack's inequality for H^σ: for every $R > 0$ and $x_0 \in \mathbb{R}^n$, there exists a positive constant $C = C_{\sigma,R,n,x_0}$ such that

$$\sup_{B_{R/2}(x_0)} f \leqslant C \inf_{B_{R/2}(x_0)} f,$$

for all nonnegative functions $f : \mathbb{R}^n \to \mathbb{R}$ that are C^2 in $B_R(x_0)$ and satisfy $H^\sigma f(x) = 0$ for all $x \in B_R(x_0)$, see **Theorem 4.10**.

To prove the Harnack's inequality for $(-\Delta)^\sigma$ the authors of [23] take advantage of the general theory of degenerate elliptic equations developed by E. Fabes, D. Jerison, C. Kenig and R. Serapioni in 1982-83. In our case such a general theory is not needed, but the Harnack's inequality for degenerate Schrödinger operators proved by C. E. Gutiérrez in [41].

1.3.4 Chapter 5: Interaction of the fractional harmonic oscillator with the Hölder spaces adapted to H and Schauder estimates

This chapter **corresponds to** [80].

To derive Schauder's estimates for H^σ (of the type (1.7)) we first need to find the right Hölder space naturally associated with H.

A new class of Hölder spaces $C_H^{k,\alpha}$, $0 < \alpha \leqslant 1$, $k \in \mathbb{N}_0$, adapted to H is introduced, that we call **Hermite-Hölder spaces**. These spaces (smaller than the classical $C^{k,\alpha}(\mathbb{R}^n)$ classes) are defined to allow some growth at infinity of the functions they contain, the moral being that the Laplacian $-\Delta$ dictates the regularity and the potential $|x|^2$ in the operator plays a relevant role only at infinity. See **Definition 5.1** for the definition of $C_H^{0,\alpha}$.

The factorization of the harmonic oscillator in terms of first order partial differential operators

$$H = \frac{1}{2}\sum_{i=1}^{n}\left[(\partial_{x_i} + x_i)(-\partial_{x_i} + x_i) + (-\partial_{x_i} + x_i)(\partial_{x_i} + x_i)\right],$$

gives us the right definition for the "derivative" operator naturally associated to H, namely

$$A_i = \partial_{x_i} + x_i, \qquad A_{-i} = -\partial_{x_i} + x_i.$$

Then the spaces $C_H^{k,\alpha}$ are defined in the usual way as the set of all differentiable functions whose $A_{\pm i}$-derivatives up to order k belong to $C_H^{0,\alpha}$ (**Definition 5.2**).

The first main theorem, **Theorem A**, basically says that

$$H^\sigma : C_H^{0,\alpha} \to C_H^{0,\alpha-2\sigma}, \qquad 2\sigma < \alpha,$$

and

$$H^\sigma : C_H^{1,\alpha} \to C_H^{0,\alpha-2\sigma+1}, \qquad 2\sigma \geqslant \alpha,$$

continuously, and similarly for higher order spaces. Hence H^σ acts as a *fractional derivative*.

The result above can be interpreted as saying that the Hölder spaces $C^{k,\alpha}_H$ are the reasonable classes for deriving Schauder estimates for H^σ. Indeed, this is what the second main result (**Theorem B**) asserts:

$$H^{-\sigma}: C^{0,\alpha}_H \to C^{0,\alpha+2\sigma}_H, \qquad \alpha + 2\sigma \leqslant 1,$$

$$H^{-\sigma}: C^{0,\alpha}_H \to C^{1,\alpha+2\sigma-1}_H, \qquad 1 < \alpha + 2\sigma \leqslant 2,$$

and

$$H^{-\sigma}: C^{0,\alpha}_H \to C^{2,\alpha+2\sigma-2}_H, \qquad 2 < \alpha + 2\sigma \leqslant 3,$$

continuously. Hence $H^{-\sigma}$ is an *inverse fractional derivative* operator in $C^{k,\alpha}_H$.

One of the main tasks in the chapter will be to derive the pointwise formula for the operators $H^\sigma u$ and $H^{-\sigma}u$ (and their derivatives) when u belongs to the Hermite-Hölder space.

To prove our results we will have to consider also the Hermite-Riesz transforms $\mathcal{R}_i$ and $\mathcal{R}_{ij}$, $i, j = 1, \ldots, n$, when acting on $C^{0,\alpha}_H$. Since $C^{0,\alpha}_H$ is the correct space, the expected result is proved:

$$\mathcal{R}_i, \mathcal{R}_{ij} : C^{0,\alpha}_H \to C^{0,\alpha}_H,$$

continuously.

In some recent papers, B. Bongioanni, E. Harboure and O. Salinas studied the boundedness of fractional integrals [15] and Riesz transforms [16] associated to a certain class of Schrödinger operators $\mathcal{L} = -\Delta + V$, in spaces of $BMO^\beta_{\mathcal{L}}$ type, $0 \leqslant \beta < 1$, using Harmonic Analysis techniques. In [15, Proposition 4] they showed that the $BMO^\beta_{\mathcal{L}}$ spaces coincide with a Hölder type space $\Lambda^\beta_{\mathcal{L}}$, $0 < \beta < 1$, with equivalent norms. In the case $V = |x|^2$, our space $C^{0,\beta}_H$ coincides with their space Λ^β_H, for $0 < \beta < 1$. The BMO spaces associated to $\mathcal{L}$ were first defined and studied in [33], see also [57]. For the boundedness of operators related to the harmonic oscillator in the classical Euclidean BMO see [76].

Very recently we were aware of R. F. Bass' paper [5] where the author, motivated by the fractional Laplacian, considers what he calls *stable-like operators* and studies its interaction with the classical Hölder spaces C^α with α not being an integer. Our fractional harmonic oscillator is not an example of such an operator because of Lemma 4.8 in Chapter 4 makes Assumption 1.1 in [5] fail. See also [66, 24, 25, 47, 21].

Notation. Throughout this dissertation $\mathcal{S}$ is the Schwartz class of rapidly decreasing $C^\infty(\mathbb{R}^n)$ functions, the letter C denotes a constant that may change in each occurrence and it will depend on the parameters involved (whenever it is necessary we point out this dependence with subscripts) and Γ stands for the Gamma function [1, 34]:

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1}\, dt, \qquad \Re z > 0.$$

Without mention it, we will repeatedly apply the inequality $r^\nu e^{-r} \leqslant C_\nu e^{-r/2}$, $\nu \geqslant 0$, $r > 0$.

Chapter 2

Preliminaries

In Section 2.1 we introduce the heat-diffusion semigroup generated by a linear second order partial differential operator L. We show how it can be used to define several operators related to L. To make the presentation more readable, in a first step the operators L considered are those which have a discrete spectrum. Secondly, the case of the Laplacian in $\mathbb{R}^n$ (continuous spectrum) is taken into account. We analyze in some detail the properties of the fractional Laplacian $(-\Delta)^\sigma$. Section 2.2 is devoted to the preliminaries on the harmonic oscillator $H = -\Delta + |x|^2$ in $\mathbb{R}^n$ and the operators associated to it. In Section 2.3 we explain the basics of spectral theory we will use to give a unified approach.

2.1 The heat semigroup and related operators

We start with a naive presentation to make clear the ideas.

Let $L = L_x$ be a positive linear second order partial differential operator, acting on functions $f = f(x)$ defined in a certain domain $\Omega \subseteq \mathbb{R}^n$, $n \geqslant 1$. Consider the diffusion equation

$$\begin{cases} v_t + Lv = 0, & \text{for } (x,t) \in \Omega \times (0,\infty), \\ v(x,0) = f(x), & \text{for } x \in \Omega. \end{cases} \tag{2.1}$$

The linear operator that maps each initial datum f to the solution v of the problem above is the **heat-diffusion semigroup generated by** L (or **heat semigroup** for short). We write

$$e^{-tL}f(x) := v(x,t), \qquad x \in \Omega, \ t \geqslant 0.$$

This notation very convenient: we can (formally) differentiate $\partial_t e^{-tL}f(x) = -Le^{-tL}f(x)$, and take the limit $\lim_{t\to 0^+} e^{-tL}f(x) = f(x)$, so $e^{-tL}f(x)$ represents the solution v. The family $\{e^{-tL}\}_{t>0}$ satisfies the semigroup property: $e^{-t_1 L}\left(e^{-t_2 L}f\right) = e^{-(t_1+t_2)L}f$, for all $t_1, t_2 > 0$.

Several operators related to L can be expressed in terms of the heat semigroup:

Poisson semigroup. The solution to the equation

$$\begin{cases} w_{yy} - Lw = 0, & \text{for } (x,y) \in \Omega \times (0,\infty), \\ w(x,0) = f(x), & \text{for } x \in \Omega. \end{cases}$$

can be written as

$$w(x,y) = e^{-y\sqrt{L}}f(x), \qquad x \in \Omega,\ y \geqslant 0.$$

Once again this notation is convenient, as formal differentiation shows. The family of linear operators $\{e^{-y\sqrt{L}}\}_{y>0}$ forms a semigroup and it is called the *Poisson semigroup generated by* L. The way to express the Poisson semigroup via the heat semigroup is by using the following *Bochner's subordination formula* which is valid for all $\lambda > 0$:

$$e^{-y\sqrt{\lambda}} = \frac{y}{2\Gamma(1/2)} \int_0^\infty e^{-t\lambda} e^{-y^2/4t} \, \frac{dt}{t^{3/2}}, \qquad y > 0. \tag{2.2}$$

Making the formal substitution $\lambda = L$ into (2.2) we get

$$e^{-y\sqrt{L}}f(x) = \frac{y}{2\Gamma(1/2)} \int_0^\infty e^{-tL}f(x) e^{-y^2/4t} \, \frac{dt}{t^{3/2}}, \qquad x \in \Omega,\ y > 0. \tag{2.3}$$

This formal computation can be made rigorous in each specific example by considering the eigenfunctions of L, or in general by means of the Spectral Theorem, as we will show later.

Negative powers of L. The operators $L^{-\sigma}$, $\sigma > 0$, are known as *fractional integrals associated to* L. As for the Poisson semigroup, we will use a formula involving the Gamma function that relates $\lambda^{-\sigma}$, for positive λ and σ, with $e^{-t\lambda}$, and then we will replace λ by L. The expression is

$$\lambda^{-\sigma} = \frac{1}{\Gamma(\sigma)} \int_0^\infty e^{-t\lambda} \, \frac{dt}{t^{1-\sigma}}, \qquad \sigma > 0. \tag{2.4}$$

Therefore,

$$L^{-\sigma}f(x) = \frac{1}{\Gamma(\sigma)} \int_0^\infty e^{-tL}f(x) \, \frac{dt}{t^{1-\sigma}}, \qquad x \in \Omega,\ \sigma > 0. \tag{2.5}$$

Fractional (positive) powers of L. The main objects of our study will be the operators L^σ, where we restrict to the range $0 < \sigma < 1$. A formula that relates λ^σ, for positive λ and $0 < \sigma < 1$, with $e^{-t\lambda}$ is

$$\lambda^\sigma = \frac{1}{\Gamma(-\sigma)} \int_0^\infty \left(e^{-t\lambda} - 1\right) \frac{dt}{t^{1+\sigma}}, \qquad 0 < \sigma < 1, \tag{2.6}$$

where

$$\Gamma(-\sigma) := \frac{\Gamma(1-\sigma)}{-\sigma} = \int_0^\infty (e^{-s} - 1) \, \frac{ds}{s^{1+\sigma}} < 0. \tag{2.7}$$

Putting $\lambda = L$ into (2.6),

$$L^\sigma f(x) = \frac{1}{\Gamma(-\sigma)} \int_0^\infty \left(e^{-tL}f(x) - f(x)\right) \frac{dt}{t^{1+\sigma}}, \qquad x \in \Omega,\ 0 < \sigma < 1. \tag{2.8}$$

Riesz transforms related to L. These operators are useful for instance to get *a priori* estimates for solutions to the Poisson problem for L. The idea is the following. Assume that L can be factorized as a finite sum $L = \sum_i D_i^* D_i$, where D_i^* and D_i are first order partial differential operators in the ith direction, that are our *derivatives*. For a given function f defined in Ω, let u be a solution of the *Poisson problem for* L *in* Ω:

$$Lu = f, \qquad \text{in } \Omega.$$

Then

$$u = L^{-1}f, \qquad \text{in } \Omega.$$

Assume that f is in some $L^p(\Omega)$ or $C^\alpha(\Omega)$ space and denote by $\|\cdot\|$ the norm in any of such spaces. Since L is a second order operator, we expect for u to have two more *derivatives* than f has or, in other words, L^{-1} *has to integrate twice.* So u should have two *derivatives* in $L^p(\Omega)$ or in $C^\alpha(\Omega)$:

$$\left\|D_iD_ju\right\| \sim \left\|D_iD_jL^{-1}f\right\|.$$

But then the question to answer is whether an estimate of the kind

$$\left\|D_iD_jL^{-1}f\right\| \sim \|f\|$$

holds, since the only information we have is that f belongs to $L^p(\Omega)$ or $C^\alpha(\Omega)$. The operators

$$R_{ij} = D_iD_jL^{-1}, \tag{2.9}$$

are the *second order Riesz transforms associated to* L. We can also define *first order Riesz transforms* as

$$R_i = D_iL^{-1/2}, \tag{2.10}$$

the moral being $L^{-1/2}$ *integrates once.* Note that the *derivatives* have to be adapted to the operator L in order for the *integration-derivation* game to be consistent. Typical examples are the classical Riesz transforms, that are associated to the Laplacian $-\Delta = -\operatorname{div}\nabla$: for $i = 1, \dots, n$,

$$\widehat{R_i f}(\xi) = \frac{-i\xi_i}{|\xi|}\widehat{f}(\xi) \quad \longleftrightarrow \quad R_if(x) = \partial_{x_i}(-\Delta)^{-1/2}f(x).$$

The ideas explained above to relate the Poisson semigroup, the negative powers and the Riesz transforms with the heat semigroup are contained in the book by E. M. Stein [74]. He studied these operators (and also the g-function, but not the fractional powers) related to the Laplacian in a compact Lie group (that has a discrete spectrum).

Other examples of Riesz transforms are those associated with orthogonal expansions. We just mention here the Hermite function expansions case that is presented in Section 2.2.

For the rest of this section we explore the heat semigroup and the related operators listed above in two particular cases. The first one is when L has a discrete spectrum in Ω. This requirement is fulfilled, for instance, by an elliptic operator on a bounded domain [35, 40] or, more generally, by operators that give rise to orthogonal expansions [81, 55]. The technique to use in this situation is the Fourier expansion method. The second case is $L = -\Delta$, the Laplacian in $\mathbb{R}^n$, which has a continuous spectrum. The natural tool here is the Fourier transform.

2.1.1 Discrete spectrum

Suppose that there exists a family of (smooth) real-valued functions $\{\phi_k : k \in \mathbb{N}_0\}$ defined in Ω that are eigenfunctions of L with positive eigenvalues $\{\lambda_k : k \in \mathbb{N}_0\}$:

$$L\phi_k(x) = \lambda_k \phi_k(x), \qquad x \in \Omega,\ k \in \mathbb{N}_0.$$

We also assume that the set $\{\phi_k : k \in \mathbb{N}_0\}$ is an orthonormal basis of $L^2(\Omega, d\eta)$, for some σ-finite positive measure $d\eta$ defined in Ω.

Let f be a finite linear combination of eigenfunctions, $f = \sum_{\text{finite}} \langle f, \phi_k\rangle \phi_k$, where we write $\langle f, \phi_k\rangle = \int_\Omega f\phi_k \, d\eta$. To solve the diffusion equation (2.1) for this f we apply Fourier's method. Set $v(x,t) = \sum_k c_k(t)\phi_k(x)$ into (2.1) to derive an equation for the coefficients:

$$\begin{cases} c_k'(t) + \lambda_k c_k(t) = 0, & \text{for } t > 0 \\ c_k(0) = \langle f, \phi_k\rangle, \end{cases}$$

for each k. The solution to this ODE is $c_k(t) = e^{-t\lambda_k}\langle f, \phi_k\rangle$, $k \in \mathbb{N}_0$. Therefore,

$$v(x,t) = e^{-tL}f(x) = \sum_k e^{-t\lambda_k}\langle f, \phi_k\rangle \phi_k(x), \qquad x \in \Omega,\ t \geqslant 0.$$

The sum above contains finitely many terms. To get an integral expression for the heat semigroup we have to assume some mild conditions on the eigenfunctions ϕ_k. Suppose that the functions ϕ_k and their derivatives have a polynomial growth with respect to λ_k, i.e. for each multi-index $\beta \in \mathbb{N}_0^n$ there exists a nonnegative integer $m = m_\beta$ such that

$$D^\beta \phi_k = O\left(\lambda_k^{m_\beta}\right), \qquad \text{for all } k \in \mathbb{N}_0.$$

Under this hypothesis, by Fubini's Theorem,

$$\begin{aligned} e^{-tL}f(x) &= \sum_{\text{finite}} e^{-t\lambda_k}\left[\int_\Omega f(z)\phi_k(z)\, d\eta(z)\right]\phi_k(x) \\ &= \int_\Omega \left[\sum_{k=0}^\infty e^{-t\lambda_k}\phi_k(x)\phi_k(z)\right] f(z)\, d\eta(z) =: \int_\Omega K_t(x,z) f(z)\, d\eta(z). \end{aligned}$$

We call the function $K_t(x,z)$ the *heat kernel associated to* L.

The Poisson semigroup can also be derived by Fourier's method:

$$w(x,y) = e^{-y\sqrt{L}}f(x) = \sum_k e^{-y\sqrt{\lambda_k}}\langle f, \phi_k\rangle\phi_k(x), \qquad x \in \Omega,\ y \geqslant 0.$$

To relate $e^{-y\sqrt{L}}$ with e^{-tL} we apply (2.2) with $\lambda = \lambda_k$ into the last sum, so that,

$$\begin{aligned} e^{-y\sqrt{L}}f(x) &= \sum_k \left[\frac{y}{2\Gamma(1/2)}\int_0^\infty e^{-t\lambda_k}e^{-y^2/4t}\,\frac{dt}{t^{3/2}}\right]\langle f, \phi_k\rangle\phi_k(x) \\ &= \frac{y}{2\Gamma(1/2)}\int_0^\infty \left[\sum_k e^{-t\lambda_k}\langle f, \phi_k\rangle \phi_k(x)\right] e^{-y^2/4t}\,\frac{dt}{t^{3/2}} \\ &= \frac{y}{2\Gamma(1/2)}\int_0^\infty e^{-tL}f(x)e^{-y^2/4t}\,\frac{dt}{t^{3/2}}, \end{aligned}$$

which is (2.3) in this case.

Since L is a linear operator, the action of L on f is given by its action on each eigenfunction:

$$Lf(x) = \sum_k \langle f, \phi_k \rangle L\phi_k(x) = \sum_k \lambda_k \langle f, \phi_k \rangle \phi_k(x), \qquad x \in \Omega.$$

It is now clear how to define L^ρ, for $\rho \in \mathbb{R}$:

$$L^\rho f(x) = \sum_k \lambda_k^\rho \langle f, \phi_k \rangle \phi_k(x), \qquad x \in \Omega.$$

Take $\rho = -\sigma$ for $\sigma > 0$ and use (2.4) with $\lambda = \lambda_k$ in the last expression to get

$$\begin{aligned} L^{-\sigma} f(x) &= \sum_k \left[\frac{1}{\Gamma(\sigma)} \int_0^\infty e^{-t\lambda_k} \, \frac{dt}{t^{1-\sigma}} \right] \langle f, \phi_k \rangle \phi_k(x) \\ &= \frac{1}{\Gamma(\sigma)} \int_0^\infty \left[\sum_k e^{-t\lambda_k} \langle f, \phi_k \rangle \phi_k(x) \right] \frac{dt}{t^{1-\sigma}} = \frac{1}{\Gamma(\sigma)} \int_0^\infty e^{-tL} f(x) \, \frac{dt}{t^{1-\sigma}}, \end{aligned}$$

and we have (2.5). For the case $\rho = \sigma$, $0 < \sigma < 1$, we use (2.6) into the sum defining L^σ to arrive to formula (2.8):

$$\begin{aligned} L^\sigma f(x) &= \sum_k \left[\frac{1}{\Gamma(-\sigma)} \int_0^\infty \left(e^{-t\lambda_k} - 1 \right) \, \frac{dt}{t^{1+\sigma}} \right] \langle f, \phi_k \rangle \phi_k(x) \\ &= \frac{1}{\Gamma(-\sigma)} \int_0^\infty \left[\sum_k e^{-t\lambda_k} \langle f, \phi_k \rangle \phi_k(x) - \sum_k \langle f, \phi_k \rangle \phi_k(x) \right] \frac{dt}{t^{1+\sigma}} \\ &= \frac{1}{\Gamma(-\sigma)} \int_0^\infty \left(e^{-tL} f(x) - f(x) \right) \, \frac{dt}{t^{1+\sigma}}, \qquad x \in \Omega. \end{aligned}$$

2.1.2 Continuous spectrum: the Laplacian in $\mathbb{R}^n$

The classical heat equation in $\mathbb{R}^n$,

$$\begin{cases} v_t = \Delta v, & \text{for } (x,t) \in \mathbb{R}^n \times (0,\infty), \\ v(x,0) = f(x), & \text{for } x \in \mathbb{R}^n, \end{cases} \tag{2.11}$$

can be solved using the Fourier transform in the x variable. For the sake of simplicity we take f in the Schwartz's class $\mathcal{S}$. The analogous representation of f as a sum of eigenfunctions is then

$$f(x) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \widehat{f}(\xi) e^{ix\cdot\xi} \, d\xi, \qquad x \in \mathbb{R}^n,$$

where the Fourier transform of f is given by

$$\widehat{f}(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} f(x) e^{-ix\cdot\xi} \, dx, \qquad \xi \in \mathbb{R}^n.$$

Denoting by $c_\xi(t)$ the Fourier transform of $v(x,t)$ in the x variable for each t, we can write

$$v(x,t) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} c_\xi(t) e^{ix\cdot\xi}\, d\xi, \qquad x \in \mathbb{R}^n,\ t > 0,$$

Plugging this expression into the heat equation (2.11), we obtain an ODE for the function $c_\xi(t)$, with initial condition $c_\xi(0) = \widehat{f}(\xi)$, for each $\xi \in \mathbb{R}^n$. Solving the ODE we get $c_\xi(t) = e^{-t|\xi|^2}\widehat{f}(\xi)$. Thus,

$$\begin{aligned}
e^{t\Delta}f(x) &= \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-t|\xi|^2}\widehat{f}(\xi) e^{ix\cdot\xi}\, d\xi = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{-t|\xi|^2} \left(\int_{\mathbb{R}^n} f(z) e^{-iz\cdot\xi}\, dz \right) e^{ix\cdot\xi}\, d\xi \\
&= \int_{\mathbb{R}^n} \left(\frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{-t|\xi|^2} e^{-iz\cdot\xi} e^{ix\cdot\xi}\, d\xi \right) f(z)\, dz = \int_{\mathbb{R}^n} W_t(x-z) f(z)\, dz,
\end{aligned}$$

where (see [32, 72])

$$W_t(x) = \frac{1}{(4\pi t)^{n/2}}\, e^{-\frac{|x|^2}{4t}}. \tag{2.12}$$

is the *Gauss-Weierstrass kernel.*

Formulas (2.3), (2.5) and (2.8) are valid if $L = -\Delta$. We can check this as we did for the discrete spectrum case: replacing the corresponding numerical formulas into the Fourier transform definition of each operator. Since the heat kernel (2.12) is available, we are going to show how the classical well known formulas can be easily recovered.

The harmonic extension of f to the upper half space is given by the Poisson semigroup acting on f. Indeed, for each $x \in \mathbb{R}^n$ and $y > 0$, applying Fubini's Theorem and the change of variables $s = \frac{|x-z|^2+y^2}{4t}$, we obtain the classical convolution formula with the Poisson kernel in the upper half space, see [32, 72]:

$$\begin{aligned}
e^{-y\sqrt{-\Delta}}f(x) &= \frac{y}{2\Gamma(1/2)} \int_0^\infty e^{t\Delta}f(x) e^{-y^2/4t}\, \frac{dt}{t^{3/2}} \\
&= \frac{y}{2\pi^{1/2}} \int_0^\infty \int_{\mathbb{R}^n} \frac{1}{(4\pi t)^{n/2}}\, e^{-\frac{|x-z|^2}{4t}} f(z)\, dz\, e^{-y^2/4t}\, \frac{dt}{t^{3/2}} \\
&= \int_{\mathbb{R}^n} \left[\frac{y}{\pi^{(n+1)/2}} \int_0^\infty \frac{1}{(4t)^{(n+1)/2}}\, e^{-\frac{|x-z|^2+y^2}{4t}}\, \frac{dt}{t} \right] f(z)\, dz \\
&= \int_{\mathbb{R}^n} \left[\frac{y}{\pi^{(n+1)/2}} \int_0^\infty \left(\frac{s}{|x-z|^2+y^2} \right)^{(n+1)/2} e^{-s}\, \frac{ds}{s} \right] f(z)\, dz \\
&= \frac{\Gamma((n+1)/2)}{\pi^{(n+1)/2}} \int_{\mathbb{R}^n} \frac{y}{(|x-z|^2+y^2)^{\frac{n+1}{2}}}\, f(z)\, dz.
\end{aligned}$$

For the fractional integrals in the case $n > 2\sigma$, we use the change of variables $s = \frac{|x-z|^2}{4t}$

to get (see [72])

$$\begin{aligned}
(-\Delta)^{-\sigma} f(x) &= \frac{1}{\Gamma(\sigma)} \int_0^\infty e^{t\Delta} f(x) \, \frac{dt}{t^{1-\sigma}} \\
&= \int_{\mathbb{R}^n} \left[\frac{1}{\pi^{n/2} 4^\sigma \Gamma(\sigma)} \int_0^\infty \frac{1}{(4t)^{(n-2\sigma)/2}} \, e^{-\frac{|x-z|^2}{4t}} \, \frac{dt}{t} \right] f(z) \, dz \\
&= \int_{\mathbb{R}^n} \left[\frac{1}{\pi^{n/2} 4^\sigma \Gamma(\sigma)} \int_0^\infty \left(\frac{s}{|x-z|^2} \right)^{(n-2\sigma)/2} e^{-s} \, \frac{ds}{s} \right] f(z) \, dz \\
&= \frac{\Gamma(n/2-\sigma)}{\pi^{n/2} 4^\sigma \Gamma(\sigma)} \int_{\mathbb{R}^n} \frac{f(z)}{|x-z|^{n-2\sigma}} \, dz.
\end{aligned}$$

Let us concentrate on the fractional Laplacian. Recall that for $f \in \mathcal{S}$ the fractional Laplacian $(-\Delta)^\sigma f$ is defined as a pseudo-differential operator

$$\widehat{(-\Delta)^\sigma} f(\xi) = |\xi|^{2\sigma} \, \widehat{f}(\xi), \qquad \xi \in \mathbb{R}^n, \; 0 < \sigma < 1. \tag{2.13}$$

The natural way to obtain a pointwise (integro-differential) formula for $(-\Delta)^\sigma f(x)$ would be by taking the inverse Fourier transform in (2.13). However, as we already mentioned in the Introduction, our heat semigroup language provide us the way to derive such a formula in an very simple way, without the aid of the inverse Fourier transform.

Lemma 2.1. *For* $f \in \mathcal{S}$ *and* $0 < \sigma < 1$,

$$(-\Delta)^\sigma f(x) = \frac{1}{\Gamma(-\sigma)} \int_0^\infty \left(e^{t\Delta} f(x) - f(x) \right) \, \frac{dt}{t^{1+\sigma}} = c_{n,\sigma} \, \mathrm{P.V.} \int_{\mathbb{R}^n} \frac{f(x) - f(z)}{|x-z|^{n+2\sigma}} \, dz, \tag{2.14}$$

with

$$c_{n,\sigma} = \frac{4^\sigma \Gamma(n/2+\sigma)}{-\pi^{n/2} \Gamma(-\sigma)} > 0.$$

Proof. Observe that by applying the Fourier transform and making the change of variables $s = t\,|\xi|^2$ we have

$$\begin{aligned}
\int_0^\infty \left| \left(e^{t\Delta} f(x) - f(x) \right) \right| \, \frac{dt}{t^{1+\sigma}} &\leqslant C_n \int_0^\infty \int_{\mathbb{R}^n} \left| (e^{-t|\xi|^2} - 1) \widehat{f}(\xi) e^{ix\cdot\xi} \right| \, d\xi \, \frac{dt}{t^{1+\sigma}} \\
&= C_n \int_{\mathbb{R}^n} \int_0^\infty \left| e^{-s} - 1 \right| \, \frac{ds}{s^{1+\sigma}} \, |\xi|^{2\sigma} \, |\widehat{f}(\xi)| \, d\xi \\
&= C_{n,\sigma} \int_{\mathbb{R}^n} |\xi|^{2\sigma} \, |\widehat{f}(\xi)| \, d\xi < \infty.
\end{aligned}$$

Hence, by Fubini's Theorem,

$$\begin{aligned}
\frac{1}{\Gamma(-\sigma)} \int_0^\infty \left(e^{t\Delta} f(x) - f(x) \right) \, \frac{dt}{t^{1+\sigma}} &= \frac{1}{\Gamma(-\sigma)} \int_{\mathbb{R}^n} \int_0^\infty (e^{-t|\xi|^2} - 1) \, \frac{dt}{t^{1+\sigma}} \widehat{f}(\xi) e^{ix\cdot\xi} \, d\xi \\
&= \frac{1}{\Gamma(-\sigma)} \int_{\mathbb{R}^n} \int_0^\infty (e^{-s} - 1) \, \frac{ds}{s^{1+\sigma}} \, |\xi|^{2\sigma} \, \widehat{f}(\xi) e^{ix\cdot\xi} \, d\xi \\
&= \int_{\mathbb{R}^n} |\xi|^{2\sigma} \, \widehat{f}(\xi) e^{-ix\cdot\xi} \, d\xi = (-\Delta)^\sigma f(x),
\end{aligned}$$

because of (2.7). This justifies the first equality in (2.14). It is easy to check that

$$e^{t\Delta}1(x) = \int_{\mathbb{R}^n} W_t(x-z)\ dz = 1, \qquad \text{for all } x \in \mathbb{R}^n,$$

where W_t is given in (2.12). Then,

$$\int_0^\infty \left(e^{t\Delta}f(x) - f(x)\right)\ \frac{dt}{t^{1+\sigma}} = \int_0^\infty \int_{\mathbb{R}^n} W_t(x-z)(f(z)-f(x))\ dz\ \frac{dt}{t^{1+\sigma}} = I_1 + I_2, \tag{2.15}$$

where

$$I_1 := \int_0^\infty \int_{|x-z|>1} W_t(x-y)(f(z)-f(x))\ dz\ \frac{dt}{t^{1+\sigma}},$$

Use the change of variables $s = \frac{|x-z|^2}{4t}$ to see that

$$\int_0^\infty \frac{1}{(4\pi t)^{n/2}}\ e^{-\frac{|x-z|^2}{4t}}\ \frac{dt}{t^{1+\sigma}} = \frac{4^\sigma \Gamma(n/2+\sigma)}{\pi^{n/2}} \cdot \frac{1}{|x-z|^{n+2\sigma}}. \tag{2.16}$$

So, since f is bounded, I_1 converges absolutely. Passing to polar coordinates,

$$I_2 = \int_0^\infty \frac{1}{(4\pi t)^{n/2}} \int_0^1 e^{-\frac{r^2}{4t}} r^{n-1} \int_{|z'|=1} (f(x+rz') - f(x))\ dS(z')\ dr\ \frac{dt}{t^{1+\sigma}}.$$

By Taylor's Theorem,

$$\int_{|z'|=1} (f(x+rz') - f(x))\ dS(z') = C_n r^2 \Delta f(x) + O(r^3),$$

thus

$$|I_2| \leqslant C_{n,\Delta f(x)} \int_0^1 r^{n+1} \int_0^\infty \frac{e^{-\frac{r^2}{4t}}}{t^{n/2+\sigma}}\ \frac{dt}{t}\ dr = C_{n,\Delta f(x),\sigma} \int_0^1 r^{1-2\sigma}\ dr = C_{n,\Delta f(x),\sigma},$$

and I_2 converges. Finally apply Fubini's Theorem in (2.15) together with (2.16) to get (2.14). □

Remark 2.2. Lemma 2.1 gives the exact value of the positive constant $c_{n,\sigma}$ in (1.6). Observe that

$$c_{n,\sigma} = \frac{\sigma 4^\sigma \Gamma(n/2+\sigma)}{\pi^{n/2}\Gamma(1-\sigma)} \to 0, \qquad \text{as } \sigma \to 0^+ \text{ or } \sigma \to 1^-. \tag{2.17}$$

When $f \in \mathcal{S}$ it is clear (by Fourier transform) that $\lim_{\sigma\to 1^-}(-\Delta)^\sigma f = -\Delta f$ and also $\lim_{\sigma\to 0^+}(-\Delta)^\sigma f = f$. The next two Propositions show that this is in fact valid for $f \in C^2$ and $f \in C^\alpha$, respectively. Even if $(-\Delta)^\sigma f \notin \mathcal{S}$ when $f \in \mathcal{S}$, we still have $(-\Delta)^\sigma f \in C^\infty$. It can be checked that for every $\beta \in \mathbb{N}_0^n$ the function $(1+|x|^{n+2\sigma})D^\beta(-\Delta)^\sigma f(x)$ is bounded. Therefore the set

$$L_\sigma := \left\{ u : \mathbb{R}^n \to \mathbb{R} : \|u\|_{L_\sigma} = \int_{\mathbb{R}^n} \frac{|u(z)|}{1+|z|^{n+2\sigma}}\ dz < \infty \right\},$$

consists of all locally integrable tempered distributions u for which $(-\Delta)^\sigma u$ can be defined. If $f \in L_\sigma$ is $C^{2\sigma+\varepsilon}$, $\varepsilon > 0$, in an open set $\mathcal{O}$ then it can be proved that $(-\Delta)^\sigma f$ is a continuous function in $\mathcal{O}$ and its values are given by the second integral in (2.14). For all the details see [67, 68].

Proposition 2.3. *Let* $f \in C^2(B_2(x)) \cap L^\infty(\mathbb{R}^n)$ *for some* $x \in \mathbb{R}^n$. *Then*

$$\lim_{\sigma \to 1^-} (-\Delta)^\sigma f(x) = -\Delta f(x).$$

Proof. Since f is bounded $(-\Delta)^\sigma f$ is well defined for all $0 < \sigma < 1$. Fix an arbitrary $\varepsilon > 0$. Since $f \in C^2(B_2(x))$ there exists $\delta = \delta_\varepsilon > 0$ such that

$$\left|D^2 f(w) - D^2 f(w')\right| < \varepsilon, \qquad \text{for all } w, w' \in \overline{B_1(x)} \text{ such that } \left|w - w'\right| < \delta. \tag{2.18}$$

Write $(-\Delta)^\sigma f(x) = c_{n,\sigma}(\mathrm{I} + \mathrm{II})$ where

$$\mathrm{I} := \int_{|x-z|>\delta} \frac{f(x) - f(z)}{|x-z|^{n+2\sigma}}\, dz.$$

We have

$$|\mathrm{I}| \leqslant \frac{C_n}{\sigma \delta^{2\sigma}} \|f\|_{L^\infty(\mathbb{R}^n)},$$

so applying (2.17), $c_{n,\sigma}\mathrm{I} \to 0$ as $\sigma \to 1^-$. Using polar coordinates, Taylor's Theorem and recalling that

$$\int_{|z'|=1} (z_1')^2\, dS(z') = \frac{(n/2+1)\pi^{n/2}}{\Gamma(n/2+2)},$$

we get

$$\begin{aligned}
\mathrm{II} &= \int_0^\delta r^{-1-2\sigma} \int_{|z'|=1} \big(f(x) - f(x - rz')\big)\, dS(z')\, dr \\
&= \int_0^\delta r^{-1-2\sigma} \int_{|z'|=1} R_1 f(x, rz')\, dS(z')\, dr \\
&= \int_0^\delta r^{-1-2\sigma} \Bigg[\frac{-\Delta f(x)(n/2+1)\pi^{n/2} r^2}{2\Gamma(n/2+2)} \\
&\qquad\qquad + \int_{|z'|=1} \left(R_1 f(x, rz') - \frac{r^2}{2}\langle D^2 f(x) z', z' \rangle \right) dS(z') \Bigg]\, dr \\
&= \frac{-\Delta f(x)(n/2+1)\pi^{n/2}\delta^{2-2\sigma}}{4\Gamma(n/2+2)(1-\sigma)} \\
&\qquad + \int_0^\delta r^{-1-2\sigma} \int_{|z'|=1} \left(R_1 f(x, rz') - \frac{r^2}{2}\langle D^2 f(x) z', z' \rangle \right) dS(z')\, dr \\
&=: \mathrm{II}_1 + \mathrm{II}_2,
\end{aligned}$$

where $R_1 f(x, rz')$ is the Taylor's remainder of first order. Then

$$c_{n,\sigma}II_1 = \frac{-\Delta f(x)\sigma(n/2+1)\Gamma(n/2+\sigma)\delta^{2-2\sigma}}{4^{1-\sigma}\Gamma(n/2+2)\Gamma(2-\sigma)} \to -\Delta f(x)\frac{(n/2+1)\Gamma(n/2+1)}{\Gamma(n/2+2)} = -\Delta f(x),$$

as $\sigma \to 1^-$. By (2.18),

$$\left| R_1 f(x, rz') - \frac{r^2}{2}\langle D^2 f(x) z', z'\rangle \right| \leqslant C_n r^2 \varepsilon,$$

and

$$|II_2| \leqslant C_n \delta^{2-2\sigma}(1-\sigma)^{-1}\varepsilon.$$

Therefore, $\lim_{\sigma\to 1^{-1}} |c_{n,\sigma} II_2| \leqslant C_n \varepsilon$. □

Remark 2.4. For $f \in C^2(B_2(x)) \cap L_\sigma$ the second identity in (2.14) is also valid. Indeed, the steps of the proof of Lemma 2.1 can be reverted since, because of the continuity of $D^2 f$, by Taylor's Theorem,

$$\int_{|z'|=1} (f(x+rz') - f(x))\, dS(z') = O(r^2), \qquad \text{as } r \to 0^+.$$

Proposition 2.5. *Let* $f \in C^{0,\alpha}(B_2(x)) \cap L_0$ *for some* $0 < \alpha \leqslant 1$ *and* $x \in \mathbb{R}^n$. *Then*

$$\lim_{\sigma\to 0^+} (-\Delta)^\sigma f(x) = f(x).$$

Proof. Since $L_0 \subset L_\sigma$ for all $\sigma > 0$, $(-\Delta)^\sigma f$ is well defined. Note that, as we take the limit $\sigma \to 0^+$, $(-\Delta)^\sigma f(x)$ is given by the second integral of (2.14), since eventually $2\sigma < \alpha$. Let $R > 0$ such that $(1+|x|)/R < 2$. Without loss of generality and for simplicity of writing we take $R = 1$. Because of the Hölder continuity of f and (2.17),

$$c_{n,\sigma}\left|\int_{|x-z|<1+|x|} \frac{f(x)-f(z)}{|x-z|^{n+2\sigma}}\, dz\right| \leqslant c_{n,\sigma}[f]_{C^\alpha}\,|\partial B_1(0)|\,\frac{(1+|x|)^{\alpha-2\sigma}}{\alpha-2\sigma} \to 0, \tag{2.19}$$

as $\sigma \to 0^+$. The kernel $|z|^{-(n+2\sigma)}$ is integrable at infinity and

$$\int_{|x-z|\geqslant 1+|x|} \frac{1}{|x-z|^{n+2\sigma}}\, dz = |\partial B_1(0)|\,\frac{(1+|x|)^{-2\sigma}}{2\sigma}.$$

Therefore,

$$c_{n,\sigma}\int_{|x-z|\geqslant 1+|x|} \frac{f(x)}{|x-z|^{n+2\sigma}}\, dz = f(x)\frac{4^\sigma(n/2)\Gamma(n/2+\sigma)}{\Gamma(n/2+1)\Gamma(1-\sigma)(1+|x|)^{2\sigma}} \to f(x), \tag{2.20}$$

as $\sigma \to 0^+$. Finally we note that $|x-z| \geqslant 1+|x|$ implies

$$1+|z| \leqslant 1+|x|+|x-z| \leqslant 2\,|x-z|\,,$$

so that

$$c_{n,\sigma}\left|\int_{|x-z|\geqslant 1+|x|} \frac{f(z)}{|x-z|^{n+2\sigma}}\, dz\right| \leqslant c_{n,\sigma}2^{n+2}\,\|f\|_{L_0} \to 0, \qquad \sigma \to 0^+. \tag{2.21}$$

From (2.19), (2.20) and (2.21) the Proposition follows. □

2.2 The harmonic oscillator and related operators

We denote by H the harmonic oscillator (also known as Hermite operator) in $\mathbb{R}^n$, $n \geqslant 1$:

$$H = -\Delta + |x|^2 .$$

For the following we refer the reader to the book by S. Thangavelu [82]. The eigenfunctions of H (with zero boundary condition at infinity) are the multi-dimensional Hermite functions defined for all $x \in \mathbb{R}^n$ by $h_\nu(x) = \Psi_\nu(x) \cdot e^{-|x|^2/2}$, for $\nu = (\nu_1, \dots, \nu_n) \in \mathbb{N}_0^n$, where Ψ_ν are the multi-dimensional Hermite polynomials. The corresponding eigenvalues are positive:

$$Hh_\nu(x) = (2|\nu| + n)h_\nu(x), \qquad x \in \mathbb{R}^n,\ \nu \in \mathbb{N}_0^n.$$

Here $|\nu| = \nu_1 + \cdots + \nu_n$. Note that $h_\nu \in \mathcal{S}$. The set of Hermite functions forms an orthonormal basis of $L^2(\mathbb{R}^n)$. Let $f \in \mathcal{S}$. The *Hermite series expansion* of f given by

$$\sum_\nu \langle f, h_\nu \rangle h_\nu(x) = \sum_{k=0}^{\infty} \sum_{|\nu|=k} \langle f, h_\nu \rangle h_\nu(x), \qquad x \in \mathbb{R}^n, \tag{2.22}$$

with $\langle f, h_\nu \rangle = \int_{\mathbb{R}^n} f h_\nu \, dx$ (which converges to f in $L^2(\mathbb{R}^n)$) converges uniformly in $\mathbb{R}^n$ to f. This uniform convergence is a consequence of the fact that $\|h_\nu\|_{L^\infty(\mathbb{R}^n)} \leqslant C$ for all $\nu \in \mathbb{N}_0^n$, and the following estimate: for every $m \in \mathbb{N}$,

$$|\langle f, h_\nu \rangle| = \frac{|\langle H^m f, h_\nu \rangle|}{(2|\nu| + n)^m} \leqslant \frac{\|H^m f\|_{L^2(\mathbb{R}^n)}}{(2|\nu| + n)^m}, \tag{2.23}$$

since H is a symmetric operator.

The heat-diffusion semigroup. If $f \in \mathcal{S}$ then

$$e^{-tH} f(x) = \sum_\nu e^{-t(2|\nu|+n)} \langle f, h_\nu \rangle h_\nu(x), \qquad x \in \mathbb{R}^n,\ t \geqslant 0, \tag{2.24}$$

the series converging uniformly in $\mathbb{R}^n$. By writing down the integral in $\langle f, h_\nu \rangle$ and applying Fubini's Theorem, the heat semigroup can be given as an integral operator, and the formula extends to $f \in \bigcup_{1 \leqslant p \leqslant \infty} L^p(\mathbb{R}^n)$ (see also [78]):

$$\begin{aligned} e^{-tH} f(x) &= \int_{\mathbb{R}^n} G_t(x,z) f(z)\, dz \\ &= \int_{\mathbb{R}^n} \left[\sum_{j=0}^{\infty} e^{-t(2j+n)} \sum_{|\nu|=j} h_\nu(x) h_\nu(z) \right] f(z)\, dz = \int_{\mathbb{R}^n} \frac{e^{-\left[\frac{1}{2}|x-z|^2 \coth 2t + x\cdot z \tanh t\right]}}{(2\pi \sinh 2t)^{n/2}} f(z)\, dz. \end{aligned} \tag{2.25}$$

In particular, (see [43])

$$e^{-tH} 1(x) = \frac{1}{(\cosh 2t)^{n/2}} \, e^{-\frac{\tanh 2t}{2}|x|^2} \leqslant 1. \tag{2.26}$$

With S. Meda's change of parameters defined by

$$t = \frac{1}{2}\log\frac{1+s}{1-s}, \qquad t \in (0,\infty),\ \ s \in (0,1), \tag{2.27}$$

the heat-diffusion kernel can be written as

$$G_{t(s)}(x,z) = \sum_{j=0}^{\infty}\left(\frac{1-s}{1+s}\right)^{j+n/2}\sum_{|\nu|=j} h_\nu(x)h_\nu(z) = \left(\frac{1-s^2}{4\pi s}\right)^{n/2} e^{-\frac{1}{4}\left[s|x+z|^2+\frac{1}{s}|x-z|^2\right]}, \tag{2.28}$$

and we also have

$$e^{-t(s)H}1(x) = \left(\frac{1-s^2}{1+s^2}\right)^{n/2} e^{-\frac{s}{1+s^2}|x|^2}, \tag{2.29}$$

for all $s \in (0,1)$. The following estimate for the size of the heat kernel $G_{t(s)}(x,z)$ has a simple but technical proof which is given in Chapter 5, Section 5.5.

Lemma 2.6. *For all* $s \in (0,1)$ *and* $x, z \in \mathbb{R}^n$,

$$G_{t(s)}(x,z) \leqslant C\left(\frac{1-s}{s}\right)^{n/2} e^{-\frac{|x||x-z|}{C}} e^{-\frac{|x-z|^2}{Cs}}. \tag{2.30}$$

In particular,

$$G_{t(s)}(x,z) \leqslant \frac{C}{|x-z|^n}\,(1-s)^{n/2} e^{-\frac{|x||x-z|}{C}} e^{-\frac{|x-z|^2}{C}} e^{-\frac{|x-z|^2}{Cs}}. \tag{2.31}$$

The negative powers $H^{-\sigma}$. For $f \in \mathcal{S}$ the fractional integral $H^{-\sigma}f$, $\sigma > 0$, is given by

$$H^{-\sigma}f(x) = \sum_{\nu}\frac{1}{(2|\nu|+n)^{\sigma}}\,\langle f, h_\nu\rangle h_\nu(x) = \frac{1}{\Gamma(\sigma)}\int_0^\infty e^{-tH}f(x)\,\frac{dt}{t^{1-\sigma}}.$$

The second identity above follows from (2.4) and Fubini's Theorem. By writing down the expression of the heat-diffusion semigroup and applying Fubini's Theorem,

$$H^{-\sigma}f(x) = \int_{\mathbb{R}^n}\left[\frac{1}{\Gamma(\sigma)}\int_0^\infty G_t(x,z)\,\frac{dt}{t^{1-\sigma}}\right] f(z)\,dz = \int_{\mathbb{R}^n} F_{-\sigma}(x,z)f(z)\,dz.$$

In [17] it is shown that the definition of $H^{-\sigma}$ extends to $f \in L^p(\mathbb{R}^n)$, $1 \leqslant p \leqslant \infty$, via the previous integral formula.

The Hermite-Riesz transforms $\mathcal{R}_i$ and $\mathcal{R}_{ij}$. We can write

$$H = \frac{1}{2}\sum_{i=1}^{n}(A_iA_{-i} + A_{-i}A_i),$$

where

$$A_i = \partial_{x_i} + x_i, \quad A_{-i} = A_i^* = -\partial_{x_i} + x_i, \qquad i = 1, \ldots, n. \tag{2.32}$$

Here A_i^* denotes the formal adjoint of A_i. The factorization above suggests the definition of the Hermite-Riesz transforms. In fact, in the Harmonic Analysis associated to H the operators A_i, $1 \leqslant |i| \leqslant n$, play the role of the classical partial derivatives ∂_{x_i} in the Euclidean Harmonic Analysis, see [82, 83, 17, 43, 76, 77, 78]. The first order Hermite-Riesz transforms are given by

$$\mathcal{R}_i = A_i H^{-1/2}, \qquad 1 \leqslant |i| \leqslant n. \tag{2.33}$$

These operators where first introduced and studied by Thangavelu [82], see also [83, 78]. The second order Hermite-Riesz transforms are (see [43, 77])

$$\mathcal{R}_{ij} = A_i A_j H^{-1}, \qquad 1 \leqslant |i|, |j| \leqslant n. \tag{2.34}$$

2.3 Spectral Theory and Functional Calculus

The general theory and all the missing details can be found in [62, Ch. 12 and 13].

Let Ω be an open subset of $\mathbb{R}^n$, $n \geqslant 1$, and let $d\eta$ be a positive σ-finite measure defined on Ω. We denote by $L^2(\Omega)$ the space $L^2(\Omega, d\eta)$, and for $f, g \in L^2(\Omega)$ we write $\langle f, g\rangle_{L^2(\Omega)}$ to denote the inner product $\int_\Omega fg \, d\eta$. Let L be a linear second order partial differential operator defined in some domain $\mathrm{Dom}(L)$, that we assume to be a dense subset of $L^2(\Omega)$. The operator L will always be taken to be nonnegative:

$$\langle Lf, f\rangle_{L^2(\Omega)} \geqslant 0, \qquad f \in \mathrm{Dom}(L).$$

Recall that this is equivalent to the fact that the spectrum of L is contained in $[0, \infty)$. Motivated by our concrete examples, we add the condition that L is self-adjoint, i.e. L equals its adjoint L^*. Under these hypotheses, the Spectral Theorem is valid: there exists a unique resolution of the identity E, supported on the spectrum of L such that

$$L = \int_0^\infty \lambda \, dE(\lambda). \tag{2.35}$$

In our context, a *resolution of the identity* E is a mapping

$$E : \{\text{Borel subsets of } (0, \infty)\} \longrightarrow \{\text{Bounded linear operators on } L^2(\Omega)\},$$

such that

1. $E(\varnothing) = 0$, $E((0, \infty)) = \mathrm{Id}$;
2. Each $E(B)$ is a self-adjoint projection ($E(B)^2 = E(B)$);
3. $E(B_1 \cap B_2) = E(B_1)E(B_2)$;
4. If $B_1 \cap B_2 = \varnothing$ then $E(B_1 \cup B_2) = E(B_1) + E(B_2)$;

5. For every $f, g \in L^2(\Omega)$, the set function $E_{f,g}$ defined by

$$E_{f,g}(B) = \langle E(B)f, g\rangle_{L^2(\Omega)},$$

is a complex measure on the Borel subsets of $(0, \infty)$.

Condition 2. above implies that $E_{f,f}(B) = \langle E(B)f, f\rangle_{L^2(\Omega)} = \|E(B)f\|^2_{L^2(\Omega)}$, for all $f \in L^2(\Omega)$, so each $E_{f,f}$ is a positive Borel measure on $(0, \infty)$ whose total variation is $\|E_{f,f}\| = E_{f,f}((0, \infty)) = \|f\|^2_{L^2(\Omega)}$.

The identity (2.35) is then a shorthand notation that means

$$\langle Lf, g\rangle_{L^2(\Omega)} = \int_0^\infty \lambda \, dE_{f,g}(\lambda), \qquad f \in \mathrm{Dom}(L),\ g \in L^2(\Omega),$$

where, by the definition above, $dE_{f,g}(\lambda)$ is a regular Borel complex measure of bounded variation concentrated on the spectrum of L, with $d\left|E_{f,g}\right|(0, \infty) \leqslant \|f\|_{L^2(\Omega)} \|g\|_{L^2(\Omega)}$.

Once the Spectral Theorem is established, the Functional Calculus can be defined. If $\phi(\lambda)$ is a real measurable function defined on $[0, \infty)$ then the operator $\phi(L)$ is given formally by

$$\phi(L) = \int_0^\infty \phi(\lambda) \, dE(\lambda). \tag{2.36}$$

That is, $\phi(L)$ is the operator with domain

$$\mathrm{Dom}(\phi(L)) = \left\{ f \in L^2(\Omega) : \int_0^\infty |\phi(\lambda)|^2 \, dE_{f,f}(\lambda) < \infty \right\},$$

defined by

$$\langle \phi(L)f, g\rangle_{L^2(\Omega)} = \left\langle \int_0^\infty \phi(\lambda) \, dE(\lambda)f, g \right\rangle_{L^2(\Omega)} = \int_0^\infty \phi(\lambda) \, dE_{f,g}(\lambda). \tag{2.37}$$

The set $\mathrm{Dom}(\phi(L))$ is dense in $L^2(\Omega)$. If $f, g \in L^2(\Omega)$ then

$$\int_0^\infty |\phi(\lambda)| \, d\left|E_{f,g}\right|(\lambda) \leqslant \|g\|_{L^2(\Omega)} \left(\int_0^\infty |\phi(\lambda)|^2 \, dE_{f,f}(\lambda) \right)^{1/2}. \tag{2.38}$$

If ϕ is a bounded function then the domain of $\phi(L)$ is all $L^2(\Omega)$.

At this point we can define the relevant operators in the general setting.

Heat-diffusion semigroup generated by L with domain $L^2(\Omega)$,

$$e^{-tL} = \int_0^\infty e^{-t\lambda} \, dE(\lambda), \qquad t \geqslant 0,$$

The contraction property in $L^2(\Omega)$ holds: $\left\|e^{-tL}f\right\|_{L^2(\Omega)} \leqslant \|f\|_{L^2(\Omega)}$.

Poisson semigroup generated by L also with domain $L^2(\Omega)$,

$$e^{-y\sqrt{L}} = \int_0^\infty e^{-y\sqrt{\lambda}}\, dE(\lambda), \qquad y \geqslant 0.$$

If in the formula above we apply (2.2) then, using (2.38), we get (2.3).

Negative powers of L can be expressed using the heat semigroup by applying (2.4) in the definition:

$$L^{-\sigma} = \int_0^\infty \lambda^{-\sigma}\, dE(\lambda) = \frac{1}{\Gamma(\sigma)} \int_0^\infty e^{-tL}\, \frac{dt}{t^{1-\sigma}}, \qquad \sigma > 0.$$

Fractional powers of L with domain $\mathrm{Dom}(L^\sigma) \subset \mathrm{Dom}(L)$. Recalling (2.6),

$$L^\sigma = \int_0^\infty \lambda^\sigma\, dE(\lambda) = \frac{1}{\Gamma(-\sigma)} \int_0^\infty \left(e^{-tL} - \mathrm{Id}\right)\, \frac{dt}{t^{1+\sigma}}, \qquad 0 < \sigma < 1.$$

Chapter 3

Extension problem for fractional operators

In this chapter we study fractional powers of second order partial differential operators L in a fairly general class. In Section 3.1 we describe any fractional operator L^σ, $0 < \sigma < 1$, as an operator that maps a Dirichlet condition to a Neumann-type condition via an extension problem. The corresponding properties are developed in Section 3.2. Some examples of operators L for which our results can be applied are given. The solution of the extension problem in terms of Fourier expansions and Bessel equations is presented in Section 3.3.

3.1 The extension problem

Let Ω be an open subset of $\mathbb{R}^n$, $n \geqslant 1$, and let $d\eta$ be a positive σ-finite measure defined on Ω. Throughout this chapter L denotes a nonnegative and self-adjoint linear second order partial differential operator densely defined in $L^2(\Omega) = L^2(\Omega, d\eta)$. As shown in Chapter 2, the fractional powers L^σ, $0 < \sigma < 1$, can be defined using the Spectral Theorem.

The main result of this chapter is the following.

Theorem 3.1. *Let* $f \in \mathrm{Dom}(L^\sigma)$. *A solution of the extension problem*

$$u(x,0) = f(x), \qquad \text{on } \Omega; \tag{3.1}$$

$$-L_x u + \frac{1-2\sigma}{y} u_y + u_{yy} = 0, \qquad \text{in } \Omega \times (0,\infty); \tag{3.2}$$

is given by

$$u(x,y) = \frac{1}{\Gamma(\sigma)} \int_0^\infty e^{-tL}(L^\sigma f)(x) e^{-\frac{y^2}{4t}} \frac{dt}{t^{1-\sigma}}, \tag{3.3}$$

and

$$\lim_{y\to 0^+} \frac{u(x,y) - u(x,0)}{y^{2\sigma}} = \frac{\Gamma(-\sigma)}{4^\sigma \Gamma(\sigma)} L^\sigma f(x) = \frac{1}{2\sigma} \lim_{y\to 0^+} y^{1-2\sigma} u_y(x,y). \tag{3.4}$$

Moreover, the following Poisson formula for u *holds:*

$$u(x,y) = \frac{y^{2\sigma}}{4^\sigma \Gamma(\sigma)} \int_0^\infty e^{-tL} f(x) e^{-\frac{y^2}{4t}} \, \frac{dt}{t^{1+\sigma}} = \frac{1}{\Gamma(\sigma)} \int_0^\infty e^{-\frac{y^2}{4r}L} f(x) e^{-r} \, \frac{dr}{r^{1-\sigma}}. \tag{3.5}$$

All identities above are understood in $L^2(\Omega)$. Note that a solution u to the degenerate boundary value problem (3.1)-(3.2) is written explicitly in terms of the heat semigroup e^{-tL} acting on $L^\sigma f$, but in the Poisson formula (3.5) no fractional power of L is involved.

When $L = -\Delta$, the extension result of [23] is recovered, see Examples 3.15. We show in Remark 3.7 that the Poisson kernel for the general case can also be derived with the method of [23].

For simplicity of reading we present first the proof of Theorem 3.1 for the discrete spectrum case and f a finite linear combination of eigenfunctions of L. Secondly, we sketch the proof for $L = -\Delta$ and $f \in \mathcal{S}$. In these contexts all identities in the statement of Theorem 3.1 also hold pointwise. Finally the general proof is given.

Proof of Theorem 3.1 (Discrete Spectrum Case). If L has discrete spectrum (as in Subsection 2.1.1 of Chapter 2) the definition of L^σ in $L^2(\Omega)$ is the natural one: if $f \in L^2(\Omega)$ has the property that

$$\sum_k \lambda_k^{2\sigma} |\langle f, \phi_k \rangle|^2 = \sum_k \lambda_k^{2\sigma} \left| \int_\Omega f \phi_k \, d\eta \right|^2 < \infty,$$

then

$$L^\sigma f = \sum_k \lambda_k^\sigma \langle f, \phi_k \rangle \phi_k, \qquad \text{sum in } L^2(\Omega). \tag{3.6}$$

Assume for the rest of the proof that f is a finite linear combination of ϕ_k's. Hence the sum in (3.6) contains finitely many terms and $L^\sigma f(x)$ is defined for all $x \in \Omega$. Then (3.3) is well defined:

$$\begin{aligned}\int_0^\infty \left| e^{-tL}(L^\sigma f)(x) e^{-\frac{y^2}{4t}} \right| \frac{dt}{t^{1-\sigma}} &\leqslant \sum_k \int_0^\infty e^{-t\lambda_k} \frac{dt}{t^{1-\sigma}} \lambda_k^\sigma |c_k| \, |\phi_k(x)| \\ &= \Gamma(\sigma) \sum_k |c_k| \, |\phi_k(x)| < \infty,\end{aligned}$$

and

$$u(x,y) = \frac{1}{\Gamma(\sigma)} \sum_k \lambda_k^\sigma c_k \phi_k(x) \int_0^\infty e^{-t\lambda_k} e^{-\frac{y^2}{4t}} \, \frac{dt}{t^{1-\sigma}}.$$

We have

$$u(x,0) = \frac{1}{\Gamma(\sigma)} \sum_k \lambda_k^\sigma c_k \phi_k(x) \int_0^\infty e^{-t\lambda_k} \, \frac{dt}{t^{1-\sigma}} = \sum_k c_k \phi_k(x) = f(x),$$

and, by integration by parts,

$$\begin{aligned}
Lu(x,y) &= \frac{1}{\Gamma(\sigma)}\sum_k \lambda_k^\sigma c_k \lambda_k \phi_k(x)\int_0^\infty e^{-t\lambda_k}e^{-\frac{y^2}{4t}}\,\frac{dt}{t^{1-\sigma}}\\
&= -\frac{1}{\Gamma(\sigma)}\sum_k \lambda_k^\sigma c_k \phi_k(x)\int_0^\infty \partial_t(e^{-t\lambda_k})e^{-\frac{y^2}{4t}}\,\frac{dt}{t^{1-\sigma}}\\
&= \frac{1}{\Gamma(\sigma)}\sum_k \lambda_k^\sigma c_k \phi_k(x)\int_0^\infty e^{-t\lambda_k}e^{-\frac{y^2}{4t}}\left(\frac{y^2}{4t^2}+\frac{\sigma-1}{t}\right)\frac{dt}{t^{1-\sigma}}\\
&= \frac{1-2\sigma}{y}\,u_y(x,y)+u_{yy}(x,y),
\end{aligned}$$

so u given by (3.3) solves (3.1)-(3.2). Note that

$$\begin{aligned}
\frac{u(x,y)-u(x,0)}{y^{2\sigma}} &= \frac{1}{4^\sigma\Gamma(\sigma)}\int_0^\infty e^{-tL}(L^\sigma f)(x)\left(\frac{e^{-\frac{y^2}{4t}}-1}{\left(\frac{y^2}{4t}\right)^\sigma}\right)\frac{dt}{t}\\
&= \frac{1}{4^\sigma\Gamma(\sigma)}\int_0^\infty e^{-\frac{y^2}{4r}L}(L^\sigma f)(x)\left(\frac{e^{-r}-1}{r^\sigma}\right)\frac{dr}{r},
\end{aligned}$$

therefore, since $\lim_{t\to 0^+} e^{-tL}(L^\sigma f)(x) = L^\sigma f(x)$, by the Dominated Convergence Theorem, we obtain the first identity in (3.4). The second equality of (3.4) follows because

$$\begin{aligned}
\frac{1}{2\sigma}\,y^{1-2\sigma}u_y(x,y) &= \frac{-1}{4^\sigma\sigma\Gamma(\sigma)}\sum_k \lambda_k^\sigma c_k\phi_k(x)\int_0^\infty e^{-t\lambda_k}e^{-\frac{y^2}{4t}}\left(\frac{y^2}{4t}\right)^{1-\sigma}\frac{dt}{t}\\
&= \frac{-1}{4^\sigma\sigma\Gamma(\sigma)}\sum_k \lambda_k^\sigma c_k\phi_k(x)\int_0^\infty e^{-\frac{y^2}{4r}\lambda_k}e^{-r}r^{1-\sigma}\frac{dr}{r},
\end{aligned}$$

implies

$$\frac{1}{2\sigma}\lim_{y\to 0^+} y^{1-2\sigma}u_y(x,y) = \frac{-\Gamma(1-\sigma)}{4^\sigma\sigma\Gamma(\sigma)}L^\sigma f(x) = \frac{\Gamma(-\sigma)}{4^\sigma\Gamma(\sigma)}L^\sigma f(x).$$

Finally, u can be written as in (3.5): indeed, using the change of variables $r = y^2/(4t\lambda_k)$,

$$\begin{aligned}
u(x,y) &= \frac{1}{\Gamma(\sigma)}\sum_k\left(\int_0^\infty e^{-t\lambda_k}(t\lambda_k)^\sigma e^{-\frac{y^2}{4t}}\,\frac{dt}{t}\right)c_k\phi_k(x)\\
&= \frac{1}{\Gamma(\sigma)}\sum_k\left(\int_0^\infty e^{-\frac{y^2}{4r}}\left(\frac{y^2}{4r}\right)^\sigma e^{-r\lambda_k}\,\frac{dr}{r}\right)c_k\phi_k(x)\\
&= \frac{y^{2\sigma}}{4^\sigma\Gamma(\sigma)}\sum_k\left(\int_0^\infty e^{-\frac{y^2}{4r}}e^{-r\lambda_k}\,\frac{dr}{r^{1+\sigma}}\right)c_k\phi_k(x)\\
&= \frac{y^{2\sigma}}{4^\sigma\Gamma(\sigma)}\int_0^\infty\left(\sum_k e^{-r\lambda_k}c_k\phi_k(x)\right)e^{-\frac{y^2}{4r}}\,\frac{dr}{r^{1+\sigma}} = \frac{y^{2\sigma}}{4^\sigma\Gamma(\sigma)}\int_0^\infty e^{-rL}f(x)e^{-\frac{y^2}{4r}}\,\frac{dr}{r^{1+\sigma}}.
\end{aligned}$$

□

Proof of Theorem 3.1 (The Laplacian in $\mathbb{R}^n$). Let f be in $\mathcal{S}$. Then u is well defined since

$$\begin{aligned}\int_0^\infty \int_{\mathbb{R}^n} \left| e^{-t|\xi|^2} |\xi|^{2\sigma} \widehat{f}(\xi) e^{ix\cdot\xi} \right| \, d\xi \, e^{-\frac{y^2}{4t}} \, \frac{dt}{t^{1-\sigma}} &\leqslant \int_{\mathbb{R}^n} |\widehat{f}(\xi)| \int_0^\infty e^{-t|\xi|^2} (t\,|\xi|^2)^\sigma \, \frac{dt}{t} \, d\xi \\ &= \Gamma(\sigma) \|\widehat{f}\|_{L^1(\mathbb{R}^n)}.\end{aligned}$$

By Fubini's Theorem we can write

$$u(x,y) = \frac{1}{(2\pi)^{n/2}\Gamma(\sigma)} \int_{\mathbb{R}^n} |\xi|^{2\sigma} \widehat{f}(\xi) e^{-ix\cdot\xi} \int_0^\infty e^{-t|\xi|^2} e^{-\frac{y^2}{4t}} \, \frac{dt}{t^{1-\sigma}} \, d\xi.$$

Now the proof follows the same lines as in the discrete spectrum case, but with the obvious modifications: the sum replaced by the integral, the coefficients c_k by the Fourier transform $\widehat{f}(\xi)$, the exponentials $e^{-ix\cdot\xi}$ by the eigenfunctions ϕ_k and $|\xi|^2$ in the place of λ_k. □

Proof of Theorem 3.1 (General Case). Before starting we suggest the reader to review Chapter 2, Section 2.3.

1. First we prove that $u(\cdot,y) \in L^2(\Omega)$ and, for all $g \in L^2(\Omega)$,

$$\langle u(\cdot,y), g(\cdot)\rangle_{L^2} = \frac{1}{\Gamma(\sigma)} \int_0^\infty \left\langle e^{-tL}(L^\sigma f), g\right\rangle_{L^2(\Omega)} e^{-\frac{y^2}{4t}} \, \frac{dt}{t^{1-\sigma}}. \tag{3.7}$$

For each $R > 0$ let

$$u_R(x,y) = \frac{1}{\Gamma(\sigma)} \int_0^R e^{-tL}(L^\sigma f)(x) e^{-\frac{y^2}{4t}} \, \frac{dt}{t^{1-\sigma}}.$$

Since $f \in \mathrm{Dom}(L^\sigma)$, $e^{-tL}(L^\sigma f) \in L^2(\Omega)$. Moreover, $e^{-\frac{y^2}{4t}}/t^{1-\sigma}$ is integrable near 0 as a function of t. Then, using Bochner's Theorem, (2.37), the fact that $dE_{f,g}(\lambda)$ is of bounded variation and the change of variables $t = r/\lambda$, we have

$$\begin{aligned}\langle u_R(\cdot,y), g(\cdot)\rangle_{L^2(\Omega)} &= \frac{1}{\Gamma(\sigma)} \int_0^R \left\langle e^{-tL} L^\sigma f, g\right\rangle_{L^2(\Omega)} e^{-\frac{y^2}{4t}} \, \frac{dt}{t^{1-\sigma}} \\ &= \frac{1}{\Gamma(\sigma)} \int_0^R \int_0^\infty e^{-t\lambda}\lambda^\sigma \, dE_{f,g}(\lambda) \, e^{-\frac{y^2}{4t}} \, \frac{dt}{t^{1-\sigma}} \\ &= \frac{1}{\Gamma(\sigma)} \int_0^\infty \int_0^R e^{-t\lambda}(t\lambda)^\sigma e^{-\frac{y^2}{4t}} \, \frac{dt}{t} \, dE_{f,g}(\lambda) \\ &= \frac{1}{\Gamma(\sigma)} \int_0^\infty \int_0^{R\lambda} e^{-r} r^\sigma e^{-\frac{y^2}{4r}\lambda} \, \frac{dr}{r} \, dE_{f,g}(\lambda),\end{aligned}$$

so that

$$\left|\langle u_R(\cdot,y), g(\cdot)\rangle_{L^2(\Omega)}\right| \leqslant \frac{1}{\Gamma(\sigma)} \int_0^\infty \int_0^\infty e^{-r} r^\sigma \, \frac{dr}{r} \, d\left|E_{f,g}\right|(\lambda) \leqslant \|f\|_{L^2(\Omega)} \|g\|_{L^2(\Omega)}.$$

Therefore, for each fixed $y > 0$, $u_R(\cdot,y)$ is in $L^2(\Omega)$, and $\|u_R(\cdot,y)\|_{L^2(\Omega)} \leqslant \|f\|_{L^2(\Omega)}$. If $0 < R_1 < R_2$ then

$$\langle u_{R_2}(\cdot,y) - u_{R_1}(\cdot,y), g(\cdot)\rangle_{L^2(\Omega)} = \frac{1}{\Gamma(\sigma)} \int_0^\infty \int_{R_1\lambda}^{R_2\lambda} e^{-r} r^\sigma e^{-\frac{y^2}{4r}\lambda} \, \frac{dr}{r} \, dE_{f,g}(\lambda).$$

Note that

$$\left|\int_{R_1\lambda}^{R_2\lambda} e^{-r} r^{\sigma} e^{-\frac{y^2}{4r}\lambda}\,\frac{dr}{r}\right| \to 0 \qquad \text{as } R_1, R_2 \to \infty,$$

and

$$\left|\int_{R_1\lambda}^{R_2\lambda} e^{-r} r^{\sigma} e^{-\frac{y^2}{4r}\lambda}\,\frac{dr}{r}\right| \leqslant \Gamma(\sigma), \qquad \text{for all } \lambda.$$

Hence, by dominated convergence,

$$\lim_{R_1,R_2\to\infty} \langle u_{R_2}(\cdot,y) - u_{R_1}(\cdot,y), g(\cdot)\rangle_{L^2(\Omega)} = 0.$$

Therefore, for any sequence $\{R^j\}_{j\in\mathbb{N}}$ of positive numbers such that $R^j \nearrow \infty$ we have that the family $\{u_{R^j}(\cdot,y)\}_{j\in\mathbb{N}}$ is a Cauchy sequence of bounded linear operators on $L^2(\Omega)$. Thus, $u_R(\cdot,y) \to u(\cdot,y)$ weakly in $L^2(\Omega)$, as $R\to\infty$, and $u(\cdot,y)\in L^2(\Omega)$. Moreover,

$$\begin{aligned}
\langle u(\cdot,y), g(\cdot)\rangle_{L^2(\Omega)} &= \lim_{R\to\infty} \langle u_R(\cdot,y), g(\cdot)\rangle_{L^2(\Omega)} \\
&= \lim_{R\to\infty} \frac{1}{\Gamma(\sigma)} \int_0^\infty \int_0^R e^{-t\lambda}(t\lambda)^{\sigma} e^{-\frac{y^2}{4t}}\,\frac{dt}{t}\,dE_{f,g}(\lambda) \\
&= \frac{1}{\Gamma(\sigma)} \int_0^\infty \int_0^\infty e^{-t\lambda}(t\lambda)^{\sigma} e^{-\frac{y^2}{4t}}\,\frac{dt}{t}\,dE_{f,g}(\lambda) \\
&= \frac{1}{\Gamma(\sigma)} \int_0^\infty \int_0^\infty e^{-t\lambda}\lambda^{\sigma}\,dE_{f,g}(\lambda)\,e^{-\frac{y^2}{4t}}\,\frac{dt}{t^{1-\sigma}} \\
&= \frac{1}{\Gamma(\sigma)} \int_0^\infty \langle e^{-tL}(L^{\sigma}f), g\rangle_{L^2(\Omega)} e^{-\frac{y^2}{4t}}\,\frac{dt}{t^{1-\sigma}},
\end{aligned}$$

where the limit can be taken inside the integral because the double integral converges absolutely. Hence, (3.7) follows.

2. Next we show that $u(\cdot,y)\in \mathrm{Dom}(L)$, that is,

$$\lim_{s\to 0^+} \left\langle \frac{e^{-sL}u(\cdot,y) - u(\cdot,y)}{s}, g(\cdot)\right\rangle_{L^2(\Omega)} \quad \text{exists for all } g\in L^2(\Omega).$$

As e^{-sL} is self-adjoint, by (3.7) we have

$$\begin{aligned}
\left\langle e^{-sL}u(\cdot,y), g(\cdot)\right\rangle_{L^2(\Omega)} &= \left\langle u(\cdot,y), e^{-sL}g(\cdot)\right\rangle_{L^2(\Omega)} \\
&= \frac{1}{\Gamma(\sigma)} \int_0^\infty \left\langle e^{-tL}L^{\sigma}f, e^{-sL}g\right\rangle_{L^2(\Omega)} e^{-\frac{y^2}{4t}}\,\frac{dt}{t^{1-\sigma}} \\
&= \frac{1}{\Gamma(\sigma)} \int_0^\infty \left\langle e^{-sL}e^{-tL}L^{\sigma}f, g\right\rangle_{L^2(\Omega)} e^{-\frac{y^2}{4t}}\,\frac{dt}{t^{1-\sigma}}.
\end{aligned}$$

Hence, (3.7), (2.37) and Fubini's Theorem give

$$\begin{aligned}\left\langle \frac{e^{-sL}u(\cdot,y)-u(\cdot,y)}{s}, g(\cdot)\right\rangle_{L^2} &= \frac{1}{\Gamma(\sigma)}\int_0^\infty \left\langle \frac{e^{-sL}e^{-tL}L^\sigma f - e^{-tL}L^\sigma f}{s}, g\right\rangle_{L^2} e^{-\frac{y^2}{4t}}\frac{dt}{t^{1-\sigma}} \\ &= \frac{1}{\Gamma(\sigma)}\int_0^\infty\int_0^\infty \frac{e^{-s\lambda}e^{-t\lambda}\lambda^\sigma - e^{-t\lambda}\lambda^\sigma}{s} dE_{f,g}(\lambda) e^{-\frac{y^2}{4t}}\frac{dt}{t^{1-\sigma}} \\ &= \frac{1}{\Gamma(\sigma)}\int_0^\infty\int_0^\infty \frac{e^{-s\lambda}e^{-t\lambda}\lambda^\sigma - e^{-t\lambda}\lambda^\sigma}{s}\, e^{-\frac{y^2}{4t}}\frac{dt}{t^{1-\sigma}} dE_{f,g}(\lambda).\end{aligned}$$

Finally, by dominated convergence

$$\begin{aligned}\left\langle \frac{e^{-sL}u(\cdot,y)-u(\cdot,y)}{s}, g(\cdot)\right\rangle_{L^2(\Omega)} &\underset{s\to0^+}{\longrightarrow} \frac{1}{\Gamma(\sigma)}\int_0^\infty\int_0^\infty \partial_t(e^{-t\lambda})\lambda^\sigma e^{-\frac{y^2}{4t}}\frac{dt}{t^{1-\sigma}}\; dE_{f,g}(\lambda) \\ &= \frac{1}{\Gamma(\sigma)}\int_0^\infty\int_0^\infty \partial_t(e^{-t\lambda})\lambda^\sigma\; dE_{f,g}(\lambda)\; e^{-\frac{y^2}{4t}}\frac{dt}{t^{1-\sigma}} \\ &= -\frac{1}{\Gamma(\sigma)}\int_0^\infty \left\langle Le^{-tL}L^\sigma f, g\right\rangle_{L^2(\Omega)} e^{-\frac{y^2}{4t}}\frac{dt}{t^{1-\sigma}}.\end{aligned}$$

3. We check the boundary condition (3.1): for $g\in L^2(\Omega)$, by (3.7),

$$\begin{aligned}\langle u(\cdot,y), g(\cdot)\rangle &= \frac{1}{\Gamma(\sigma)}\int_0^\infty\int_0^\infty e^{-t\lambda}(t\lambda)^\sigma\; dE_{f,g}(\lambda)\; e^{-\frac{y^2}{4t}}\frac{dt}{t^{1-\sigma}} \\ &= \frac{1}{\Gamma(\sigma)}\int_0^\infty\int_0^\infty e^{-r}r^\sigma e^{-\frac{y^2\lambda}{4r}}\;\frac{dr}{r}\; dE_{f,g}(\lambda) \underset{y\to0}{\longrightarrow} \langle f, g\rangle_{L^2(\Omega)}.\end{aligned}$$

4. The function u is differentiable with respect to y and

$$\begin{aligned}u_y(x,y) &= \frac{1}{\Gamma(\sigma)}\int_0^\infty e^{-tL}(L^\sigma f)(x)\; \partial_y(e^{-\frac{y^2}{4t}})\;\frac{dt}{t^{1-\sigma}} \\ &= \frac{-1}{\Gamma(\sigma)}\int_0^\infty e^{-tL}(L^\sigma f)(x)\;\frac{ye^{-\frac{y^2}{4t}}}{2t}\;\frac{dt}{t^{1-\sigma}}.\end{aligned}\tag{3.8}$$

Take y from the interval (y_1, y_2) and fix $0<h_0<y_1$. For all h such that $0<|h|\leqslant h_0$, by (3.7),

$$\left\langle \frac{u(\cdot,y+h)-u(\cdot,y)}{h}, g(\cdot)\right\rangle_{L^2(\Omega)} = \frac{1}{\Gamma(\sigma)}\int_0^\infty \left\langle e^{-tL}(L^\sigma f), g\right\rangle_{L^2(\Omega)} \frac{e^{-\frac{(y+h)^2}{4t}} - e^{-\frac{y^2}{4t}}}{h}\;\frac{dt}{t^{1-\sigma}}.$$

Applying the Mean Value Theorem to $e^{-\frac{(y+h)^2}{4t}}$ as a function of h, we get

$$\left|\frac{e^{-\frac{(y+h)^2}{4t}} - e^{-\frac{y^2}{4t}}}{h}\right| \leqslant C_{y_1,y_2,h_0}\frac{1}{1+t}.$$

Hence, by dominated convergence and Bochner's Theorem,

$$\begin{aligned}\lim_{h\to 0}\left\langle \frac{u(\cdot,y+h)-u(\cdot)}{h}, g(\cdot)\right\rangle_{L^2(\Omega)} &= \frac{1}{\Gamma(\sigma)}\int_0^\infty \langle e^{-tL}(L^\sigma f), g\rangle_{L^2(\Omega)}\,\partial_y(e^{-\frac{y^2}{4t}})\,\frac{dt}{t^{1-\sigma}}\\ &= \left\langle \frac{1}{\Gamma(\sigma)}\int_0^\infty e^{-tL}(L^\sigma f)\partial_y(e^{-\frac{y^2}{4t}})\,\frac{dt}{t^{1-\sigma}}, g\right\rangle_{L^2(\Omega)}.\end{aligned}$$

5. The function u verifies the extension equation (3.2). Observe that the integral defining u_y in (3.8) is absolutely convergent as a Bochner integral, and it can be differentiated again with respect to y. Hence,

$$\begin{aligned}&\left\langle \frac{1-2\sigma}{y}\,u_y(\cdot,y)+u_{yy}(\cdot,y), g(\cdot)\right\rangle_{L^2(\Omega)}\\ &= \frac{1}{\Gamma(\sigma)}\int_0^\infty \langle e^{-tL}L^\sigma f, g\rangle_{L^2(\Omega)}\left(\frac{\sigma-1}{t}+\frac{y^2}{4t^2}\right)e^{-\frac{y^2}{4t}}\,\frac{dt}{t^{1-\sigma}}\\ &= -\frac{1}{\Gamma(\sigma)}\int_0^\infty \partial_t\Big[\langle e^{-tL}L^\sigma f, g\rangle\Big]e^{-\frac{y^2}{4t}}\,\frac{dt}{t^{1-\sigma}}\\ &= -\frac{1}{\Gamma(\sigma)}\int_0^\infty \partial_t\left[\int_0^\infty e^{-t\lambda}\lambda^\sigma\,dE_{f,g}(\lambda)\right]e^{-\frac{y^2}{4t}}\,\frac{dt}{t^{1-\sigma}}\\ &= \frac{1}{\Gamma(\sigma)}\int_0^\infty \lambda\int_0^\infty e^{-t\lambda}\lambda^\sigma e^{-\frac{y^2}{4t}}\,\frac{dt}{t^{1-\sigma}}\,dE_{f,g}(\lambda)\\ &= \left\langle L\int_0^\infty e^{-tL}(L^\sigma f)e^{-\frac{y^2}{4t}}\,\frac{dt}{t^{1-\sigma}}, g\right\rangle = \langle Lu(\cdot,y), g(\cdot)\rangle_{L^2(\Omega)}.\end{aligned}$$

6. Let us check (3.4). Note that, for all $g\in L^2(\Omega)$, by (3.7) and the change of variables $t=y^2/(4r)$,

$$\left\langle \frac{u(\cdot,y)-u(\cdot,0)}{y^{2\sigma}}, g(\cdot)\right\rangle_{L^2(\Omega)} = \frac{1}{4^\sigma\Gamma(\sigma)}\int_0^\infty \left\langle e^{-\frac{y^2}{4r}L}L^\sigma f, g\right\rangle_{L^2(\Omega)}\left(\frac{e^{-r}-1}{r^\sigma}\right)\frac{dr}{r},$$

therefore, since $\lim_{t\to 0^+}\langle e^{-tL}L^\sigma f, g\rangle_{L^2(\Omega)} = \langle L^\sigma f, g\rangle_{L^2(\Omega)}$, by dominated convergence, we obtain the first identity in (3.4). Using (3.8) and the same change of variables, the second equality of (3.4) also follows because

$$\begin{aligned}\frac{1}{2\sigma}\langle y^{1-2\sigma}u_y(\cdot,y), g(\cdot)\rangle_{L^2(\Omega)} &= \frac{-1}{4^\sigma\sigma\Gamma(\sigma)}\int_0^\infty \langle e^{-tL}L^\sigma f, g\rangle_{L^2(\Omega)}\,e^{-\frac{y^2}{4t}}\left(\frac{y^2}{4t}\right)^{1-\sigma}\frac{dt}{t}\\ &= \frac{-1}{4^\sigma\sigma\Gamma(\sigma)}\int_0^\infty \left\langle e^{-\frac{y^2}{4r}L}L^\sigma f, g\right\rangle_{L^2(\Omega)} e^{-r}r^{1-\sigma}\frac{dr}{r},\end{aligned}$$

implies that

$$\frac{1}{2\sigma}\lim_{y\to 0^+}\langle y^{1-2\sigma}u_y(\cdot,y), g(\cdot)\rangle_{L^2(\Omega)} = \frac{-\Gamma(1-\sigma)}{4^\sigma\sigma\Gamma(\sigma)}\langle L^\sigma f, g\rangle_{L^2(\Omega)} = \frac{\Gamma(-\sigma)}{4^\sigma\Gamma(\sigma)}\langle L^\sigma f, g\rangle_{L^2(\Omega)}.$$

7. Let us derive the Poisson formula (3.5). By (3.7), (2.37), Fubini's Theorem and the change of variables $t = y^2/(4r\lambda)$, we get

$$\begin{aligned}
\langle u(\cdot, y), g(\cdot)\rangle_{L^2(\Omega)} &= \frac{1}{\Gamma(\sigma)} \int_0^\infty \int_0^\infty e^{-t\lambda}(t\lambda)^\sigma e^{-\frac{y^2}{4t}} \frac{dt}{t}\, dE_{f,g}(\lambda) \\
&= \frac{1}{\Gamma(\sigma)} \int_0^\infty \int_0^\infty e^{-\frac{y^2}{4r}} \left(\frac{y^2}{4r}\right)^\sigma e^{-r\lambda} \frac{dr}{r}\, dE_{f,g}(\lambda) \\
&= \frac{y^{2\sigma}}{4^\sigma \Gamma(\sigma)} \int_0^\infty \langle e^{-tL} f, g\rangle_{L^2(\Omega)}\, e^{-\frac{y^2}{4r}} \frac{dr}{r^{1+\sigma}} \\
&= \left\langle \frac{y^{2\sigma}}{4^\sigma \Gamma(\sigma)} \int_0^\infty e^{-tL} f\, e^{-\frac{y^2}{4r}} \frac{dr}{r^{1+\sigma}}, g \right\rangle_{L^2(\Omega)}.
\end{aligned}$$

The last equality is due to Bochner's Theorem.

The second identity of (3.5) follows from the first one via the change of variables $r = y^2/(4t)$. □

3.2 Poisson formula, fundamental solution and Cauchy-Riemann equations

In what follows we assume that the heat-diffusion semigroup generated by L, that is defined in a spectral way, is given by integration against a nonnegative heat kernel $K_t(x, z)$, that is, for $f \in L^2(\Omega)$,

$$e^{-tL} f(x) = \int_\Omega K_t(x, z) f(z)\, d\eta(z), \qquad t > 0.$$

Since e^{-tL} is self-adjoint, $K_t(x, z) = K_t(z, x)$. The second assumption we make is that the heat kernel belongs to the domain of L and $\partial_t K_t(x, z) = LK_t(x, z)$, the derivative with respect to t is understood in the classical sense. This implies that

$$\partial_t \int_\Omega K_t(x, z) f(z)\, d\eta(z) = \int_\Omega \partial_t K_t(x, z) f(z)\, d\eta(z), \qquad f \in L^2(\Omega).$$

Motivated by concrete examples, we add the hypotheses that given x there exists a constant C_x and $\varepsilon > 0$ such that

$$\|K_t(x, \cdot)\|_{L^2(\Omega)} + \|\partial_t K_t(x, \cdot)\|_{L^2(\Omega)} \leqslant C_x (1 + t^\varepsilon) t^{-\varepsilon}.$$

Theorem 3.2 (Poisson formula). *Denote by $\mathcal{P}_y^\sigma f(x)$ the function $u(x, y)$ given in* (3.5). *Then:*

(1) We have

$$\mathcal{P}_y^\sigma f(x) = \int_\Omega P_y^\sigma(x, z) f(z)\, d\eta(z),$$

where the Poisson kernel

$$P_y^\sigma(x,z) := \frac{y^{2\sigma}}{4^\sigma\Gamma(\sigma)}\int_0^\infty K_t(x,z)e^{-\frac{y^2}{4t}}\,\frac{dt}{t^{1+\sigma}}, \tag{3.9}$$

is, for each fixed $z \in \Omega$, *an* $L^2(\Omega)$*-function that verifies* (3.2)*.*

(2) $\sup_{y\geqslant 0}\left|\mathcal{P}_y^\sigma f\right| \leqslant \sup_{t\geqslant 0}\left|e^{-tL}f\right|$*, in* Ω*.*

(3) If e^{-tL} *has the contraction property in* $L^p(\Omega)$ *then* $\left\|\mathcal{P}_y^\sigma f\right\|_{L^p(\Omega)} \leqslant \|f\|_{L^p(\Omega)}$*, for all* $y \geqslant 0$*.*

(4) If $\lim_{t\to 0^+} e^{-tL}f = f$ *in* $L^p(\Omega)$ *then* $\lim_{y\to 0^+}\mathcal{P}_y^\sigma f = f$ *in* $L^p(\Omega)$*.*

Proof. The integral formula in *(1)* can be verified by using (3.5), Bochner's and Fubini's Theorems:

$$\begin{aligned}\left\langle\mathcal{P}_y^\sigma f, g\right\rangle &= \frac{y^{2\sigma}}{4^\sigma\Gamma(\sigma)}\int_0^\infty \left\langle e^{-tL}f, g\right\rangle_{L^2(\Omega)} e^{-\frac{y^2}{4t}}\,\frac{dt}{t^{1+\sigma}} \\ &= \frac{y^{2\sigma}}{4^\sigma\Gamma(\sigma)}\int_0^\infty\int_\Omega\int_\Omega K_t(x,z)f(z)g(x)\,d\eta(z)\,d\eta(x)\,e^{-\frac{y^2}{4t}}\,\frac{dt}{t^{1+\sigma}} \\ &= \int_\Omega\int_\Omega\left[\frac{y^{2\sigma}}{4^\sigma\Gamma(\sigma)}\int_0^\infty K_t(x,z)e^{-\frac{y^2}{4t}}\,\frac{dt}{t^{1+\sigma}}\right]f(z)\,d\eta(z)\,g(x)\,d\eta(x).\end{aligned}$$

The last identity above is justified by observing that the triple integral of the modulus of the integrand is bounded by $C_{\sigma,y}\,\|f\|_{L^2(\Omega)}\,\|g\|_{L^2(\Omega)}$.

In order to see that the Poisson kernel satisfies (3.2) we begin by showing that it belongs to the domain of L. By the assumptions established on the L^2-norm of the heat kernel, $P_y^\sigma(\cdot,z) \in L^2(\Omega)$, for each z, and, by Bochner's Theorem,

$$e^{-sL}P_y^\sigma(\cdot,z) = \frac{y^{2\sigma}}{4^\sigma\Gamma(\sigma)}\int_0^\infty e^{-sL}K_t(x,z)e^{-\frac{y^2}{4t}}\,\frac{dt}{t^{1+\sigma}}, \qquad s \geqslant 0.$$

With this,

$$\frac{e^{-sL}P_y^\sigma(\cdot,z) - P_y^\sigma(\cdot,z)}{s} = \frac{y^{2\sigma}}{4^\sigma\Gamma(\sigma)}\int_0^\infty \frac{e^{-sL}K_t(\cdot,z) - K_t(\cdot,z)}{s}\,e^{-\frac{y^2}{4r}}\,\frac{dr}{r^{1+\sigma}}. \tag{3.10}$$

Using the Mean Value Theorem, the fact that $K_t(\cdot,z) \in \mathrm{Dom}(L)$, and the contraction property of e^{-sL}, we get

$$\begin{aligned}\left\|\frac{e^{-sL}K_t(\cdot,z) - K_t(\cdot,z)}{s}\right\|_{L^2(\Omega)} &= \left\|Le^{-\theta L}K_t(\cdot,z)\right\|_{L^2(\Omega)} = \left\|e^{-\theta L}LK_t(\cdot,z)\right\|_{L^2(\Omega)} \\ &\leqslant \|LK_t(\cdot,z)\|_{L^2(\Omega)} = \|\partial_t K_t(\cdot,z)\|_{L^2(\Omega)} \leqslant C_z(1+t^\varepsilon)t^{-\varepsilon}.\end{aligned}$$

Hence, the Dominated Convergence Theorem (for Bochner integrals) can be applied in (3.10) to see that the limit as $s \to 0^+$ of both sides exists and

$$-L_x P_y^\sigma(x,z) = \frac{y^{2\sigma}}{4^\sigma\Gamma(\sigma)}\int_0^\infty \partial_t\left(K_t(x,z)\right)e^{-\frac{y^2}{4t}}\,\frac{dt}{t^{1+\sigma}}.$$

Now we are in position to check that $P_y^\sigma(x,z)$ verifies (3.2). Note that, by dominated convergence, the derivatives with respect to y of $P_y^\sigma(x,z)$ exist and can be computed by differentiation inside the integral sign in (3.9). Then, using integration by parts,

$$\begin{aligned}\frac{1-2\sigma}{y}\,\partial_y P_y^\sigma(x,z)+\partial_{yy}P_y^\sigma(x,z) &= \frac{y^{2\sigma}}{4^\sigma\Gamma(\sigma)}\int_0^\infty K_t(x,z)e^{-\frac{y^2}{4t}}\left(\frac{y^2}{4t^2}-\frac{1+\sigma}{t}\right)\frac{dt}{t^{1+\sigma}}\\ &= -\frac{y^{2\sigma}}{4^\sigma\Gamma(\sigma)}\int_0^\infty \partial_t(K_t(x,z))e^{-\frac{y^2}{4t}}\,\frac{dt}{t^{1+\sigma}} = L_x P_y^\sigma(x,z),\end{aligned}$$

thus *(1)* is proved. *(2)* follows from the second identity of (3.5). The contraction property of the heat semigroup gives *(3)*:

$$\left\|\mathcal{P}_y^\sigma f\right\|_{L^p(\Omega)} \leqslant \frac{1}{\Gamma(\sigma)}\int_0^\infty \left\|e^{-\frac{y^2}{4r}L}f\right\|_{L^p(\Omega)}e^{-r}\,\frac{dr}{r^{1-\sigma}} \leqslant \|f\|_{L^p(\Omega)}.$$

Observe that

$$\left\|\mathcal{P}_y^\sigma f-f\right\|_{L^p(\Omega)} \leqslant \frac{1}{\Gamma(\sigma)}\int_0^\infty \left\|e^{-\frac{y^2}{4t}L}f-f\right\|_{L^p(\Omega)}e^{-r}\,\frac{dr}{r^{1-\sigma}},$$

so *(4)* follows. □

Remark 3.3. Note in (3.5) that, when $\sigma = 1/2$, $\mathcal{P}_y^{1/2}f = e^{-y\sqrt{L}}f$ is the Poisson semigroup generated by L acting on f, see Chapter 2.

Proposition 3.4 (Fundamental solution of (3.2)). *The function*

$$\Psi_x^\sigma(z,y) = \frac{1}{\Gamma(\sigma)}\int_0^\infty K_t(x,z)e^{-\frac{y^2}{4t}}\,\frac{dt}{t^{1-\sigma}}, \tag{3.11}$$

satisfies equation (3.2), $\Psi_x^\sigma(z,y) = \Psi_z^\sigma(x,y)$, *and*

$$\lim_{y\to 0^+}\left\langle \frac{1}{2\sigma}\,y^{1-2\sigma}\partial_y\Psi_x^\sigma(\cdot,y),f(\cdot)\right\rangle_{L^2(\Omega)} = \frac{\Gamma(-\sigma)}{4^\sigma\Gamma(\sigma)}f(x). \tag{3.12}$$

Proof. As in the proof of Theorem 3.1, it can be checked that for each x,

$$\begin{aligned}\lim_{R\to\infty}\left\langle\frac{1}{\Gamma(\sigma)}\int_0^R K_t(x,\cdot)e^{-\frac{y^2}{4t}}\,\frac{dt}{t^{1-\sigma}},g(\cdot)\right\rangle_{L^2(\Omega)} &= \langle\Psi_x^\sigma(\cdot,y),g(\cdot)\rangle_{L^2(\Omega)}\\ &= \frac{1}{\Gamma(\sigma)}\int_0^\infty e^{-tL}g(x)e^{-\frac{y^2}{4t}}\,\frac{dt}{t^{1-\sigma}},\end{aligned}$$

and that (3.11) satisfies (3.2). Differentiation with respect to y inside the integral in (3.11) can be performed to get

$$\begin{aligned}\frac{y^{1-2\sigma}}{2\sigma}\,\partial_y\Psi_x^\sigma(z,y) &= \frac{-1}{4^\sigma\sigma\Gamma(\sigma)}\int_0^\infty K_t(x,z)e^{-\frac{y^2}{4t}}\left(\frac{y^2}{4t}\right)^{1-\sigma}\frac{dt}{t}\\ &= \frac{-1}{4^\sigma\sigma\Gamma(\sigma)}\int_0^\infty K_{\frac{y^2}{4r}}(x,z)e^{-r}\,\frac{dr}{r^\sigma}.\end{aligned}$$

Therefore we obtain (3.12):

$$\begin{aligned}\frac{y^{1-2\sigma}}{2\sigma}\int_\Omega \partial_y\Psi_x^\sigma(z,y)f(z)\;d\eta(z) &= \frac{-1}{4^\sigma\sigma\Gamma(\sigma)}\int_0^\infty e^{-\frac{y^2}{4r}}f(x)e^{-r}\,\frac{dr}{r^\sigma}\\ &\to \frac{-\Gamma(1-\sigma)}{4^\sigma\sigma\Gamma(\sigma)}f(x) = \frac{\Gamma(-\sigma)}{4^\sigma\Gamma(\sigma)}f(x), \qquad y\to 0^+.\end{aligned}$$

□

Remark 3.5. It can also be proved that

$$\lim_{y\to 0^+}\left\langle\frac{\Psi_x^\sigma(\cdot,y)-\Psi_x^\sigma(\cdot,0)}{y^{2\sigma}},f(\cdot)\right\rangle_{L^2(\Omega)} = \frac{\Gamma(-\sigma)}{4^\sigma\Gamma(\sigma)}f(x).$$

Proposition 3.6. *Let* $v(x,y) := y^{1-2\sigma}u_y(x,y)$*, where* u *solves* (3.2)*. Then* v *is a solution of the following "conjugate equation"*

$$-Lv - \frac{1-2\sigma}{y}\,v_y + v_{yy} = 0, \qquad \text{in } \Omega\times(0,\infty). \tag{3.13}$$

Proof. The computation is analogous to the one given in [23], with the obvious modifications:

$$\begin{aligned}-Lv - \frac{1-2\sigma}{y}\,v_y + v_{yy} &= -y^{1-2\sigma}Lu_y - \frac{1-2\sigma}{y}\left((1-2\sigma)y^{-2\sigma}u_y + y^{1-2\sigma}u_{yy}\right)\\ &\quad - 2\sigma(1-2\sigma)y^{-2\sigma-1}u_y + 2(1-2\sigma)y^{-2\sigma}u_{yy} + y^{1-2\sigma}u_{yyy}\\ &= y^{1-2\sigma}\left(-Lu_y - \frac{(1-2\sigma)^2}{y^2}u_y - \frac{1-2\sigma}{y}u_{yy} - \frac{2\sigma(1-2\sigma)}{y^2}u_y + \frac{2(1-2\sigma)}{y}u_{yy} + u_{yyy}\right)\\ &= y^{1-2\sigma}\left((-Lu)_y - \frac{1-2\sigma}{y^2}u_y + \frac{1-2\sigma}{y}u_{yy} + u_{yyy}\right)\\ &= y^{1-2\sigma}\partial_y\left(-Lu + \frac{1-2\sigma}{y}u_y + u_{yy}\right) = 0.\end{aligned}$$

□

Remark 3.7. As in [23] the fundamental solution (3.11) and the "conjugate equation" (3.13) (which coincides with the conjugate equation given in [23] when $L = -\Delta$) can help us to find the Poisson kernel (3.9). Indeed, we want to write

$$u(x,y) = \mathcal{P}_y^\sigma f(x) = \int_\Omega P_y^\sigma(x,z)f(z)\;d\eta(z),$$

where the Poisson kernel $P_y^\sigma(x,z)$ must be a solution of (3.2) for all z and $\lim_{y\to 0^+} \mathcal{P}_y^\sigma f(x) = f(x)$. The right choice would be

$$P_y^\sigma(x,z) = \frac{4^{1-\sigma}\Gamma(1-\sigma)}{2(1-\sigma)\Gamma(-(1-\sigma))}\, y^{1-2(1-\sigma)}\partial_y \Psi_x^{1-\sigma}(z,y) = C_{1-\sigma} y^{1-2(1-\sigma)}\partial_y \Psi_x^{1-\sigma}(z,y), \tag{3.14}$$

since it solves the "conjugate equation" (3.13) with $1-\sigma$ in the place of σ (thus it verifies (3.2)) and by (3.12) and the choice of $C_{1-\sigma}$,

$$\lim_{y\to 0^+} C_{1-\sigma}\int_\Omega y^{1-2(1-\sigma)}\partial_y \Psi_x^{1-\sigma}(z,y) f(z)\ d\eta(z) = f(x).$$

A simple calculation shows that (3.14) coincides with (3.9).

For the following discussion we shall assume that the operator L can be factorized as $L = D_i^* D_i$, where $D_i = a_i(x_i)\partial_{x_i} + b_i(x_i)$, is a one dimensional (in the ith direction) partial differential operator and D_i^* is the formal adjoint (with respect to $d\eta$) of D_i. In this case we give a definition of n conjugate functions related to the Poisson formula for u.

Let $E_\sigma := -L + \frac{1-2\sigma}{y}\,\partial_y + \partial_{yy}$. Then the factorization

$$E_\sigma = -\sum_{i=1}^n D_i^* D_i + y^{-(1-2\sigma)}\partial_y(y^{1-2\sigma}\partial_y),$$

suggests the following definition of Cauchy-Riemann equations for a system of functions $u, v_1, \ldots, v_n : \Omega\times(0,\infty)\to\mathbb{R}$ such that $E_\sigma u = 0$:

$$\begin{cases} y^{1-2\sigma}\partial_y u = D_1^* v_1 + \cdots + D_n^* v_n, & \\ D_i u = y^{-(1-2\sigma)}\partial_y v_i, & i = 1,\ldots,n, \\ D_k v_i = D_i v_k, & i,k = 1,\ldots,n. \end{cases} \tag{3.15}$$

Proposition 3.8. *Let* u *be a solution of* $E_\sigma u = 0$ *in* $\Omega\times(0,\infty)$. *If* $v_1,\ldots,v_n$ *verify* (3.15) *then each* v_i *solves the* ith conjugate equation

$$E_{1-\sigma}^i v_i = -Lv_i + [D_i^*, D_i]v_i - \frac{1-2\sigma}{y}\,\partial_y v_i + \partial_{yy} v_i = 0, \qquad i = 1,\ldots,n, \tag{3.16}$$

where $[D_i^*, D_i] = D_i^* D_i - D_i D_i^*$.

Proof.

$$\begin{aligned} -Lv_i + [D_i^*, D_i]v_i &= -\sum_{k\neq i} D_k^* D_k v_i - D_i D_i^* v_i \\ &= -\sum_{k\neq i} D_k^* D_i v_k - D_i D_i^* v_i = -D_i\left(\sum_{k=1}^n D_k^* v_k\right) \\ &= -D_i\left(y^{1-2\sigma}\partial_y u\right) = -y^{1-2\sigma}\partial_y(D_i u) \\ &= -y^{1-2\sigma}\partial_y(y^{-(1-2\sigma)}\partial_y v_i) = \frac{1-2\sigma}{y}\,\partial_y v_i - \partial_{yy} v_i. \end{aligned}$$

□

Remark 3.9. The ith conjugate equation (3.16) is not the same as the "conjugate equation" (3.13). They will coincide only when $[D_i^*, D_i] = 0$. This is the case if $L = -\Delta$: the conjugate equation established in [23] is equal to each ith conjugate equation (3.16).

Proposition 3.10. *Fix $z \in \Omega$ and choose $u(x,y) = P_y^\sigma(x,z)$. Then a solution to* (3.15) *is given by the* n conjugate Poisson kernels *defined by*

$$\nu_i(x,y) = Q_y^{\sigma,i}(x,z) := \frac{-2}{4^\sigma\Gamma(\sigma)} D_i \int_0^\infty K_t(x,z) e^{-\frac{y^2}{4t}} \frac{dt}{t^\sigma}, \qquad i = 1, \ldots, n. \tag{3.17}$$

Proof. From (3.14) and the second equation of (3.15),

$$C_{1-\sigma} D_i \partial_y \Psi_x^{1-\sigma}(z,y) = \partial_y Q_y^{\sigma,i}(x,z),$$

so in view of (3.11), $Q_y^{\sigma,i}(x,z)$ can be chosen as in (3.17). Clearly

$$D_k Q_y^{\sigma,i}(x,z) = D_i Q_y^{\sigma,k}(x,z).$$

Moreover, the first equation of (3.15) holds:

$$\begin{aligned}
D_1^* Q_y^{\sigma,1}(x,z) + \cdots + D_n^* Q_y^{\sigma,n}(x,z) &= \frac{-2}{4^\sigma\Gamma(\sigma)} \sum_{i=1}^n D_i^* D_i \int_0^\infty K_t(x,z) e^{-\frac{y^2}{4t}} \frac{dt}{t^\sigma} \\
&= \frac{-2}{4^\sigma\Gamma(\sigma)} \int_0^\infty L K_t(x,z) e^{-\frac{y^2}{4t}} \frac{dt}{t^\sigma} \\
&= \frac{2}{4^\sigma\Gamma(\sigma)} \int_0^\infty \partial_t K_t(x,z) e^{-\frac{y^2}{4t}} \frac{dt}{t^\sigma} \\
&= \frac{-2}{4^\sigma\Gamma(\sigma)} \int_0^\infty K_t(x,z) e^{-\frac{y^2}{4t}} \left(\frac{y^2}{4t} - \sigma\right) \frac{dt}{t^{1+\sigma}} \\
&= y^{1-2\sigma} \partial_y P_y^\sigma(x,z).
\end{aligned}$$

□

Corollary 3.11. *The* Poisson integral of f, $u(x,y) = \mathcal{P}_y^\sigma f(x)$, *and the* n conjugate Poisson integrals *of* f *defined by*

$$\nu_i(x,y) \equiv \mathcal{Q}_y^{\sigma,i} f(x) := \int_\Omega Q_y^{\sigma,i}(x,z) f(z)\, d\eta(z) = \frac{-2}{4^\sigma\Gamma(\sigma)} D_i \int_0^\infty e^{-tL} f(x) e^{-\frac{y^2}{4t}} \frac{dt}{t^\sigma}, \tag{3.18}$$

for $i = 1, \ldots, n$, *solve* (3.15).

Remark 3.12. When $\sigma = 1/2$, $\mathcal{Q}_y^{1/2,i} f(x)$ is the ith conjugate function of f associated to L, see [74, 82, 83, 78]. A natural question arises: what is the limit of $\mathcal{Q}_y^{\sigma,i} f(x)$ as $y \to 0^+$? The answer is contained in the next result.

Theorem 3.13. *For each $x \in \Omega$,*

$$\lim_{y\to 0^+} \mathcal{Q}_y^{\sigma,i} f(x) = \frac{-2\Gamma(1-\sigma)}{4^\sigma\Gamma(\sigma)} D_i L^{-(1-\sigma)} f(x).$$

Proof. From the expression of $\mathcal{Q}_y^{\sigma,i}f(x)$ in (3.18) and (2.5),

$$\begin{aligned}\lim_{y\to 0^+}\mathcal{Q}_y^{\sigma,i}f(x) &= \frac{-2\Gamma(1-\sigma)}{4^\sigma\Gamma(\sigma)}D_i\frac{1}{\Gamma(1-\sigma)}\int_0^\infty e^{-tL}f(x)\frac{dt}{t^{1-(1-\sigma)}}\\ &= \frac{-2\Gamma(1-\sigma)}{4^\sigma\Gamma(\sigma)}D_iL^{-(1-\sigma)}f(x).\end{aligned}$$

□

Remark 3.14. The conclusion of Theorem 3.13 can also be obtained from the following observation: except for a multiplicative constant, the last formula of (3.18) is just the D_i-derivative of the solution of the extension problem (3.2) for $L^{1-\sigma}$ with boundary value $L^{-(1-\sigma)}f(x)$, see (3.3). For $\sigma = 1/2$, Theorem 3.13 establishes the boundary convergence to the Riesz transforms $D_iL^{-1/2}$ (which in case $L = -\Delta$ are the classical Riesz transforms $\partial_{x_i}(-\Delta)^{-1/2}$). See [74, 72] and [82, 83, 78].

Examples 3.15. *We present some examples of operators* L *for which our results apply.*

The Laplacian in $\mathbb{R}^n$ *Recall that the heat semigroup* $e^{t\Delta}f(x)$ *is given by convolution with the Gauss-Weierstrass kernel* (2.12). *We can recover the Poisson formula given in [23]: use the change of variables* $\frac{|x-z|^2+y^2}{4t} = r$ *in* (3.9) *to see that the Poisson kernel in this case is*

$$P_y^{\sigma,-\Delta}(x,z) = \frac{y^{2\sigma}}{4^\sigma\Gamma(\sigma)}\int_0^\infty\frac{e^{-\frac{|x-z|^2+y^2}{4t}}}{(4\pi t)^{n/2}}\frac{dt}{t^{1+\sigma}} = \frac{\Gamma(n/2+\sigma)}{\pi^{n/2}\Gamma(\sigma)}\cdot\frac{y^{2\sigma}}{\left(|x-z|^2+y^2\right)^{\frac{n+2\sigma}{2}}}.$$

The function $P_y^{1/2,-\Delta}(x,z)$ *coincides with the Poisson kernel for the harmonic extension of a function* f *to the upper half space, see Chapter 2. The Cauchy-Riemann equations read*

$$\begin{cases} y^{1-2\sigma}\partial_y u = -\left(\partial_{x_1}v_1+\cdots+\partial_{x_n}v_n\right), & \\ \partial_{x_i}u = y^{-(1-2\sigma)}\partial_y v_i, & i=1,\ldots,n,\\ \partial_{x_k}v_i = \partial_{x_i}v_k, & i,k=1,\ldots,n.\end{cases}\tag{3.19}$$

The case $\sigma = 1/2$ *is the classical Cauchy-Riemann system for the* n *conjugate harmonic functions to* u, *see [72, 75]. In dimension one* (3.19) *reduces to*

$$\begin{cases} y^{1-2\sigma}\partial_y u = -\partial_x v,\\ \partial_x u = y^{-(1-2\sigma)}\partial_y v,\end{cases}$$

which already appeared in a paper by B. Muckenhoupt and E. M. Stein [59].

Classical expansions L *can be each one of the operators arising in orthogonal expansions, like the Ornstein-Uhlenbeck operator (Hermite polynomials arise under the Gaussian measure* $d\eta(x) = e^{-|x|^2}\, dx$*, see also [69]),*

$$-\Delta + 2x\cdot\nabla = \sum_i \left(-\partial_{x_i} + 2x_i\right)\left(\partial_{x_i}\right);$$

the harmonic oscillator (Hermite functions and Lebesgue measure $d\eta(x) = dx$*),*

$$-\Delta + |x|^2 = \frac{1}{2}\sum_i \left[\left(-\partial_{x_i} + x_i\right)\left(\partial_{x_i} + x_i\right) + \left(\partial_{x_i} + x_i\right)\left(-\partial_{x_i} + x_i\right) \right];$$

the Laguerre operator (Laguerre polynomials on the cartesian product of n *half lines* $(0,\infty)^n$ *and measure* $d\eta(x) = \prod_i x_i^{\alpha_i} e^{-x_i}\, dx$*),*

$$\sum_i x_i \partial^2_{x_i,x_i} + (\alpha_i + 1 - x_i)\partial_{x_i} = \sum_i \sqrt{x_i}\left(\partial_{x_i} + \left(\frac{\alpha_i + 1/2}{x_i} - 1\right)\right)\sqrt{x_i}\ \partial_{x_i};$$

Jacobi and ultraspherical on $(-1,1)$*; etc. For classical orthogonal expansions see [81] and [55].*

Elliptic operators *Let* L *be a positive self-adjoint linear elliptic partial differential operator on* $L^2(\Omega)$ *with Dirichlet boundary condition and bounded measurable coefficients. Then its associated heat kernel exists and it verifies our assumptions stated at the beginning of this section. Moreover, such a heat kernel has Gaussian bounds [30, p. 89]. We can also consider Schrödinger operators with nonnegative potentials in a large class [30, Section 4.5].*

3.3 Existence and uniqueness results for the extension problem

In this section we derive the solution of the extension problem in the case of discrete spectrum. We also find solutions with zero Neumann-type condition. This is done in an elementary way: using Fourier's method.

The same kind of computations can be carried out if $L = -\Delta$ in $\mathbb{R}^n$, by using the Fourier transform.

3.3.1 L^2 theory

Recall the definition of L^σ in $L^2(\Omega)$ given in (3.6). Let $f \in L^2(\Omega)$ and look for solutions u to (3.1)-(3.2) of the form

$$u(x,y) = \sum_k c_k(y)\phi_k(x). \tag{3.20}$$

Then for each $k \in \mathbb{N}_0$ we have to solve the following ordinary differential equation:

$$\begin{cases} -\lambda_k c_k + \frac{1-2\sigma}{y}\, c_k' + c_k'' = 0, & \text{for } y > 0, \\ c_k(0) = \langle f, \phi_k \rangle. \end{cases}$$

According to the book by N. N. Lebedev [55, p. 106] (the reader may also consult [34, 87]), this last equation has a general solution of the form

$$c_k(y) = y^\sigma Z_\sigma(\pm i\lambda_k^{1/2} y), \tag{3.21}$$

where Z_σ is a linear combination of Bessel functions of order σ. To have uniqueness of the solution we include the boundary condition $\lim_{y\to\infty} u(x,y) = 0$ weakly in $L^2(\Omega)$, which translates to the coefficients as

$$\lim_{y\to\infty} c_k(y) = 0. \tag{3.22}$$

From [55, p. 104] we see that Z_σ can be written as

$$Z_\sigma(z) = A_1 J_\sigma(z) + A_2 H_\sigma^{(1)}(z) = B_1 J_\sigma(z) + B_2 H_\sigma^{(2)}(z) \tag{3.23}$$

$$= C_1 J_\sigma(z) + C_2 J_{-\sigma}(z) = D_1 H_\sigma^{(1)}(z) + D_2 H_\sigma^{(2)}(z), \tag{3.24}$$

where $J_{\pm\sigma}$ denotes the Bessel function of the first kind and $H_\sigma^{(1)}$ and $H_\sigma^{(2)}$ are the Hankel functions. To fulfill condition (3.22) we need to review the asymptotic behavior of the Bessel functions. When $|\arg z| \leqslant \pi - \delta$,

$$\begin{aligned} J_\sigma(z) &= \left(\frac{2}{\pi z}\right)^{1/2} \left[\cos\left(z - \frac{2\sigma\pi + \pi}{4}\right)\left(1 + O(|z|^{-2})\right)\right. \\ &\qquad \left. - \sin\left(z - \frac{2\sigma\pi + \pi}{4}\right)\left(\frac{4\sigma^2 - 1}{8z} + O(|z|^{-3})\right)\right], \\ H_\sigma^{(1)}(z) &= \left(\frac{2}{\pi z}\right)^{1/2} e^{i\left(z - \frac{2\sigma\pi+\pi}{4}\right)}\left(1 + O(|z|^{-1})\right), \\ H_\sigma^{(2)}(z) &= \left(\frac{2}{\pi z}\right)^{1/2} e^{-i\left(z - \frac{2\sigma\pi+\pi}{4}\right)}\left(1 + O(|z|^{-1})\right). \end{aligned} \tag{3.25}$$

Note that for purely imaginary $z \to \infty$, $J_\sigma(z) \to \infty$ exponentially and $H_\sigma^{(1)}(z) \to 0$ or ∞ depending on the sign of the imaginary part of z. Putting $z = i\lambda_k^{1/2} y$ in (3.21) we see that the only possible choice as solution is the first linear combination of (3.23) as soon as $A_1 = 0$:

$$c_k(y) = A_{2,k} y^\sigma H_\sigma^{(1)}(i\lambda_k^{1/2} y).$$

If K_σ denotes the modified Bessel function of the third kind then $H_\sigma^{(1)}(iz) = 2\pi^{-1} i^{-\sigma-1} K_\sigma(z)$ and

$$c_k(y) = A_{2,k} y^\sigma \frac{2i^{-\sigma-1}}{\pi}\, K_\sigma(\lambda_k^{1/2} y).$$

To determine $A_{2,k}$ we use the initial condition. The asymptotic behavior of $K_\sigma(z)$ as $z \to 0$ reads

$$K_\sigma(z) \approx \Gamma(\sigma)2^{\sigma-1}\frac{1}{z^\sigma}. \tag{3.26}$$

So that, when $y \to 0$, $c_k(y) \approx A_{2,k}2^\sigma\pi^{-1}i^{-\sigma-1}\Gamma(\sigma)\lambda_k^{-\sigma/2}$. Therefore

$$A_{2,k} = \frac{\pi i^{1+\sigma}}{2^\sigma\Gamma(\sigma)}\lambda_k^{\sigma/2}\langle f, \phi_k\rangle.$$

Thus

$$c_k(y) = y^\sigma\frac{2^{1-\sigma}}{\Gamma(\sigma)}\lambda_k^{\sigma/2}\langle f, \phi_k\rangle K_\sigma(\lambda_k^{1/2}y). \tag{3.27}$$

Since as $|z| \to \infty$,

$$K_\sigma(z) \approx \left(\frac{\pi}{2z}\right)^{1/2} e^{-z}\left(1 + O(|z|^{-1})\right),$$

the series in (3.20), with c_k as in (3.27), converges in $L^2(\Omega)$ for each $y \in (0, \infty)$. Finally, (3.26) implies that (3.1) is fulfilled in the $L^2(\Omega)$ sense.

On the other hand, by using the properties of the derivatives of K_σ [55, p. 110] and (3.26), as $y \to 0$ we have

$$\begin{aligned}
\frac{1}{2\sigma}y^{1-2\sigma}c_k'(y) &= \frac{1}{2\sigma}y^{1-2\sigma}\frac{2^{1-\sigma}}{\Gamma(\sigma)}\langle f, \phi_k\rangle\frac{d}{d(\lambda_k^{1/2}y)}\left[(\lambda_k^{1/2}y)^\sigma K_\sigma(\lambda_k^{1/2}y)\right]\frac{d(\lambda_k^{1/2}y)}{dy}\\
&= \frac{1}{2\sigma}y^{1-2\sigma}\frac{2^{1-\sigma}}{\Gamma(\sigma)}\langle f, \phi_k\rangle(-1)(\lambda_k^{1/2}y)^\sigma K_{\sigma-1}(\lambda_k^{1/2}y)\lambda_k^{1/2}\\
&= \frac{2^{-\sigma}}{-\sigma\Gamma(\sigma)}\langle f, \phi_k\rangle\lambda_k^{\sigma/2}\lambda_k^{1/2}y^{1-\sigma}K_{1-\sigma}(\lambda_k^{1/2}y)\\
&\approx \frac{2^{-\sigma}}{-\sigma\Gamma(\sigma)}\langle f, \phi_k\rangle\lambda_k^{\sigma/2}\lambda_k^{1/2}y^{1-\sigma}\Gamma(1-\sigma)2^{-\sigma}\frac{1}{(\lambda_k^{1/2}y)^{1-\sigma}}\\
&= \frac{2^{-2\sigma}\Gamma(1-\sigma)}{-\sigma\Gamma(\sigma)}\lambda_k^\sigma\langle f, \phi_k\rangle = \frac{\Gamma(-\sigma)}{4^\sigma\Gamma(\sigma)}\lambda_k^\sigma\langle f, \phi_k\rangle.
\end{aligned}$$

As a consequence,

$$\frac{1}{2\sigma}\lim_{y\to0^+} y^{1-2\sigma}u_y(x, y) = \frac{\Gamma(-\sigma)}{4^\sigma\Gamma(\sigma)}\sum_k \lambda_k^\sigma\langle f, \phi_k\rangle\phi_k(x) = \frac{\Gamma(-\sigma)}{4^\sigma\Gamma(\sigma)}L^\sigma f(x),$$

the limit taken in $L^2(\Omega)$, see (3.6).

Remark 3.16. The Fourier's method also gives us the explicit formula (3.3). Indeed, applying the representation formula [55, p. 119]

$$K_\sigma(z) = \frac{z^\sigma}{2^{1+\sigma}}\int_0^\infty e^{-\frac{z^2}{4r}}e^{-r}\,\frac{dr}{r^{1+\sigma}}, \qquad |\arg z| < \frac{\pi}{4},$$

to (3.27) we get

$$\begin{aligned} u(x,y) &= \sum_k c_k(y)\phi_k(x) = \frac{y^{2\sigma}}{4^\sigma\Gamma(\sigma)}\sum_k \lambda_k^\sigma\langle f,\phi_k\rangle\phi_k(x)\int_0^\infty e^{-\frac{y^2}{4r}\lambda_k}e^{-r}\,\frac{dr}{r^{1+\sigma}} \\ &= \frac{1}{\Gamma(\sigma)}\sum_k \lambda_k^\sigma\langle f,\phi_k\rangle\phi_k(x)\int_0^\infty e^{-t\lambda_k}e^{-\frac{y^2}{4t}}\,\frac{dt}{t^{1-\sigma}} \\ &= \frac{1}{\Gamma(\sigma)}\int_0^\infty\left(\sum_k e^{-t\lambda_k}\lambda_k^\sigma\langle f,\phi_k\rangle\phi_k(x)\right)e^{-\frac{y^2}{4t}}\,\frac{dt}{t^{1-\sigma}} \\ &= \frac{1}{\Gamma(\sigma)}\int_0^\infty e^{-tL}(L^\sigma f)(x)e^{-\frac{y^2}{4t}}\,\frac{dt}{t^{1-\sigma}}. \end{aligned}$$

In particular, the unique solution of (3.1)-(3.2) with zero boundary condition at infinity is the one given by Theorem 3.1.

3.3.2 Local Neumann solutions

Let us find a solution to (3.2) such that

$$\frac{1}{2\sigma}\lim_{y\to 0^+} y^{1-2\sigma}u_y(x,y) = 0, \qquad \text{for all } x\in\Omega. \tag{3.28}$$

Writing $u(x,y) = \sum_k d_k(y)\phi_k(x)$, condition (3.28) implies that $\lim_{y\to 0^+} y^{1-2\sigma}d_k'(y) = 0$. Therefore, as (see (3.21))

$$d_k'(y) = (i\lambda_k)^{1-\sigma}\frac{d}{d(i\lambda_k^{1/2}y)}\left[(i\lambda_k^{1/2}y)^\sigma Z_\sigma(i\lambda_k^{1/2}y)\right] = i\lambda_k^{1/2}y^\sigma Z_{\sigma-1}(i\lambda_k^{1/2}y),$$

we require

$$y^{1-2\sigma}d_k'(y) = i\lambda_k^{1/2}y^{1-\sigma}Z_{\sigma-1}(i\lambda_k^{1/2}y)\to 0, \qquad y\to 0.$$

When $z\to 0$ (see [55]),

$$J_\sigma(z)\approx\frac{z^\sigma}{2^\sigma\Gamma(1+\sigma)}, \quad H_\sigma^{(1)}(z)\approx\frac{2^\sigma\Gamma(\sigma)}{i\pi}\frac{1}{z^\sigma}, \quad \text{and} \quad H_\sigma^{(2)}(z)\approx-\frac{2^\sigma\Gamma(\sigma)}{i\pi}\frac{1}{z^\sigma}.$$

Then, as $y\to 0$,

$$\begin{aligned} y^{1-\sigma}J_{\sigma-1}(i\lambda_k^{1/2}y) &\to \frac{(i\lambda_k^{1/2})^{\sigma-1}}{2^{\sigma-1}\Gamma(\sigma)}, \\ y^{1-\sigma}H_{\sigma-1}^{(1)}(i\lambda_k^{1/2}y) = y^{1-\sigma}i^{2\sigma}H_{1-\sigma}^{(1)}(i\lambda_k^{1/2}y) &\to \frac{2^{1-\sigma}\Gamma(1-\sigma)i^{2\sigma-1}}{\pi}\,(i\lambda_k^{1/2})^{1-\sigma}, \\ y^{1-\sigma}H_{\sigma-1}^{(2)}(i\lambda_k^{1/2}y) = y^{1-\sigma}i^{-2\sigma}H_{1-\sigma}^{(2)}(i\lambda_k^{1/2}y) &\to -\frac{2^{1-\sigma}\Gamma(1-\sigma)i^{-(2\sigma+1)}}{\pi}\,(i\lambda_k^{1/2})^{1-\sigma}, \end{aligned}$$

but

$$y^{1-\sigma}J_{1-\sigma}(i\lambda_k^{1/2}y) \approx \frac{(i\lambda_k^{1/2})^{1-\sigma}}{2^{1-\sigma}\Gamma(2-\sigma)}\, y^{2-2\sigma} \to 0.$$

Consequently, we choose the first linear combination in (3.24) with $C_1 = 0$. Thus

$$d_k(y) = C_{2,k}y^{\sigma}J_{-\sigma}(i\lambda_k^{1/2}y),$$

verifies $\lim_{y\to 0} y^{1-2\sigma}d_k'(y) = 0$ and u formally reads

$$u(x,y) = y^{\sigma}\sum_k C_{2,k}J_{-\sigma}(i\lambda_k^{1/2}y)\phi_k(x).$$

In order to have a convergent series (at least for small y) let us determine $C_{2,k}$: taking into account (3.25) it is enough to fix $R > 0$ and put $C_{2,k} = Ce^{-\lambda_k^{1/2}R}$.

In this way we obtained a solution u to equation (3.2) in $\Omega \times (0,R)$ that satisfies the required property (3.28).

Chapter 4

The fractional harmonic oscillator: pointwise formula and Harnack's inequality

In Section 4.1 we study the definition and basic properties of the fractional harmonic oscillator H^σ. We obtain the following pointwise formula

$$H^\sigma f(x) = \int_{\mathbb{R}^n} (f(x) - f(z))F_\sigma(x, z)\,dz + f(x)B_\sigma(x), \qquad x \in \mathbb{R}^n,$$

where the suitable kernel $F_\sigma(x, z)$ and the function $B_\sigma(x)$ are given in terms of the heat kernel for H. Some maximum and comparison principles are derived. Using the extension problem of Chapter 3 the Harnack's inequality for H^σ is proved in Section 4.2.

4.1 Pointwise formula for H^σ and some of its consequences

We suggest the reader to recall the notation and basic facts related to the harmonic oscillator H established in Chapter 2.

Let $f \in \mathcal{S}$ and $0 < \sigma < 1$. Since $\|h_\nu\|_{L^\infty(\mathbb{R}^n)} < C$ for all $\nu \in \mathbb{N}_0^n$ and (2.23) holds, the series defining the fractional harmonic oscillator

$$H^\sigma f(x) = \sum_\nu (2|\nu| + n)^\sigma \langle f, h_\nu\rangle h_\nu(x), \qquad x \in \mathbb{R}^n, \tag{4.1}$$

converges uniformly in $\mathbb{R}^n$. Note that, by using Hermite series expansions, we can check that

$$\langle H^\sigma f, g\rangle = \langle f, H^\sigma g\rangle,$$

for all $f, g \in \mathcal{S}$, and $H^1 f = Hf$, $H^0 f = f$. Moreover, $H^{-\sigma} f = (H^\sigma)^{-1} f$, $f \in \mathcal{S}$.

Lemma 4.1. *For* $f \in \mathcal{S}$,

$$H^\sigma f(x) = \frac{1}{\Gamma(-\sigma)} \int_0^\infty \left(e^{-tH} f(x) - f(x)\right) \frac{dt}{t^{1+\sigma}}, \qquad x \in \mathbb{R}^n.$$

Proof. Let $c_\nu = \langle f, h_\nu \rangle$. Because of the uniform convergence of the series of (2.22), (2.24) and (4.1) we get

$$\begin{aligned}\int_0^\infty \left(e^{-tH}f(x) - f(x)\right) \frac{dt}{t^{1+\sigma}} &= \int_0^\infty \left(\sum_\nu e^{-t(2|\nu|+n)} c_\nu h_\nu(x) - \sum_\nu c_\nu h_\nu(x)\right) \frac{dt}{t^{1+\sigma}} \\ &= \sum_\nu c_\nu h_\nu(x) \int_0^\infty \left[e^{-t(2|\nu|+n)} - 1\right] \frac{dt}{t^{1+\sigma}} \\ &= \Gamma(-\sigma) \sum_\nu (2|\nu| + n)^\sigma c_\nu h_\nu(x) = \Gamma(-\sigma) H^\sigma f(x).\end{aligned}$$

□

Next we show that the fractional harmonic oscillator is a bounded operator on $\mathcal{S}$. This fact contrasts with the case of the fractional Laplacian, that does not preserve the class $\mathcal{S}$.

Lemma 4.2. H^σ *is a continuous operator on* $\mathcal{S}$.

Proof. Recall the operators A_i and A_{-i} defined in (2.32). It is well known that

$$A_i h_\nu(x) = (2\nu_i)^{1/2} h_{\nu - e_i}(x), \quad A_{-i} h_\nu(x) = (2\nu_i + 2)^{1/2} h_{\nu + e_i}(x), \qquad x \in \mathbb{R}^n, \tag{4.2}$$

where e_i is the ith coordinate vector in $\mathbb{N}_0^n$, see [82]. Hence $H^\sigma f \in C^\infty(\mathbb{R}^n)$ and for all $k \in \mathbb{N}$,

$$A_{i_1} \cdots A_{i_k} H^\sigma f(x) = \sum_\nu (2|\nu| + n)^\sigma \langle f, h_\nu \rangle A_{i_1} \cdots A_{i_k} h_\nu(x), \qquad i_l = \pm 1,\ l = 1, \dots, k, \tag{4.3}$$

the series converging uniformly on $\mathbb{R}^n$. Since

$$\frac{A_i + A_{-i}}{2} = x_i, \quad \frac{A_i - A_{-i}}{2} = \partial_{x_i}, \qquad i = 1, \dots, n, \tag{4.4}$$

for each multi-index $\gamma, \beta \in \mathbb{N}_0^n$ we can write $x^\gamma D^\beta = x_1^{\gamma_1} \cdots x_n^{\gamma_n} \frac{\partial^{|\beta|}}{\partial x_1^{\beta_1} \cdots \partial x_n^{\beta_n}}$ as a finite linear combination of operators A_i and A_{-i}. Therefore, to check that $x^\gamma D^\beta H^\sigma f \in L^\infty(\mathbb{R}^n)$ it is enough to verify that for each $k \in \mathbb{N}$, $A_{i_1} \cdots A_{i_k} H^\sigma f \in L^\infty(\mathbb{R}^n)$, where $\{i_1, \dots, i_k\} \subset \{-1, 1\}$. The identities in (4.2) easily imply the following commutation relations for Hermite functions and thus for $f \in \mathcal{S}$:

$$\begin{cases} A_i H^\sigma f = (H + 2)^\sigma A_i f, & 1 \leqslant i \leqslant n; \\ A_i H^\sigma f = (H - 2)^\sigma A_i f, & -n \leqslant i \leqslant -1. \end{cases}$$

Here $(H \pm 2)^\sigma A_i f := \sum_\nu (2|\nu| + n \pm 2)^\sigma \langle A_i f, h_\nu \rangle h_\nu$. Hence, in (4.3),

$$A_{i_1} \cdots A_{i_k} H^\sigma f = \sum_\nu (2|\nu| + n + 2j)^\sigma \langle g, h_\nu \rangle h_\nu,$$

for some $j \in \mathbb{Z}$ and $g := A_{i_1} \cdots A_{i_k} f \in \mathcal{S}$. For $m \in \mathbb{N}$ sufficiently large, using the symmetry of H as in (2.23), we have

$$\left|\sum_\nu (2|\nu| + n + 2j)^\sigma \langle g, h_\nu \rangle h_\nu(x)\right| \leqslant \|H^m g\|_{L^2(\mathbb{R}^n)} \sum_\nu \frac{(2|\nu| + n + 2j)^\sigma}{(2|\nu| + n)^m} = C \|H^m g\|_{L^2(\mathbb{R}^n)}.$$

Therefore $x^\gamma D^\beta H^\sigma f \in L^\infty(\mathbb{R}^n)$. Moreover,

$$\begin{aligned} \left|x^\gamma D^\beta H^\sigma f(x)\right| &= \left|\sum c_{i,k} A_{i_1}\cdots A_{i_k} H^\sigma f(x)\right| \leqslant C \sum |(H+2j)^\sigma A_{i_1}\cdots A_{i_k} f(x)| \\ &\leqslant C\,(\text{seminorms in } \mathcal{S} \text{ of } (A_{i_1}\cdots A_{i_k} f)) = C\,(\text{seminorms in } \mathcal{S} \text{ of } f). \end{aligned}$$

□

Lemma 4.2 together with the symmetry of H^σ on $\mathcal{S}$ allow us to give a distributional definition of H^σ: for $u \in \mathcal{S}'$, define $H^\sigma u \in \mathcal{S}'$ through

$$\langle H^\sigma u, f\rangle := \langle u, H^\sigma f\rangle, \qquad f \in \mathcal{S}.$$

Therefore, H^σ is well defined for all functions u that are tempered distributions. In particular, u can be taken from the space L^p_N defined for $1 \leqslant p < \infty$ and $N > 0$ as

$$L^p_N := \left\{ u : \mathbb{R}^n \to \mathbb{R} : \|u\|_{L^p_N} = \left(\int_{\mathbb{R}^n} \frac{|u(z)|^p}{(1+|z|^2)^{Np}}\, dz \right)^{1/p} < \infty \right\}. \tag{4.5}$$

Using Lemma 4.1 and the formula for $e^{-tH}f(x)$ given in terms of a heat kernel (2.25) we can sketch the derivation of the pointwise formula for $H^\sigma f(x)$, for $f \in \mathcal{S}$ and $x \in \mathbb{R}^n$. Since $e^{-tH}1(x)$ is not a constant function (see (2.26)), we have

$$\begin{aligned} &\frac{1}{\Gamma(-\sigma)} \int_0^\infty \left(e^{-tH}f(x) - f(x)\right) \frac{dt}{t^{1+\sigma}} = \frac{1}{\Gamma(-\sigma)} \int_0^\infty \left(\int_{\mathbb{R}^n} G_t(x,z) f(z)\, dz - f(x) \right) \frac{dt}{t^{1+\sigma}} \\ &= \frac{1}{\Gamma(-\sigma)} \int_0^\infty \left[\int_{\mathbb{R}^n} G_t(x,z)(f(z)-f(x))\, dz + f(x) \left(\int_{\mathbb{R}^n} G_t(x,z)\, dz - 1 \right) \right] \frac{dt}{t^{1+\sigma}} \\ &= \frac{1}{\Gamma(-\sigma)} \int_0^\infty \int_{\mathbb{R}^n} G_t(x,z)(f(z)-f(x))\, dz \frac{dt}{t^{1+\sigma}} + f(x) \frac{1}{\Gamma(-\sigma)} \int_0^\infty \left(e^{-tH}1(x) - 1\right) \frac{dt}{t^{1+\sigma}} \\ &= \int_{\mathbb{R}^n} (f(x)-f(z)) F_\sigma(x,z)\, dz + f(x) B_\sigma(x), \end{aligned}$$

where we have defined the nonnegative functions

$$F_\sigma(x,z) := \frac{1}{-\Gamma(-\sigma)} \int_0^\infty G_t(x,z) \frac{dt}{t^{1+\sigma}}, \quad B_\sigma(x) := \frac{1}{\Gamma(-\sigma)} \int_0^\infty \left(e^{-tH}1(x) - 1\right) \frac{dt}{t^{1+\sigma}}. \tag{4.6}$$

The subtle point in the chain of computations above is to justify the last equality. If $0 < \sigma < 1/2$ the last integral is absolutely convergent and Fubini's Theorem can be applied directly. In the case $1/2 \leqslant \sigma < 1$ a cancelation is involved that allows to show that the integral converges as a principal value.

Theorem 4.3. *Let f be a function in L^p_N that is C^2 in some open subset $\mathcal{O} \subseteq \mathbb{R}^n$. Then $H^\sigma f$ is a continuous function in $\mathcal{O}$ and*

$$H^\sigma f(x) = S_\sigma f(x) + f(x) B_\sigma(x), \qquad x \in \mathcal{O},$$

where

$$S_\sigma f(x) = \mathrm{P.V.}\int_{\mathbb{R}^n} (f(x)-f(z))F_\sigma(x,z)\ dz, \qquad x \in \mathcal{O}. \tag{4.7}$$

Remark 4.4. From (4.7) we see that H^σ is a nonlocal integro-differential operator.

Remark 4.5. The condition $f \in C^2$ ensures that the integral in (4.7) is convergent. In general, for f being just continuous (4.7) will diverge. In fact, roughly speaking, the borderline convergent case is $f \in C^{2\sigma}$ around x, as happens for the fractional Laplacian $(-\Delta)^\sigma$. See Chapter 5.

Before giving the proof of Theorem 4.3 we establish some easy consequences.

Theorem 4.6 (Maximum principle for H^σ). *Let f be a function in L^p_N that is C^2 in an open set $\mathcal{O} \subseteq \mathbb{R}^n$. Assume that $f \geqslant 0$ and $f(x_0) = 0$ for some $x_0 \in \mathcal{O}$. Then $H^\sigma f(x_0) \leqslant 0$. Moreover, $H^\sigma f(x_0) = 0$ only when $f \equiv 0$.*

Proof. By Theorem 4.3, since $f, F_\sigma \geqslant 0$,

$$H^\sigma f(x_0) = \int_{\mathbb{R}^n} (f(x_0)-f(z))F_\sigma(x_0,z)\ dz + f(x_0)B_\sigma(x_0) = -\int_{\mathbb{R}^n} f(z)F_\sigma(x_0,z)\ dz \leqslant 0.$$

If $f(z) > 0$ in some set of positive measure, then the last inequality is strict. $\square$

Corollary 4.7 (Comparison principle for H^σ). *Let $f, g \in L^p_N \cap C^2(\mathcal{O})$ be such that $f \geqslant g$ and $f(x_0) = g(x_0)$ at some $x_0 \in \mathcal{O}$. Then $H^\sigma f(x_0) \leqslant H^\sigma g(x_0)$. Moreover, $H^\sigma f(x_0) = H^\sigma g(x_0)$ only when $f \equiv g$.*

For the proof of Theorem 4.3 we need some estimates on F_σ and B_σ. First we derive some equivalent formulas for these kernels. For $\rho \in \mathbb{R}$, S. Meda's change of parameters (2.27) produces

$$\frac{dt}{t^{1+\rho}} = d\mu_\rho(s) := \frac{ds}{(1-s^2)\left(\frac{1}{2}\log\frac{1+s}{1-s}\right)^{1+\rho}}, \qquad t \in (0,\infty),\ s \in (0,1). \tag{4.8}$$

Then, from (4.6) and (4.8) with $\rho = \sigma$,

$$F_\sigma(x,z) = \frac{1}{-\Gamma(-\sigma)}\int_0^1 G_{t(s)}(x,z)\ d\mu_\sigma(s), \quad B_\sigma(x) = \frac{1}{\Gamma(-\sigma)}\int_0^1 \left(e^{-t(s)H}1(x) - 1\right) d\mu_\sigma(s).$$

Observe in (4.8) that when $\rho \in \mathbb{R}$,

$$d\mu_\rho(s) \sim \frac{ds}{s^{1+\rho}},\ s \sim 0, \qquad d\mu_\rho(s) \sim \frac{ds}{(1-s)(-\log(1-s))^{1+\rho}},\ s \sim 1. \tag{4.9}$$

Lemma 4.8. *For all $x, z \in \mathbb{R}^n$,*

$$F_\sigma(x,z) \leqslant \frac{C}{|x-z|^{n+2\sigma}}\ e^{-\frac{|x||x-z|}{C}} e^{-\frac{|x-z|^2}{C}}, \quad \text{and} \quad B_\sigma(x) \leqslant C\left(1+|x|^{2\sigma}\right). \tag{4.10}$$

Moreover, $B_\sigma \in C^\infty(\mathbb{R}^n)$.

Proof. Estimate (2.31) gives

$$F_\sigma(x,z) \leqslant C\frac{e^{-\frac{|x||x-z|}{C}}}{|x-z|^n}\int_0^1 (1-s)^{n/2} e^{-\frac{|x-z|^2}{Cs}}\, d\mu_\sigma(s).$$

Then (4.9) implies

$$\int_0^{1/2} (1-s)^{n/2} e^{-\frac{|x-z|^2}{Cs}}\, d\mu_\sigma(s) \leqslant C\int_0^{1/2} e^{-\frac{|x-z|^2}{Cs}}\frac{ds}{s^{1+\sigma}} \leqslant C\begin{cases} \frac{1}{|x-z|^{2\sigma}}, & \text{if } |x-z|<1; \\ e^{-\frac{|x-z|^2}{C}}, & \text{if } |x-z|\geqslant 1; \end{cases}$$

and

$$\int_{1/2}^1 (1-s)^{n/2} e^{-\frac{|x-z|^2}{Cs}}\, d\mu_\sigma(s) \leqslant Ce^{-\frac{|x-z|^2}{C}}\int_{1/2}^1 \frac{ds}{(1-s)(-\log(1-s))^{1+\sigma}} = Ce^{-\frac{|x-z|^2}{C}}.$$

Thus the first inequality in (4.10) follows.

Up to the factor $1/\Gamma(-\sigma)$, by (2.29), we can write

$$B_\sigma(x) = \int_0^1 \left[\left(\frac{1-s^2}{1+s^2}\right)^{n/2} - 1\right] e^{-\frac{s}{1+s^2}|x|^2}\, d\mu_\sigma(s) + \int_0^1 \left(e^{-\frac{s}{1+s^2}|x|^2} - 1\right)\, d\mu_\sigma(s) = I + II.$$

To estimate I and II we use (4.9) and the Mean Value Theorem. For I we have

$$|I| \leqslant C\int_0^{1/2}\left|\left(\frac{1-s^2}{1+s^2}\right)^{n/2} - 1\right|\frac{ds}{s^{1+\sigma}} + \int_{1/2}^1 d\mu_\sigma(s) \leqslant C\int_0^{1/2} s^2\,\frac{ds}{s^{1+\sigma}} + C \;=\; C.$$

In II we consider two cases. Assume first that $|x|^2 \leqslant 2$. Then

$$|II| \leqslant C\int_0^{1/2}\left|e^{-\frac{s}{1+s^2}|x|^2} - 1\right|\,\frac{ds}{s^{1+\sigma}} + \int_{1/2}^1 d\mu_\sigma(s) \leqslant C\int_0^{1/2} |x|^2\, s\,\frac{ds}{s^{1+\sigma}} + C \leqslant C.$$

In the case $|x|^2 > 2$,

$$\begin{aligned} |II| &\leqslant |x|^2\int_0^{\frac{1}{|x|^2}} s\,\frac{ds}{s^{1+\sigma}} + \int_{\frac{1}{|x|^2}}^1 d\mu_\sigma(s) \leqslant |x|^2\int_0^{\frac{1}{|x|^2}} s^{-\sigma}\, ds + \int_{\frac{1}{|x|^2}}^1 \frac{ds}{(1-s)\,(-\log(1-s))^{1+\sigma}} \\ &= C\,|x|^{2\sigma} + C\left[-\log\left(1-\frac{1}{|x|^2}\right)\right]^{-\sigma} \leqslant C\,|x|^{2\sigma}, \end{aligned}$$

since $-\log(1-s) \sim s$ as $s \to 0$. Therefore the second estimate of (4.10) follows. The function B_σ is differentiable since the gradient of the integrand in its definition is bounded by

$$2\,|x|\,\frac{s}{1+s^2}\left(\frac{1-s^2}{1+s^2}\right)^{n/2} e^{-\frac{s}{1+s^2}|x|^2} \leqslant C\,|x|\,s \in L^1\left((0,1); d\mu_\sigma(s)\right),$$

thus we can differentiate inside the integral:

$$\nabla B_\sigma(x) = 2x\int_0^1 \frac{s}{1+s^2}\left(\frac{1-s^2}{1+s^2}\right)^{n/2} e^{-\frac{s}{1+s^2}|x|^2}\, d\mu_\sigma(s).$$

For higher order derivatives we can proceed similarly. □

Proof of Theorem 4.3. Take first $f \in \mathcal{S}$. By the computation preceding Theorem 4.3 and (4.8),

$$\int_0^\infty \left(e^{-tH}f(x) - f(x)\right) \frac{dt}{t^{1+\sigma}} = \int_0^1 \int_{\mathbb{R}^n} G_{t(s)}(x,z)(f(z)-f(x))\,dz\,d\mu_\sigma(s) + f(x)B_\sigma(x).$$

Due to Lemma 4.1, the integral in the left hand side above is well defined and converges absolutely. Write the double integral of the right hand side as $I_\delta + I_{\delta^c}$ with

$$I_{\delta^c} := \int_0^1 \int_{|x-z|>\delta} G_{t(s)}(x,z)(f(z)-f(x))\,dz\,d\mu_\sigma(s),$$

for some $\delta > 0$ (in this step δ is arbitrary, but we will fix it later). Estimate (4.10) implies that I_{δ^c} is absolutely convergent and $|I_{\delta^c}| \leqslant C\,\|f\|_{L^\infty(\mathbb{R}^n)}$. Pass to polar coordinates in I_δ to get

$$\begin{aligned} I_\delta &= \int_0^1 \left(\frac{1-s^2}{4\pi s}\right)^{n/2} \int_{|x-z|<\delta} e^{-\frac{1}{4}\left[s|x+z|^2+\frac{1}{s}|x-z|^2\right]}(f(z)-f(x))\,dz\,d\mu_\sigma(s) \\ &= \int_0^1 \left(\frac{1-s^2}{4\pi s}\right)^{n/2} \int_0^\delta r^{n-1}e^{-\frac{r^2}{4s}} \int_{|z'|=1} e^{-\frac{s}{4}|2x+rz'|^2}(f(x+rz')-f(x))\,dS(z')\,dr\,d\mu_\sigma(s). \end{aligned}$$

To estimate

$$I_{S^{n-1}} := \int_{|z'|=1} e^{-\frac{s}{4}|2x+rz'|^2}(f(x+rz')-f(x))\,dS(z'),$$

use the Taylor expansions of f and $\psi_s(w) := e^{-\frac{s}{4}|w|^2}$ and cancel out terms:

$$\begin{aligned} I_{S^{n-1}} &= \int_{|z'|=1} \left(e^{-\frac{s}{4}|2x|^2} + R_0\psi_s(x,rz')\right)\left(\nabla f(x)(rz') + R_1 f(x,rz')\right)\,dS(z') \\ &= \int_{|z'|=1} \left[e^{-\frac{s}{4}|2x|^2}R_1 f(x,rz') + R_0\psi_s(x,rz')\nabla f(x)(rz') + R_0\psi_s(x,rz')R_1 f(x,rz')\right] dS(z'). \end{aligned}$$

Since $|R_0\psi_s(x,rz')| \leqslant s^{1/2}r$, and $|R_1 f(x,rz')| \leqslant \left\|D^2 f\right\|_{L^\infty(B_\delta(x))} r^2$, we have $|I_{S^{n-1}}| \leqslant Cr^2$. Thus

$$\begin{aligned} |I_\delta| &\leqslant \int_0^1 \left(\frac{1-s^2}{4\pi s}\right)^{n/2} \int_0^\delta r^{n-1}e^{-\frac{r^2}{4s}}\,|I_{S^{n-1}}|\,dr\,d\mu_\sigma(s) \\ &\leqslant C\int_0^\delta r^{n+1}\int_0^1 \frac{1}{s^{n/2}}\,e^{-\frac{r^2}{4s}}\,d\mu_\sigma(s)\,dr \leqslant C\int_0^\delta r^{n+1}\frac{1}{r^{n+2\sigma}}\,dr = C\delta^{2-2\sigma}. \end{aligned}$$

Hence I_δ converges. The conclusion follows, for $f \in \mathcal{S}$, by Fubini's Theorem.

Now assume that $f \in L^p_N \cap C^2(\mathcal{O})$, for some $1 \leqslant p < \infty$ and $N > 0$. Then $H^\sigma f$ is well defined as a tempered distribution. Fix an arbitrary $x \in \mathcal{O}$ and take $\delta > 0$ so that $B_\delta(x) \subset \mathcal{O}$. Observe that the integral in (4.7) is well defined: just apply Taylor's Theorem (as above) in I_δ, and the L^p_N condition together with (4.10) in I_{δ^c}. Let f_k, $k \in \mathbb{N}$, be as in Lemma 4.9

bellow and fix $\varepsilon > 0$. Since B_σ is a continuous function, $f_k B_\sigma$ converges uniformly to fB_σ in $B_\delta(x)$. Let $0 < \rho < \delta/2$ be such that for all k

$$\left| \int_{B_\rho(x)} (f_k(x) - f_k(z)) F_\sigma(x,z)\ dz \right| < \frac{\varepsilon}{3}, \quad \text{and} \quad \left| \int_{B_\rho(x)} (f(x) - f(z)) F_\sigma(x,z)\ dz \right| < \frac{\varepsilon}{3}.$$

For k sufficiently large, by Hölder's inequality,

$$\begin{aligned} &\left| \int_{B_\rho^c(x)} (f_k(x) - f_k(z)) F_\sigma(x,z)\ dz - \int_{B_\rho^c(x)} (f(x) - f(z)) F_\sigma(x,z)\ dz \right| \\ &\qquad \leqslant |f_k(x) - f(x)| \int_{B_\rho^c(x)} F_\sigma(x,z)\ dz + \int_{B_\rho^c(x)} |f_k(z) - f(z)|\, F_\sigma(x,z)\ dz \\ &\qquad \leqslant C \left(|f_k(x) - f(x)| + \|f_k - f\|_{L^p_N} \right) < \frac{\varepsilon}{3}. \end{aligned}$$

Thus

$$S_\sigma f_k(x) \rightrightarrows \int_{\mathbb{R}^n} (f(x) - f(z)) F_\sigma(x,z)\ dz$$

in $B_\delta(x)$. But $H^\sigma f_k \to H^\sigma f$ in $\mathcal{S}'$. By uniqueness of the limits, $S_\sigma f(x)$ coincides with the integral in (4.7). Moreover, $H^\sigma f$ is continuous in $B_\delta(x)$ because it is the uniform limit of continuous functions. $\square$

Lemma 4.9. *Let* $f \in L^p_N \cap C^2(\mathcal{O})$. *Take* $\delta > 0$ *such that* $B_\delta(x) \subset \mathcal{O}$, $x \in \mathcal{O}$. *Then there exists a sequence* $f_k \in C_c^\infty(\mathbb{R}^n)$, $k \in \mathbb{N}$, *such that:*

(i) $\|D^2 f_k\|_{L^\infty(B_\delta(x))} \leqslant \|D^2 f\|_{L^\infty(B_\delta(x))}$ *for all* k,

(ii) f_k *converges uniformly to* f *in* $B_\delta(x)$ *and*

(iii) $f_k \to f$ *in the norm of* L^p_N, *as* $k \to \infty$.

Proof. Let ψ be a bump function at the origin, that is, ψ is a nonnegative $C_c^\infty(B_1(0))$ function defined in $\mathbb{R}^n$ with integral 1. Set $\psi_k(x) := k^n \psi(kx)$, $k \in \mathbb{N}$. Note that $\psi_k \in C_c^\infty(B_{1/k}(0))$ for all k. The convolution $g_k(x) := \psi_k * f(x)$ is well defined and

$$g_k(x) = \int_{B_{1/k}(0)} \psi_k(z) f(x - z)\ dz, \qquad x \in \mathbb{R}^n.$$

Let ζ be a smooth nonnegative cutoff function: $0 \leqslant \zeta \leqslant 1$, $\zeta = 1$ in $B_1(0)$, $\zeta = 0$ in $B_2^c(0)$ and $|\nabla \zeta| \leqslant C$. Put $\zeta_k(x) := \zeta(x/k)$. Define

$$f_k(x) = \zeta_k(x) g_k(x), \qquad x \in \mathbb{R}^n,\ k \in \mathbb{N}.$$

Each f_k is a smooth function with compact support. By the choice of ζ and since ψ_k has integral 1, for large k,

$$\|D^2 f_k\|_{L^\infty(B_\delta(x))} = \|D^2 g_k\|_{L^\infty(B_\delta(x))} \leqslant \|D^2 f\|_{L^\infty(B_\delta(x))},$$

and *(i)* holds. Let $\varepsilon > 0$. Since f is uniformly continuous in $B_\delta(x)$, there exists $\tau > 0$ such that $|f(y-z) - f(y)| < \varepsilon$ for all $y, z \in B_\delta(x)$ such that $|z| < \tau$. Set $k_0 \in \mathbb{N}$ so that $1/k < \tau$ and $\zeta_k(x) = 1$ in $B_\delta(0)$ for all $k \geqslant k_0$. Then

$$|f_k(y) - f(y)| \leqslant \int_{B_{1/k}(0)} \psi_k(z)\,|f(y-z) - f(y)|\ dz < \varepsilon, \qquad \text{for all } y \in B_\delta(x),$$

for $k \geqslant k_0$, and we have *(ii)*. To prove *(iii)* we first show that $g_k \to f$ as $k \to \infty$ in the norm of L^p_N. By Minkowski's integral inequality

$$\begin{aligned}
\left(\int_{\mathbb{R}^n} \frac{|g_k(x) - f(x)|^p}{(1+|x|^2)^{Np}}\ dx\right)^{1/p} &= \left(\int_{\mathbb{R}^n} \left|\int_{B_{1/k}(0)} \psi_k(z) \frac{f(x-z) - f(x)}{(1+|x|^2)^N}\ dz\right|^p\ dx\right)^{1/p} \\
&\leqslant \int_{|z|<1/k} \psi_k(z) \left(\int_{\mathbb{R}^n} \frac{|f(x-z) - f(x)|^p}{(1+|x|^2)^{Np}}\ dx\right)^{1/p} dz \\
&= \int_{|z|<1/k} \psi_k(z) \left(\int_{\mathbb{R}^n} \frac{|f(x) - f(x+z)|^p}{(1+|x+z|^2)^{Np}}\ dx\right)^{1/p} dz \\
&=: \int_{|z|<1/k} \psi_k(z) \cdot I(z)\ dz.
\end{aligned}$$

As

$$\frac{f(x) - f(x+z)}{(1+|x+z|^2)^N} = f(x)\left[\frac{1}{(1+|x+z|^2)^N} - \frac{1}{(1+|x|^2)^N}\right] + \frac{f(x)}{(1+|x|^2)^N} - \frac{f(x+z)}{(1+|x+z|^2)^N},$$

we have

$$\begin{aligned}
I(z) \leqslant &\left(\int_{\mathbb{R}^n} |f(x)|^p \left|\frac{1}{(1+|x+z|^2)^N} - \frac{1}{(1+|x|^2)^N}\right|^p\ dx\right)^{1/p} \\
&+ \left(\int_{\mathbb{R}^n} \left|\frac{f(x)}{(1+|x|^2)^N} - \frac{f(x+z)}{(1+|x+z|^2)^N}\right|^p\ dx\right)^{1/p} =: I_1(z) + I_2(z).
\end{aligned}$$

Let $\varepsilon > 0$. There exists $k_1 \in \mathbb{N}$ such that for all $k \geqslant k_1$,

$$|I_2(z)| < \frac{\varepsilon}{2}, \qquad \text{for all } |z| < \frac{1}{k}, \tag{4.11}$$

because of the continuity in L^p of the translations. For $I_1(z)$ we note that

$$\begin{aligned}
|f(x)|^p &\left|\frac{1}{(1+|x+z|^2)^N} - \frac{1}{(1+|x|^2)^N}\right|^p \\
&= \frac{|f(x)|^p}{(1+|x|^2)^{Np}}\left|\frac{(1+|x|^2)^N-(1+|x+z|^2)^N}{(1+|x+z|^2)^N}\right|^p \\
&\leqslant C_{p,N}\frac{|f(x)|^p}{(1+|x|^2)^{Np}}\left|\frac{1+|x+z|^{2N}+|z|^{2N}+1+|x+z|^{2N}}{(1+|x+z|^2)^N}\right|^p \\
&\leqslant C_{p,N}\frac{|f(x)|^p}{(1+|x|^2)^{Np}}\left|\frac{2+2\,|x+z|^{2N}+k^{-2N}}{(1+|x+z|^2)^N}\right|^p \leqslant C_{p,N}\frac{|f(x)|^p}{(1+|x|^2)^{Np}} \in L^1(\mathbb{R}^n),
\end{aligned}$$

and that

$$|f(x)|^p\left|\frac{1}{(1+|x+z|^2)^N} - \frac{1}{(1+|x|^2)^N}\right|^p \to 0, \qquad z\to 0.$$

Therefore, by the Dominated Convergence Theorem, $\lim_{z\to 0}|I_1(z)| = 0$, so there exists $k_2\in\mathbb{N}$ such that for all $k\geqslant k_2$,

$$|I_1(z)| < \frac{\varepsilon}{2}, \qquad \text{for all } |z| < \frac{1}{k}. \tag{4.12}$$

Let $\bar{k} = \max\{k_1,k_2\}$. Then for all $k\geqslant\bar{k}$, by (4.11) and (4.12),

$$\begin{aligned}
\left(\int_{\mathbb{R}^n}\frac{|g_k(x)-f(x)|^p}{(1+|x|^2)^{Np}}\,dx\right)^{1/p} &\leqslant \int_{|z|<1/k}\psi_k(z)(I_1(z)+I_2(z))\,dz \\
&< \varepsilon\int_{|z|<1/k}\psi_k(z)\,dz = \varepsilon.
\end{aligned}$$

We finish the argument noting that

$$\|f_k - g_k\|_{L^p_N} \leqslant \|\chi_{B^c_{2k}(0)}g_k\|_{L^p_N} \to 0, \qquad \text{as } k\to\infty,$$

and the same for f in the place of g_k, because of dominated convergence. □

4.2 The Harnack's inequality for H^σ

The result we are going to prove in this section is the following.

Theorem 4.10. *Let* $x_0\in\mathbb{R}^n$ *and* $R>0$. *Then there exists a positive constant* C *depending only on* n, σ, x_0 *and* R *such that*

$$\sup_{B_{R/2}(x_0)} f \leqslant C \inf_{B_{R/2}(x_0)} f,$$

for all nonnegative functions $f:\mathbb{R}^n\to\mathbb{R}$ *that are* C^2 *in* $B_R(x_0)$ *and such that* $H^\sigma f(x) = 0$ *for all* $x\in B_R(x_0)$.

The Harnack's inequality is valid for $0 < \sigma < 1$ and the proof we give is based on the extension problem and the Harnack's inequality for degenerate Schrödinger operators proved by C. E. Gutiérrez in [41]. The idea, that is also contained in [23] for the case of the fractional Laplacian, is the following: let u be the solution of the extension problem (3.1)-(3.2) posed for $L = H$, $d\eta = dx$ and $\Omega = \mathbb{R}^n$. Then the extension $\tilde{u}$ of u by reflection to $\mathbb{R}^{n+1}$ satisfies a degenerate Schrödinger equation in $B_R(x_0)$. We apply Gutiérrez's result to $\tilde{u}$ and we get the estimate for $\tilde{u}(x,0) = f(x)$.

We point out that the Harnack's inequality for H ($\sigma = 1$) follows from general results, see the classical paper by N. S. Trudinger [85].

Let us first study the problem (3.1)-(3.2) for the harmonic oscillator. In order to do that, we collect some useful facts about e^{-tH} in the next Proposition.

Proposition 4.11. *For $f \in L^p_N$ the heat semigroup $e^{-tH}f(x)$ is well defined and*

$$\left|e^{-tH}f(x)\right| \leqslant C\frac{(1+|x|^\rho)\,\|f\|_{L^p_N}}{t^{n/2}}, \qquad x \in \mathbb{R}^n,\ t > 0, \tag{4.13}$$

where $\rho > 0$ depends on p and N. Moreover, $(\partial_t + H)e^{-tH}f(x) = 0$ for all $x \in \mathbb{R}^n$ and $t > 0$, and for $i, j = 1, \dots, n$,

$$\left|\partial_{x_i}(e^{-tH}f)(x)\right| \leqslant C\frac{(1+|x|^\rho)\,\|f\|_{L^p_N}}{t^{(n+1)/2}}, \qquad \left|\partial_{x_i x_j}(e^{-tH}f)(x)\right| \leqslant C\frac{(1+|x|^\rho)\,\|f\|_{L^p_N}}{t^{(n+2)/2}}. \tag{4.14}$$

If f is also a C^2 function in some open subset $\mathcal{O} \subseteq \mathbb{R}^n$ then $\lim_{t\to 0} e^{-tH}f(x) = f(x)$ for all $x \in \mathcal{O}$.

Proof. By (2.27), (2.30) and Hölder's inequality,

$$\left|e^{-t(s)H}f(x)\right| \leqslant \frac{C\,\|f\|_{L^p_N}}{s^{n/2}}\left(\int_{\mathbb{R}^n} e^{-\frac{p'|x-z|^2}{C}}(1+|z|^2)^{Np'}\,dz\right)^{1/p'} \leqslant C\frac{(1+|x|^\rho)\,\|f\|_{L^p_N}}{s^{n/2}}.$$

For (4.13) note that if $0 < s < \frac{1}{2}$, then $s < t(s) < \frac{4}{3}s$. The equality

$$\partial_t e^{-tH}f(x) = \int_{\mathbb{R}^n} \partial_t G_t(x,z)f(z)\,dz$$

is valid if the last integral is absolutely convergent for all t in some interval. But

$$\partial_t G_t(x,z)f(z) = -H_x G_t(x,z)f(z),$$

therefore we have to verify that the integral $\int_{\mathbb{R}^n} H_x G_{t(s)}(x,z)f(z)\,dz$ converges absolutely for all s in some interval. This last statement is true since

$$\left|\nabla_x G_{t(s)}(x,z)\right| \leqslant \left(\frac{1-s}{s}\right)^{n/2}\frac{1}{s^{1/2}}\,e^{-C\left[s|x+z|^2+\frac{1}{s}|x-z|^2\right]},$$

and

$$\left|D_x^2 G_{t(s)}(x,z)\right| \leqslant \left(\frac{1-s}{s}\right)^{n/2} \frac{1}{s}\; e^{-C\left[s|x+z|^2+\frac{1}{s}|x-z|^2\right]},$$

which give estimates similar to (2.30) for $\nabla G_{t(s)}$ and $D^2 G_{t(s)}$ (see Lemma 5.20 in Chapter 5). Hence $\partial_t e^{-tH} f(x) = -H_x e^{-tH} f(x)$ and (4.14) follows. Observe that $t(s) \to 0$ if and only if $s \to 0$. For $x \in \mathcal{O}$ we have

$$\left|e^{-t(s)H} f(x) - f(x)\right| \leqslant \left|\int_{\mathbb{R}^n} G_{t(s)}(x,z)(f(z)-f(x))\; dz\right| + |f(x)|\left|e^{-t(s)H}1(x) - 1\right|.$$

The last term above tends to 0 as $t(s) \to 0$ because of (2.29). Let $\delta > 0$ be such that $B_\delta(x) \subset \mathcal{O}$. Then, as $f \in C^1(\overline{B_\delta(x)})$,

$$\begin{aligned}\left|\int_{B_\delta(x)} G_{t(s)}(x,z)(f(z)-f(x))\; dz\right| &\leqslant C\int_{B_\delta(x)} \frac{e^{-\frac{|x-y|^2}{Cs}}}{s^{n/2}}\,|z-x|\,dz\\ &\leqslant C\int_{B_\delta(x)} \frac{e^{-\frac{|x-y|^2}{Cs}}}{|z-x|^{n-1}}\; dz \to 0,\end{aligned}$$

when $s \to 0$, by the Dominated Convergence Theorem. On the other hand, by Hölder's inequality,

$$\begin{aligned}\left|\int_{B^c_\delta(x)} G_{t(s)}(x,z)(f(z)-f(x))\; dz\right| &\leqslant \left[\int_{B^c_\delta(x)} e^{-\frac{p|x-z|^2}{2C}}\left(\frac{|f(z)|^p}{(1+|z|^2)^{Np}} + \frac{|f(x)|^p}{(1+|z|^2)^{Np}}\right) dz\right]^{1/p}\\ &\quad\times \frac{C}{s^{n/2}}\left[\int_{B^c_\delta(x)} e^{-\frac{p'|x-z|^2}{Cs}} e^{-\frac{p'|x-z|^2}{2C}}(1+|z|^2)^{Np'}\; dz\right]^{1/p'}\\ &=: \mathrm{I}\times\mathrm{II}.\end{aligned}$$

Clearly $\mathrm{I} < \infty$ and, by dominated convergence,

$$\mathrm{II} \leqslant C\left(\int_{B^c_\delta(x)} \frac{e^{-\frac{p'|x-z|^2}{Cs}}}{|x-z|^{np'}}\; e^{-\frac{p'|x-z|^2}{C}}(1+|z|^2)^{Np'}\; dz\right)^{1/p'} \to 0, \qquad \text{as } s \to 0.$$

□

Remark 4.12. If $f \in L^p_N \cap C^2(\mathcal{O})$ then, for each $x \in \mathcal{O}$,

$$\int_0^\infty \left|e^{-tH} f(x) - f(x)\right| \frac{dt}{t^{1+\sigma}} = \int_0^1 \left|e^{-t(s)H} f(x) - f(x)\right|\; d\mu_\sigma(s) < \infty.$$

Indeed, by (4.13),

$$\int_{\frac{1}{2}\log 3}^\infty \left|e^{-tH} f(x) - f(x)\right| \frac{dt}{t^{1+\sigma}} \leqslant C(x)\int_{\frac{1}{2}\log 3}^\infty \frac{dt}{t^{1+\sigma}} < \infty,$$

and

$$\int_0^{\frac{1}{2}\log 3} \left|e^{-tH}f(x) - f(x)\right| \frac{dt}{t^{1+\sigma}} = \int_0^{1/2} \left|e^{-t(s)H}f(x) - f(x)\right| d\mu_\sigma(s)$$
$$\leqslant C\int_0^{1/2} \left|\int_{\mathbb{R}^n} G_{t(s)}(x,z)(f(z)-f(x))dz\right| \frac{ds}{s^{1+\sigma}}$$
$$+ C\,|f(x)| \int_0^{1/2} \left[1 - \left(\frac{1-s^2}{1+s^2}\right)^{n/2} e^{-\frac{s}{1+s^2}|x|}\right] \frac{ds}{s^{1+\sigma}}.$$

Both integrals above are finite: the first one by the arguments of the proof of Theorem 4.3 in the previous section (Taylor's Theorem) and the second one because of the Mean Value Theorem. Moreover, Lemma 4.9 allows us to show that

$$H^\sigma f(x) = \frac{1}{\Gamma(-\sigma)} \int_0^\infty \left(e^{-tH}f(x) - f(x)\right) \frac{dt}{t^{1+\sigma}}, \qquad \text{for } f \in L^p_N \cap C^2(\mathcal{O}),\ x \in \mathcal{O}.$$

The relevant observation in the following statement is that all identities are understood in the classical sense.

Theorem 4.13. *If $f \in L^p_N$ is a C^2 function in some open subset $\mathcal{O} \subseteq \mathbb{R}^n$ then*

$$u(x,y) := \frac{y^{2\sigma}}{4^\sigma\Gamma(\sigma)} \int_0^\infty e^{-tH}f(x)e^{-\frac{y^2}{4t}} \frac{dt}{t^{1+\sigma}}, \tag{4.15}$$

is well defined for all $x \in \mathbb{R}^n$, $y > 0$, and

$$-H_x u + \frac{1-2\sigma}{y}\, u_y + u_{yy} = 0, \qquad \textit{in } \mathbb{R}^n \times (0,\infty);$$
$$\lim_{y\to 0^+} u(x,y) = f(x), \qquad \textit{for } x \in \mathcal{O}.$$

In addition, for all $x \in \mathcal{O}$,

$$\frac{1}{2\sigma} \lim_{y\to 0^+} y^{1-2\sigma} u_y(x,y) = \frac{1}{4^\sigma\Gamma(\sigma)} \int_0^\infty \left(e^{-tH}f(x) - f(x)\right) \frac{dt}{t^{1+\sigma}} = H^\sigma f(x). \tag{4.16}$$

Proof. Estimate (4.13) implies that the integral defining u is absolutely convergent and u_y and u_{yy} can be computed by taking the derivatives inside the integral sign. Moreover, by using (4.14), we have

$$H_x u(x,y) = \frac{y^{2\sigma}}{4^\sigma\Gamma(\sigma)} \int_0^\infty He^{-tH}f(x)e^{-\frac{y^2}{4t}} \frac{dt}{t^{1+\sigma}},$$

in the classical sense. Hence, for each $x \in \mathbb{R}^n$, u verifies the extension problem in the classical sense. To check that identity (4.16) is also classical, we begin by recalling that integration by parts gives us

$$\int_0^\infty e^{-\frac{y^2}{4t}} \left(2\sigma - \frac{y^2}{2t}\right) \frac{dt}{t^{1+\sigma}} = 0.$$

Thus

$$\begin{aligned}\frac{1}{2\sigma}\, y^{1-2\sigma}u_y(x,y) &= \frac{1}{2\sigma 4^\sigma\Gamma(\sigma)}\int_0^\infty e^{-tH}f(x)e^{-\frac{y^2}{4t}}\left(2\sigma-\frac{y^2}{2t}\right)\,\frac{dt}{t^{1+\sigma}}\\ &= \frac{1}{2\sigma 4^\sigma\Gamma(\sigma)}\int_0^\infty \left(e^{-tH}f(x)-f(x)\right)e^{-\frac{y^2}{4t}}\left(2\sigma-\frac{y^2}{2t}\right)\,\frac{dt}{t^{1+\sigma}}.\end{aligned}$$

As we already pointed out in Remark 4.12, the integral in (4.16) is absolutely convergent for all $x \in \mathcal{O}$ and f as in the hypotheses. Therefore the first identity of (4.16) follows by dominated convergence and, by Remark 4.12, also the second one. $\square$

Remark 4.14. Theorem 4.13 is valid if H is replaced by $-\Delta$ and the function f, having the same smoothness in $\mathcal{O}$, belongs to $L_\sigma = L^1_{n/2+\sigma}$. See the discussion on $(-\Delta)^\sigma$ given in Subsection 2.1.2 of Chapter 2.

Lemma 4.15 (Reflection extension). *Fix* $R > 0$ *and* $x_0 \in \mathbb{R}^n$*. Let* u *be a solution of*

$$-H_x u + \frac{1-2\sigma}{y}\,u_y + u_{yy} = 0, \qquad \text{in } \mathbb{R}^n\times(0,R),$$

with

$$\lim_{y\to 0^+} y^{1-2\sigma}u_y(x,y) = 0, \qquad \text{for every } x \text{ such that } |x-x_0|<R. \tag{4.17}$$

Then the extension to $\mathbb{R}^n\times(-R,R)$ *defined by*

$$\tilde{u}(x,y) = \begin{cases} u(x,y), & y \geqslant 0;\\ u(x,-y), & y<0;\end{cases} \tag{4.18}$$

verifies the degenerate Schrödinger equation

$$\operatorname{div}(|y|^{1-2\sigma}\nabla\tilde{u}) - |y|^{1-2\sigma}|x|^2\,\tilde{u} = 0, \tag{4.19}$$

in the weak sense in $B := \left\{(x,y)\in\mathbb{R}^{n+1} : |x-x_0|^2+y^2<R^2\right\}$.

Proof. A nontrivial solution u can be found with the method of Chapter 3, Subsection 3.3.2. Given $\varphi\in C_c^\infty(B)$ we want to prove that

$$I := \int_B \left(\nabla\tilde{u}\cdot\nabla\varphi + |x|^2\,\tilde{u}\varphi\right)|y|^{1-2\sigma}\;dx\;dy = 0.$$

For $\delta>0$ we have

$$\begin{aligned} I &= \int_{B\cap\{|y|\geqslant\delta\}} \operatorname{div}(|y|^{1-2\sigma}\,\varphi\nabla\tilde{u})\;dx\;dy + \int_{B\cap\{|y|<\delta\}}\left(\nabla\tilde{u}\cdot\nabla\varphi+|x|^2\,\tilde{u}\varphi\right)|y|^{1-2\sigma}\;dx\;dy\\ &= \int_{B\cap\{|y|=\delta\}} \varphi\delta^{1-2\sigma}\tilde{u}_y(x,\delta)\;dx + \int_{B\cap\{|y|<\delta\}}\left(\nabla\tilde{u}\cdot\nabla\varphi+|x|^2\,\tilde{u}\varphi\right)|y|^{1-2\sigma}\;dx\;dy.\end{aligned}$$

As $\delta\to 0$, the first term above goes to zero because of (4.17) and the second term goes to zero because $\left(|\nabla\tilde{u}|^2+|x|^2\,\tilde{u}\right)|y|^{1-2\sigma}$ is a locally integrable function. $\square$

The last ingredient we need to give the proof of Theorem 4.10 is Gutiérrez's Harnack's inequality for degenerate Schrödinger equations [41]. The version we will use reads as follows. Consider a degenerate Schrödinger equation of the form

$$Lu - Vu = \partial_{x_i}\left(a_{ij}(x)\partial_{x_j}u\right) - V(x)u = 0, \tag{4.20}$$

where $x \in \mathbb{R}^n$, $a = (a_{ij})$ is a symmetric matrix of measurable real-valued coefficients and

$$\lambda^{-1}\omega(x)\,|\xi|^2 \leqslant \langle a(x)\xi, \xi\rangle \leqslant \lambda\omega(x)\,|\xi|^2,$$

for some $\lambda > 0$, for all $\xi \in \mathbb{R}^n$. The function ω is an A_2-weight and the requirement for the potential V is that $V/\omega \in L^p_\omega$ locally, for some large p. Let Ω be an open, bounded subset of $\mathbb{R}^n$. Then there exist positive constants r_0 and γ (depending on Ω, ω, λ and n) such that if u is any nonnegative weak solution of (4.20) in Ω then for every ball B_r with $B_{2r} \subset \Omega$ and $0 < r \leqslant r_0$ we have

$$\sup_{B_{r/2}} u \leqslant \gamma \inf_{B_{r/2}} u.$$

Proof of Theorem 4.10. Let u be as in Theorem 4.13. Since f is a nonnegative function, from (2.25) and (4.15) we see that $u \geqslant 0$. By virtue of (4.16) its reflection extension (4.18) satisfies Lemma 4.15. Note that (4.19) is a degenerate Schrödinger equation with A_2 weight $w = |y|^{1-2\sigma}$ and potential $V = |y|^{1-2\sigma}\,|x|^2$ such that $V/w \in L^p_w$ locally for p large enough. So Gutiérrez's result just explained above applies to obtain the Harnack's inequality for $\tilde{u}$ and thus for f. $\square$

Chapter 5

Regularity theory for the fractional harmonic oscillator

We prove regularity estimates in Hölder spaces for the fractional harmonic oscillator H^σ. The new classes of Hölder spaces $C_H^{k,\alpha}$ adapted to H are defined. The main results are Theorem A, where the action of H^σ on $C_H^{k,\alpha}$ is studied, and Theorem B, that contains the Schauder estimates for H^σ. The proof of both statements require to show that the Hermite-Riesz transforms are bounded in $C_H^{0,\alpha}$.

5.1 Hermite-Hölder spaces and main results

We introduce our spaces of smooth functions adapted to H.

Definition 5.1. Let $0 < \alpha \leqslant 1$. A continuous function $u : \mathbb{R}^n \to \mathbb{R}$ belongs to the *Hermite-Hölder space* $C_H^{0,\alpha}$ *associated to* H if there exists a constant C, depending only on u and α, such that

$$|u(x_1) - u(x_2)| \leqslant C\,|x_1 - x_2|^\alpha, \quad \text{and} \quad |u(x)| \leqslant \frac{C}{(1+|x|)^\alpha},$$

for all $x_1, x_2, x \in \mathbb{R}^n$. Defining

$$[u]_{C^{0,\alpha}} := \sup_{\substack{x_1,x_2\in\mathbb{R}^n \\ x_1\neq x_2}} \frac{|u(x_1)-u(x_2)|}{|x_1-x_2|^\alpha}, \quad \text{and} \quad [u]_{M^\alpha} := \sup_{x\in\mathbb{R}^n} |(1+|x|)^\alpha u(x)|,$$

the norm in the spaces $C_H^{0,\alpha}$ is given by

$$\|u\|_{C_H^{0,\alpha}} := [u]_{C^{0,\alpha}} + [u]_{M^\alpha}.$$

Using the natural *derivatives* (2.32) associated to H:

$$A_i = \partial_{x_i} + x_i, \quad A_{-i} = -\partial_{x_i} + x_i, \qquad i = 1, \dots, n,$$

the classes $C_H^{k,\alpha}$ are defined in the natural way.

Definition 5.2. For each $k \in \mathbb{N}$, we define the *Hermite-Hölder space* $C_H^{k,\alpha}$, $0 < \alpha \leqslant 1$, as the set of all functions $u \in C^k(\mathbb{R}^n)$ such that the following norm is finite:

$$\|u\|_{C_H^{k,\alpha}} := [u]_{M^\alpha} + \sum_{\substack{1\leqslant|i_1|,\ldots,|i_m|\leqslant n\\ 1\leqslant m\leqslant k}} [A_{i_1}\cdots A_{i_m}u]_{M^\alpha} + \sum_{1\leqslant|i_1|,\ldots,|i_k|\leqslant n} [A_{i_1}\cdots A_{i_k}u]_{C^{0,\alpha}}.$$

Remark 5.3. Clearly $C_H^{k,\alpha}$ is continuously embedded into $C^{k,\alpha}(\mathbb{R}^n)$, $k \geqslant 0$, $0 < \alpha \leqslant 1$.

Let us present the two main results. First the interaction of H^σ with $C_H^{k,\alpha}$.

Theorem A. *Let* $\alpha \in (0,1]$ *and* $\sigma \in (0,1)$.

(A1) Let $u \in C_H^{0,\alpha}$ *and* $2\sigma < \alpha$. *Then* $H^\sigma u \in C_H^{0,\alpha-2\sigma}$ *and*

$$\|H^\sigma u\|_{C_H^{0,\alpha-2\sigma}} \leqslant C\,\|u\|_{C_H^{0,\alpha}}.$$

(A2) Let $u \in C_H^{1,\alpha}$ *and* $2\sigma < \alpha$. *Then* $H^\sigma u \in C_H^{1,\alpha-2\sigma}$ *and*

$$\|H^\sigma u\|_{C_H^{1,\alpha-2\sigma}} \leqslant C\,\|u\|_{C_H^{1,\alpha}}.$$

(A3) Let $u \in C_H^{1,\alpha}$ *and* $2\sigma \geqslant \alpha$, *with* $\alpha - 2\sigma + 1 \neq 0$. *Then* $H^\sigma u \in C_H^{0,\alpha-2\sigma+1}$ *and*

$$\|H^\sigma u\|_{C_H^{0,\alpha-2\sigma+1}} \leqslant C\,\|u\|_{C_H^{1,\alpha}}.$$

(A4) Let $u \in C_H^{k,\alpha}$ *and assume that* $k + \alpha - 2\sigma$ *is not an integer. Then* $H^\sigma u \in C_H^{l,\beta}$ *where* l *is the integer part of* $k + \alpha - 2\sigma$ *and* $\beta = k + \alpha - 2\sigma - l$.

Schauder's estimates read as follows.

Theorem B. *Let* $u \in C_H^{0,\alpha}$, *for some* $0 < \alpha \leqslant 1$, *and* $0 < \sigma \leqslant 1$.

(B1) If $\alpha + 2\sigma \leqslant 1$, *then* $H^{-\sigma}u \in C_H^{0,\alpha+2\sigma}$ *and*

$$\left\|H^{-\sigma}u\right\|_{C_H^{0,\alpha+2\sigma}} \leqslant C\,\|u\|_{C_H^{0,\alpha}}.$$

(B2) If $1 < \alpha + 2\sigma \leqslant 2$, *then* $H^{-\sigma}u \in C_H^{1,\alpha+2\sigma-1}$ *and*

$$\left\|H^{-\sigma}u\right\|_{C_H^{1,\alpha+2\sigma-1}} \leqslant C\,\|u\|_{C_H^{0,\alpha}}.$$

(B3) If $2 < \alpha + 2\sigma \leqslant 3$, *then* $H^{-\sigma}u \in C_H^{2,\alpha+2\sigma-2}$ *and*

$$\left\|H^{-\sigma}u\right\|_{C_H^{2,\alpha+2\sigma-2}} \leqslant C\,\|u\|_{C_H^{0,\alpha}}.$$

Remark 5.4. If in Theorem B *(B3)* we take $\sigma = 1$, we get the Schauder estimate for the solution to

$$Hu = f, \qquad \text{in } \mathbb{R}^n,$$

for $f \in C_H^{0,\alpha}$.

The Hermite-Riesz transforms preserve the spaces $C_H^{0,\alpha}$:

Theorem 5.5. *The Hermite-Riesz transforms $\mathcal{R}_i$ and $\mathcal{R}_{ij}$, $1 \leqslant |i|, |j| \leqslant n$, are bounded operators on the spaces $C_H^{0,\alpha}$: if $u \in C_H^{0,\alpha}$, for some $0 < \alpha < 1$, then $\mathcal{R}_i u, \mathcal{R}_{ij} u \in C_H^{0,\alpha}$, and*

$$\|\mathcal{R}_i u\|_{C_H^{0,\alpha}} + \left\|\mathcal{R}_{ij} u\right\|_{C_H^{0,\alpha}} \leqslant C \, \|u\|_{C_H^{0,\alpha}}.$$

The first main task will be to obtain explicit pointwise expressions for all the operators involved when they are applied to functions belonging to the spaces $C_H^{k,\alpha}$ and the second one is to actually prove the estimates. Section 5.2 contains two abstract Propositions dealing with these two aspects: Proposition 5.6 takes care of the pointwise formulas and Proposition 5.8 contains a regularity result. We will apply, in a systematic way, both Propositions in order to reach our objectives: see Section 5.3 for all the pointwise formulas, and Section 5.4 for the proofs of Theorems 5.5, A and B. In Section 5.5 we collect all the computational Lemmas used in the previous sections.

5.2 Two abstract results

Proposition 5.6. *Let T be a bounded operator on $\mathcal{S}$ such that $\langle Tf, g\rangle = \langle f, Tg\rangle$ for all $f, g \in \mathcal{S}$. Assume that*

$$Tf(x) = \int_{\mathbb{R}^n} (f(x) - f(z))K(x,z)\; dz + f(x)B(x), \qquad x \in \mathbb{R}^n,$$

where the kernel K verifies the size condition

$$|K(x,z)| \leqslant \frac{C}{|x-z|^{n+\gamma}}\; e^{-\frac{|x||x-z|}{C}} e^{-\frac{|x-z|^2}{C}}, \qquad x, z \in \mathbb{R}^n, \tag{5.1}$$

for some $-n \leqslant \gamma < 1$ and B is a continuous function with polynomial growth at infinity. Let $u \in C_H^{0,\gamma+\varepsilon}$ with $0 < \gamma + \varepsilon \leqslant 1$ and $\varepsilon > 0$. Then Tu is well defined as a tempered distribution and it coincides with the continuous function

$$Tu(x) = \int_{\mathbb{R}^n} (u(x) - u(z))K(x,z)\; dz + u(x)B(x), \qquad x \in \mathbb{R}^n. \tag{5.2}$$

Proof. By (5.1) and the smoothness of u, the integral in (5.2) is absolutely convergent. Since B has polynomial growth at infinity, the right hand side of (5.2) defines a tempered distribution. Let us take $\frac{n}{\gamma+\varepsilon} < p < \infty$. Then the finiteness of $[u]_{M^{\gamma+\varepsilon}}$ implies that $u \in L^p(\mathbb{R}^n)$ and Tu is well defined as a tempered distribution. Fix arbitrary positive numbers η and R. We use an approximation argument parallel to the one given in the proof of Theorem 4.3, Chapter 4. Let $f_j(x) := \zeta(x/j)\left(u * \mathcal{W}_{1/j}\right)(x)$, $j \in \mathbb{N}$, where $\mathcal{W}_t(x)$ is the Gauss-Weierstrass kernel (2.12) and ζ is a nonnegative smooth cutoff function (see the proof of Lemma 4.9 in Chapter 4). Note that each f_j belongs to $\mathcal{S}$. It is easy to check that the sequence $\{f_j : j \in \mathbb{N}\}$ converges to u in $L^p(\mathbb{R}^n)$ and uniformly in $B_R(x)$ for each $x \in \mathbb{R}^n$, and $[f_j]_{C^{0,\gamma+\varepsilon}} \leqslant C \|u\|_{C_H^{0,\gamma+\varepsilon}} =: M$. As $j \to \infty$, $Tf_j \to Tu$ in $\mathcal{S}'$. Since B is a continuous function, $f_j B$ converges uniformly to uB in $B_R(x_0)$, $x_0 \in \mathbb{R}^n$. There exists $0 < \delta < R/2$ such that

$$M \int_{B_\delta(0)} |z|^{\varepsilon - n} \, dz \leqslant \frac{\eta}{3}.$$

For $x \in B_{R/2}(x_0)$ we write

$$\int_{\mathbb{R}^n} (f_j(x) - f_j(z)) K(x,z) \, dz = \int_{B_\delta(x)} + \int_{B_\delta^c(x)} = I + II.$$

Then, by the choice of δ,

$$|I| + \left| \int_{B_\delta(x)} (u(x) - u(z)) K(x,z) \, dz \right| \leqslant \frac{2}{3}\, \eta.$$

We also have

$$\begin{aligned} \left| II - \int_{B_\delta^c(x)} (u(x) - u(z)) K(x,z) \, dz \right| &\leqslant C \left| f_j(x) - u(x) \right| + C \left(\int_{B_\delta^c(x)} \left| f_j(z) - u(z) \right|^p \, dz \right)^{1/p} \\ &\leqslant \frac{\eta}{3}, \end{aligned}$$

for sufficiently large j, uniformly in $x \in B_{R/2}(x_0)$. Therefore,

$$\int_{\mathbb{R}^n} (f_j(x) - f_j(z)) K(x,z) \, dz \rightrightarrows \int_{\mathbb{R}^n} (u(x) - u(z)) K(x,z) \, dz, \quad j \to \infty,$$

in $B_{R/2}(x_0)$. Hence, by uniqueness of the limits, Tu is a function that coincides with (5.2), and it is a continuous function, because it is the uniform limit of continuous functions. $\square$

Remark 5.7. In the context of Proposition 5.6 assume that, instead of having estimate (5.1) on the kernel, we just know that $|K(x,z)| \leqslant \Phi(x-z)$, where $\Phi \in L^{p'}(\mathbb{R}^n)$ and p' is the conjugate exponent of some p such that $\frac{n}{\gamma+\varepsilon} < p < \infty$. Then, it is enough to take $u \in C_H^{0,\alpha}$, for some $0 < \alpha \leqslant 1$, to get the same conclusion, since the approximation procedure given in the proof above can also be applied in this situation.

Proposition 5.8. *Let T be an operator satisfying the hypotheses of Proposition 5.6, with $0 \leqslant \gamma < 1$ and $0 < \gamma + \varepsilon \leqslant 1$ for some $0 < \varepsilon < 1$. Assume that the kernel K and the function B also satisfy:*

(a) $|K(x_1,z)-K(x_2,z)| \leqslant C\dfrac{|x_1-x_2|}{|x_2-z|^{n+1+\gamma}}\ e^{-\frac{|z||x_2-z|^2}{C}}e^{-\frac{|x_2-z|^2}{C}}$, *when* $|x_1-z|>2|x_1-x_2|$.

(b) There exists a constant C *such that* $\left|\int_{|x-z|>r} K(x,z)\ dz\right| \leqslant Cr^{-\gamma}$, *for all* $x\in\mathbb{R}^n$.

(c) For all $x\in\mathbb{R}^n$, $|B(x)|\leqslant C(1+|x|)^{\gamma}$, *and* $\nabla B\in L^{\infty}(\mathbb{R}^n)$.

Then T *maps* $C_H^{0,\gamma+\varepsilon}$ *into* $C_H^{0,\varepsilon}$ *continuously.*

Proof. Given $x_1,x_2\in\mathbb{R}^n$, let us denote $B=B(x_1,2|x_1-x_2|)$, $\widetilde{B}=B(x_2,4|x_1-x_2|)$ and $B'=B(x_2,|x_1-x_2|)$. We write

$$\begin{aligned}Tu(x) &= \int_B (u(x)-u(z))K(x,z)dz+\int_{B^c}(u(x)-u(z))K(x,z)dz+u(x)B(x)\\ &= I(x)+II(x)+III(x).\end{aligned}$$

By (5.1) we have

$$\begin{aligned}|I(x_1)-I(x_2)| &\leqslant \int_B |(u(x_1)-u(z))K(x_1,z)|\ dz+\int_{\widetilde{B}}|(u(x_2)-u(z))K(x_2,z)|\ dz\\ &\leqslant C[u]_{C^{0,\gamma+\varepsilon}}\left[\int_B \frac{|x_1-z|^{\gamma+\varepsilon}}{|x_1-z|^{n+\gamma}}\ dz+\int_{\widetilde{B}}\frac{|x_2-z|^{\gamma+\varepsilon}}{|x_2-z|^{n+\gamma}}\ dz\right]\\ &= C[u]_{C^{0,\gamma+\varepsilon}}|x_1-x_2|^{\varepsilon}.\end{aligned}$$

For the difference $II(x_1)-II(x_2)$, we add the term $\pm u(x_2)K(x_1,z)$ and we use the smoothness and cancelation properties of the kernel $K(x,z)$ (hypotheses *(a)* and *(b)*) to get:

$$\begin{aligned}&|II(x_1)-II(x_2)|\\ &\quad\leqslant \int_{B^c}|(u(x_2)-u(z))(K(x_1,z)-K(x_2,z))|\,dz+\left|\int_{B^c}(u(x_1)-u(x_2))K(x_1,z)dz\right|\\ &\quad\leqslant C[u]_{C^{0,\gamma+\varepsilon}}\left[\int_{B^c}|x_2-z|^{\gamma+\varepsilon}\frac{|x_1-x_2|}{|x_2-z|^{n+1+\gamma}}\ dz+|x_1-x_2|^{\gamma+\varepsilon}\left|\int_{B^c}K(x_1,z)dz\right|\right]\\ &\quad\leqslant C[u]_{C^{0,\gamma+\varepsilon}}\left[\int_{(B')^c}\frac{|x_1-x_2|}{|x_2-z|^{n+1-\varepsilon}}\ dz+|x_1-x_2|^{\varepsilon}\right]\\ &\quad= C[u]_{C^{0,\gamma+\varepsilon}}|x_1-x_2|^{\varepsilon}.\end{aligned}$$

If $|x_1-x_2|<\frac{1}{1+|x_1|}$, by *(c)*,

$$\begin{aligned}\frac{|III(x_1)-III(x_2)|}{|x_1-x_2|^{\varepsilon}} &\leqslant \frac{|u(x_1)-u(x_2)|}{|x_1-x_2|^{\gamma+\varepsilon}}|B(x_1)|\,|x_1-x_2|^{\gamma}+|u(x_2)|\,\frac{|B(x_1)-B(x_2)|}{|x_1-x_2|^{\varepsilon}}\\ &\leqslant C[u]_{C^{0,\gamma+\varepsilon}}+[u]_{M^{\gamma+\varepsilon}}\|\nabla B\|_{L^{\infty}(\mathbb{R}^n)}|x_1-x_2|^{1-\varepsilon}\leqslant C\|u\|_{C_H^{0,\gamma+\varepsilon}}.\end{aligned}$$

Assume that $|x_1 - x_2| \geqslant \frac{1}{1+|x_1|}$. Then $1 + |x_1| \leqslant 1 + |x_2| + |x_1 - x_2|$, which implies

$$\frac{1+|x_1|}{1+|x_2|} \leqslant 1 + \frac{|x_1 - x_2|}{1+|x_2|},$$

and hence,

$$\frac{1}{1+|x_2|} \leqslant \frac{1}{1+|x_1|} + \frac{|x_1 - x_2|}{(1+|x_1|)(1+|x_2|)} \leqslant 2\,|x_1 - x_2|.$$

With this and hypothesis *(c)* we have

$$\frac{|III(x_1) - III(x_2)|}{|x_1 - x_2|^{\varepsilon}} \leqslant |u(x_1)B(x_1)|\,(1+|x_1|)^{\varepsilon} + |u(x_2)B_{\sigma}(x_2)|\,2^{\varepsilon}(1+|x_2|)^{\varepsilon} \leqslant C[u]_{M^{\gamma+\varepsilon}}.$$

Let us finally study the growth of $Tu(x)$. For the multiplicative term uB we clearly have $|u(x)B(x)| \leqslant C[u]_{M^{\gamma+\varepsilon}}(1+|x|)^{-\varepsilon}$. Consider next the integral part in the formula for $Tu(x)$, (5.2). Since Tu and B are continuous functions, it is enough to consider $|x| > 2$. We write

$$\left|\int_{\mathbb{R}^n} (u(x) - u(z))K(x,z)\ dz\right| = \left|\left(\int_{|x-z|<\frac{1}{1+|x|}} + \int_{|x-z|\geqslant\frac{1}{1+|x|}}\right)\ dz\right|.$$

On one hand,

$$\begin{aligned}
\int_{|x-z|<\frac{1}{1+|x|}} |u(x) - u(z)|\,|K(x,z)|\ dz &\leqslant C[u]_{C^{0,\gamma+\varepsilon}} \int_{|x-z|<\frac{1}{1+|x|}} \frac{|x-z|^{\gamma+\varepsilon}}{|x-z|^{n+\gamma}}\ dz \\
&= [u]_{C^{0,\gamma+\varepsilon}} \frac{C}{(1+|x|)^{\varepsilon}}.
\end{aligned}$$

On the other hand, by *(b)*,

$$|u(x)|\left|\int_{|x-z|\geqslant\frac{1}{1+|x|}} K(x,z)\ dz\right| \leqslant \frac{[u]_{M^{\gamma+\varepsilon}}}{(1+|x|)^{\gamma+\varepsilon}}\ C(1+|x|)^{\gamma} = [u]_{M^{\gamma+\varepsilon}} \frac{C}{(1+|x|)^{\varepsilon}}.$$

Since $|x-z| \geqslant \frac{1}{1+|x|}$ implies that $\frac{1}{1+|z|} \leqslant 2\,|x-z|$, applying (5.1) we get

$$\begin{aligned}
\int_{|x-z|\geqslant\frac{1}{1+|x|}} |u(z)|\,|K(x,z)|\ dz &\leqslant C[u]_{M^{\gamma+\varepsilon}} \int_{|x-z|\geqslant\frac{1}{1+|x|}} \frac{1}{(1+|z|)^{\gamma+\varepsilon}}\ \frac{e^{-\frac{|x||x-z|}{C}} e^{-\frac{|x-z|^2}{C}}}{|x-z|^{n+\gamma}}\ dz \\
&\leqslant C[u]_{M^{\gamma+\varepsilon}} \int_{|x-z|\geqslant\frac{1}{1+|x|}} |x-z|^{\gamma+\varepsilon}\ \frac{e^{-\frac{|x||x-z|}{C}} e^{-\frac{|x-z|^2}{C}}}{|x-z|^{n+\gamma}}\ dz \\
&= C[u]_{M^{\gamma+\varepsilon}} \sum_{j=0}^{\infty} \int_{|x-z|\sim\frac{2^j}{1+|x|}} \frac{e^{-\frac{|x||x-z|}{C}} e^{-\frac{|x-z|^2}{C}}}{|x-z|^{n-\varepsilon}}\ dz \\
&\leqslant [u]_{M^{\gamma+\varepsilon}} \frac{C}{(1+|x|)^{\varepsilon}} \sum_{j=0}^{\infty} 2^{j\varepsilon} e^{-\frac{2^j}{C'}} = [u]_{M^{\gamma+\varepsilon}} \frac{C}{(1+|x|)^{\varepsilon}},
\end{aligned}$$

where in the last line the constant C' appearing in the exponential is independent of x because $|x|\,|x-z| \sim 2^j$. Therefore, by pasting the estimates above, the result is proved. □

5.3 The operators

In this section we give the pointwise definitions in $C_H^{k,\alpha}$ of all the operators involved.

5.3.1 The fractional operators H^σ and $(H \pm 2k)^\sigma$, $k \in \mathbb{N}$

Since $C_H^{k,\alpha} \subset \mathcal{S}'$, by the results of Chapter 4, $H^\sigma u$ is well defined for u in $C_H^{k,\alpha}$. The following Theorem says that in this case the pointwise formula for $H^\sigma u$ is the same as the one given in Theorem 4.3 of Chapter 4.

Theorem 5.9. *Let* $0 < \alpha \leqslant 1$ *and* $0 < \sigma < 1$.

(1) If $0 < \alpha - 2\sigma < 1$ *and* $u \in C_H^{0,\alpha}$ *then*

$$H^\sigma u(x) = \int_{\mathbb{R}^n} (u(x) - u(z)) F_\sigma(x,z)\ dz + u(x) B_\sigma(x), \qquad x \in \mathbb{R}^n, \tag{5.3}$$

and the integral converges absolutely.

(2) If $-1 < \alpha - 2\sigma \leqslant 0$ *and* $u \in C_H^{1,\alpha}$ *then* $H^\sigma u(x)$ *is given by* (5.3)*, where the integral converges as a principal value.*

(3) When $-2 < \alpha - 2\sigma \leqslant -1$*, it is enough to take* $u \in C_H^{1,1}$ *to have the conclusion of (2).*

In the three cases $H^\sigma u \in C(\mathbb{R}^n)$.

Proof. If $0 < \alpha - 2\sigma < 1$ then $\sigma < 1/2$. The properties of F_σ and B_σ established in Lemmas 5.21 and 5.22 (see Section 5.5) allow us to apply Proposition 5.6 with $K(x,z) = F_\sigma(x,z)$, $B = B_\sigma$ and $\gamma = 2\sigma < 1$, to get *(1)*.

Under the hypotheses of *(2)*, we will take advantage of a cancelation to show that the integral in (5.3) is well defined. Let $\delta > 0$. By Lemma 5.21,

$$\int_{|x-z|\geqslant\delta} |u(x) - u(z)|\, F_\sigma(x,z)\ dz \leqslant C_\delta \left\|u\right\|_{L^\infty(\mathbb{R}^n)} \int_{|x-z|\geqslant\delta} e^{-\frac{|x-z|^2}{C}}\ dz < \infty.$$

Recall that Meda's change of parameters (2.27) gives (2.28) and (4.8). Hence

$$F_\sigma(x,z) = \frac{1}{-\Gamma(-\sigma)} \int_0^1 \left(\frac{1-s^2}{4\pi s}\right)^{n/2} e^{-\frac{1}{4}\left[s|x+z|^2 + \frac{1}{s}|x-z|^2\right]}\ d\mu_\sigma(s). \tag{5.4}$$

Up to the multiplicative constant $1/(-\Gamma(-\sigma))$ we have

$$\begin{aligned} I &:= \int_{|z|<\delta} (u(x) - u(x-z)) F_\sigma(x, x-z)\ dz \\ &= \int_0^\delta r^{n-1} \int_{|z'|=1} (u(x) - u(x - rz')) \int_0^1 \left(\frac{1-s^2}{4\pi s}\right)^{n/2} e^{-\frac{1}{4}\left[s|2x-rz'|^2 + \frac{r^2}{s}\right]}\ d\mu_\sigma(s)\ dS(z')\ dr. \end{aligned}$$

By the smoothness of u,

$$u(x) - u(x - rz') = \nabla u(x)(rz') + R_1 u(x, rz'), \quad \text{with} \quad \left|R_1 u(x, rz')\right| \leqslant [\nabla u]_{C^{0,\alpha}} r^{1+\alpha}.$$

We apply the Mean Value Theorem to the function $\psi(x) = \psi_{s,r}(x) = e^{-\frac{1}{4}\left[s|x|^2 + \frac{r^2}{s}\right]}$, to see that

$$e^{-\frac{1}{4}\left[s|2x - rz'|^2 + \frac{r^2}{s}\right]} = e^{-\frac{1}{4}\left[s|2x|^2 + \frac{r^2}{s}\right]} + R_0\psi(x, rz'), \quad \text{with} \quad \left|R_0\psi(x, rz')\right| \leqslant C s^{1/2} r e^{-\frac{r^2}{8s}}.$$

Therefore,

$$\begin{aligned}
I &= \int_0^\delta r^{n-1} \int_{|z'|=1} \nabla u(x)(rz') \int_0^1 \left(\frac{1-s^2}{4\pi s}\right)^{n/2} R_0\psi(x, rz')\, d\mu_\sigma(s)\, dS(z')\, dr \\
&+ \int_0^\delta r^{n-1} \int_{|z'|=1} R_1 u(x, rz') \int_0^1 \left(\frac{1-s^2}{4\pi s}\right)^{n/2} \left[e^{-\frac{1}{4}\left[s|2x|^2 + \frac{r^2}{s}\right]} + R_0\psi(x, rz')\right] d\mu_\sigma\, dS(z')\, dr \\
&=: I_1 + I_2.
\end{aligned}$$

With the estimates on $R_1 u$ and $R_0\psi$ given above and (4.9), we obtain

$$\begin{aligned}
|I_1| &\leqslant C\,|\nabla u(x)| \int_0^\delta r^{n+1} \int_0^1 \left(\frac{1-s}{s}\right)^{n/2} s^{1/2} e^{-\frac{r^2}{8s}} d\mu_\sigma(s)\, dr \\
&\leqslant C\,|\nabla u(x)| \int_0^\delta \frac{r^{n+1}}{r^{n-1+2\sigma}}\, dr = C\delta^{3-2\sigma},
\end{aligned}$$

and

$$\begin{aligned}
|I_2| &\leqslant C[\nabla u]_{C^{0,\alpha}} \int_0^\delta r^{n+\alpha} \int_0^1 \left(\frac{1-s}{s}\right)^{n/2} e^{-\frac{r^2}{4s}}\, d\mu_\sigma(s)\, dr \\
&\leqslant C[\nabla u]_{C^{0,\alpha}} \int_0^\delta \frac{r^{n+\alpha}}{r^{n+2\sigma}}\, dr = C\delta^{\alpha-2\sigma+1}.
\end{aligned}$$

Thus, the integral in (5.3) converges as a principal value. The same happens if we take $u \in C_H^{1,1}$: we repeat the argument above, but applying in I_2 the estimate $|R_1 u(x, rz')| \leqslant [\nabla u]_{C^{0,1}} \cdot r^2$.

To obtain the conclusions of *(2)* and *(3)* we note that the approximation procedure used in the proof of Proposition 5.6 can be applied here (with the estimate $[\nabla f_j]_{C^{0,\alpha}} \leqslant C\,\|u\|_{C_H^{1,\alpha}} = M$). □

Remark 5.10. As in Chapter 4 the maximum and comparison principles for H^σ can be derived from Theorem 5.9. For instance, take $0 < \alpha \leqslant 1$, $0 < \sigma < 1$, and u, v in the class $C_H^{0,\alpha}$ (or $C_H^{1,\alpha}$, depending on the value of $\alpha - 2\sigma$) such that $u \geqslant v$ in $\mathbb{R}^n$ with $u(x_0) = v(x_0)$ for some $x_0 \in \mathbb{R}^n$. Then $H^\sigma u(x_0) \leqslant H^\sigma v(x_0)$. Moreover, $H^\sigma u(x_0) = H^\sigma v(x_0)$ only when $u \equiv v$.

In order to prove the regularity estimates for H^σ, we will have to work with the derivatives of H^σ, that is, with operators of the type $A_i H^\sigma$, $1 \leqslant |i| \leqslant n$. Recall that by (4.2) we have

$$A_i f = \sum_\nu (2\nu_i)^{1/2} \langle f, h_\nu \rangle h_{\nu - e_i}, \qquad A_{-i} f = \sum_\nu (2\nu_i + 2)^{1/2} \langle f, h_\nu \rangle h_{\nu + e_i},$$

for all $f \in \mathcal{S}$ and $1 \leqslant i \leqslant n$, and both series converge uniformly in $\mathbb{R}^n$.

Remark 5.11. Let $b \in \mathbb{R}$. Then, by using Hermite series expansions, it is easy to check that for all $f \in \mathcal{S}$ and $1 \leqslant i \leqslant n$, we have

$$\begin{aligned} A_i H^b f &= (H+2)^b A_i f, & H^b A_i f &= A_i (H-2)^b f, \\ A_{-i} H^b f &= (H-2)^b A_{-i} f, & H^b A_{-i} f &= A_{-i} (H+2)^b f, \end{aligned}$$

where we defined $(H \pm 2)^b h_\nu := (2|\nu| + n \pm 2)^b h_\nu$.

Consequently, we need to study the operators $(H \pm 2k)^\sigma$, $k \in \mathbb{N}$.

Let us start with $(H + 2k)^\sigma$, k a positive integer. For $f \in \mathcal{S}$ and $k \in \mathbb{N}$ we define

$$(H + 2k)^\sigma f(x) = \sum_\nu (2|\nu| + n + 2k)^\sigma \langle f, h_\nu \rangle h_\nu(x), \qquad x \in \mathbb{R}^n.$$

The series above converges in $L^2(\mathbb{R}^n)$ and uniformly in $\mathbb{R}^n$, it defines a Schwartz's class function and

$$(H + 2k)^\sigma f(x) = \frac{1}{\Gamma(-\sigma)} \int_0^\infty \left(e^{-2kt} e^{-tH} f(x) - f(x) \right) \frac{dt}{t^{1+\sigma}}, \qquad x \in \mathbb{R}^n.$$

By using Lemmas 5.21 and 5.22 stated in Section 5.5, the following result can be proved in a parallel way to Theorem 5.9.

Theorem 5.12. *Let u be as in Theorem 5.9. Then $(H + 2k)^\sigma u \in \mathcal{S}' \cap C(\mathbb{R}^n)$ and*

$$(H + 2k)^\sigma u(x) = \int_{\mathbb{R}^n} (u(x) - u(z)) F_{2k,\sigma}(x, z)\, dz + u(x) B_{2k,\sigma}(x), \qquad x \in \mathbb{R}^n,$$

where

$$F_{2k,\sigma}(x, z) = \frac{1}{-\Gamma(-\sigma)} \int_0^\infty e^{-2kt} G_t(x, z) \frac{dt}{t^{1+\sigma}} = \frac{1}{-\Gamma(-\sigma)} \int_0^1 \left(\frac{1-s}{1+s} \right)^k G_{t(s)}(x, z)\, d\mu_\sigma(s),$$

and

$$B_{2k,\sigma}(x) = \frac{1}{\Gamma(-\sigma)} \int_0^1 \left[\left(\frac{1-s}{1+s} \right)^k \left(\frac{1-s^2}{2\pi(1+s^2)} \right)^{n/2} e^{-\frac{s}{1+s^2}|x|^2} - 1 \right] d\mu_\sigma(s).$$

Consider next the operators $(H-2k)^\sigma$, $k\in\mathbb{N}$. We say that a function $f\in\mathcal{S}$ belongs to the space $\mathcal{S}_k$ if

$$\int_{\mathbb{R}^n} f(z)h_\nu(z)\,dz = 0, \qquad \text{for all } \nu\in\mathbb{N}_0^n \text{ such that } |\nu|<k.$$

For $f\in\mathcal{S}_k$ we define

$$(H-2k)^\sigma f(x) = \sum_{|\nu|\geqslant k}(2|\nu|+n-2k)^\sigma\langle f,h_\nu\rangle h_\nu(x).$$

Note that on $\mathcal{S}_k$ the operator $(H-2k)^\sigma$ is positive. Let

$$\phi_{2k}(x)=\phi_{2k}(x,z,s)=\left[\sum_{j=0}^{k-1}\left(\frac{1-s}{1+s}\right)^{j+n/2}\sum_{|\nu|=j}h_\nu(x)h_\nu(z)\right]\chi_{(1/2,1)}(s),$$

the sum of the first $(k-1)$-terms of the series defining $G_{t(s)}(x,z)$ for $s\in(1/2,1)$, see (2.28). Then the heat-diffusion semigroup generated by $H-2k$:

$$\begin{aligned}e^{-t(H-2k)}f(x)&=\int_{\mathbb{R}^n}e^{2kt}G_t(x,z)f(z)\,dz\\&=\int_{\mathbb{R}^n}\left(\frac{1+s}{1-s}\right)^k G_{t(s)}(x,z)f(z)\,dz=e^{-t(s)(H-2k)}f(x),\end{aligned}$$

can be written as

$$e^{-t(s)(H-2k)}f(x)=\int_{\mathbb{R}^n}\left(\frac{1+s}{1-s}\right)^k\left[G_{t(s)}(x,z)-\phi_{2k}(x,z,s)\right]f(z)\,dz, \qquad f\in\mathcal{S}_k.$$

Moreover,

$$\begin{aligned}(H-2k)^\sigma f(x)&=\frac{1}{\Gamma(-\sigma)}\int_0^\infty\left(e^{-t(H-2k)}f(x)-f(x)\right)\frac{dt}{t^{1+\sigma}}\\&=\frac{1}{\Gamma(-\sigma)}\int_0^1\left(e^{-t(s)(H-2k)}f(x)-f(x)\right)d\mu_\sigma(s).\end{aligned}$$

The following idea is taken from [43]. By the n-dimensional Mehler's formula (see [81, p. 380] or [82, p. 6]),

$$M_r(x,z):=\sum_{j=0}^\infty r^j\sum_{|\nu|=j}h_\nu(x)h_\nu(z)=\frac{1}{\pi^{n/2}(1-r^2)^{n/2}}\,e^{-\frac14\left[\frac{1-r}{1+r}|x+z|^2+\frac{1+r}{1-r}|x-z|^2\right]}, \tag{5.5}$$

for $r\in(0,1)$. Then for all $r\in(0,1/3)$,

$$\left|\frac{d^k}{dr^k}\,M_r(x,z)\right|\leqslant C\left(1+|x+z|^2+|x-z|^2\right)^k e^{-\frac14\left[\frac{1-r}{1+r}|x+z|^2+\frac{1+r}{1-r}|x-z|^2\right]}\leqslant Ce^{-\frac{|x||x-z|}{C}}e^{-\frac{|x-z|^2}{C}},$$

where in the second inequality we applied Lemma 5.19 of Section 5.5, with $s = \frac{1-r}{1+r}$. Thus, by Taylor's formula,

$$\left|\sum_{j=k}^{\infty} r^j \sum_{|\nu|=j} h_\nu(x)h_\nu(z)\right| \leqslant Cr^k e^{-\frac{|x||x-z|}{C}} e^{-\frac{|x-z|^2}{C}}, \qquad r \in (0,1/3).$$

Therefore, letting $r = \frac{1-s}{1+s}$ above, we obtain

$$\left|G_{t(s)}(x,z) - \phi_{2k}(x,z,s)\right| = \left|\sum_{j=k}^{\infty} \left(\frac{1-s}{1+s}\right)^{j+n/2} \sum_{|\nu|=j} h_\nu(x)h_\nu(z)\right| \tag{5.6}$$

$$\leqslant C\left(\frac{1-s}{1+s}\right)^{k+n/2} e^{-\frac{|x||x-z|}{C}} e^{-\frac{|x-z|^2}{C}}, \qquad s \in (1/2,1). \tag{5.7}$$

If $u \in C_H^{k,\alpha}$ then we have

$$\int_{\mathbb{R}^n} A_{-i_1}\cdots A_{-i_k}u(x)h_\nu(x)\,dx = 0, \qquad 1 \leqslant i_1,\ldots,i_k \leqslant n,\ |\nu| < k.$$

Theorem 5.13. *Let* $0 < \alpha \leqslant 1$ *and* $0 < \sigma < 1$. *Assume that* $0 < \alpha - 2\sigma < 1$ *and take* $u \in C_H^{k,\alpha}$. *If* $v(x) = (A_{-i_1}\cdots A_{-i_k}u)(x)$, $1 \leqslant i_1,\ldots,i_k \leqslant n$, *then* $A_{-i_1}\cdots A_{-i_k}H^\sigma u \in \mathcal{S}' \cap C(\mathbb{R}^n)$ *and for all* $x \in \mathbb{R}^n$,

$$\begin{aligned} A_{-i_1}\cdots A_{-i_k}H^\sigma u(x) &= (H-2k)^\sigma v(x) \\ &= \int_{\mathbb{R}^n} (v(x)-v(z))F_{-2k,\sigma}(x,z)\,dz + v(x)B_{-2k,\sigma}(x), \end{aligned} \tag{5.8}$$

where

$$F_{-2k,\sigma}(x,z) = \frac{1}{-\Gamma(-\sigma)}\int_0^1 \left(\frac{1+s}{1-s}\right)^k \left[G_{t(s)}(x,z) - \phi_{2k}(x,z,s)\right]\,d\mu_\sigma(s),$$

and

$$B_{-2k,\sigma}(x) = \frac{1}{\Gamma(-\sigma)}\int_0^1 \left[\left(\frac{1+s}{1-s}\right)^k \int_{\mathbb{R}^n} \left[G_{t(s)}(x,z) - \phi_{2k}(x,z,s)\right]\,dz - 1\right]\,d\mu_\sigma(s).$$

The integral in (5.8) *is absolutely convergent.*

Proof. Even if we have good estimates for $F_{-2k,\sigma}$ and $B_{-2k,\sigma}$ (see Lemmas 5.21 and 5.22), we can not apply directly Proposition 5.6 here because the test space for $(H-2k)^\sigma$ is not $\mathcal{S}$ but $\mathcal{S}_k$. Nevertheless, the same ideas will work. Indeed, using Lemmas 5.21 and 5.22 it can be checked that the conclusion is valid when u is a Schwartz's class function (and then $v \in \mathcal{S}_k$), and, for the general result, we can apply the approximation procedure given in the proof of Proposition 5.6 noting that $(A_{-i_1}\cdots A_{-i_k}f_j)(x)$ can be used to approximate $v(x)$. □

5.3.2 The negative powers $H^{-\sigma}$

Observe that

$$H^{-\sigma}f(x) = \int_{\mathbb{R}^n} (f(z) - f(x))F_{-\sigma}(x,z)\ dz + f(x)H^{-\sigma}1(x), \qquad \text{for } f \in \mathcal{S},\ x \in \mathbb{R}^n,$$

where

$$H^{-\sigma}1(x) = \frac{1}{\Gamma(\sigma)}\int_0^\infty e^{-tH}1(x)\ \frac{dt}{t^{1-\sigma}} = \int_{\mathbb{R}^n} F_{-\sigma}(x,z)\ dz, \qquad x \in \mathbb{R}^n. \tag{5.9}$$

The fractional integral operator $H^{-\sigma}$ can be defined in $C_H^{0,\alpha}$ precisely by this formula:

Theorem 5.14. *For* $u \in C_H^{0,\alpha}$, $0 < \alpha \leqslant 1$ *and* $0 < \sigma \leqslant 1$, $H^{-\sigma}u \in \mathcal{S}' \cap C(\mathbb{R}^n)$ *and*

$$H^{-\sigma}u(x) = \int_{\mathbb{R}^n} (u(z) - u(x))F_{-\sigma}(x,z)\ dz + u(x)H^{-\sigma}1(x), \qquad x \in \mathbb{R}^n.$$

Proof. Using Hermite series expansion and (4.4) it can be checked that $H^{-\sigma}$ is a symmetric and continuous operator in $\mathcal{S}$. In Lemmas 5.23 and 5.24 we collect the properties of the kernel $F_{-\sigma}(x,z)$ and the function $H^{-\sigma}1(x)$. When $n > 2\sigma$, an application of Proposition 5.6 with $\gamma = -2\sigma$ and $\varepsilon = \alpha + 2\sigma$ gives the result. For the case $n \leqslant 2\sigma$ we use Remark 5.7. □

We shall also need to work with the derivatives of $H^{-\sigma}u$.

Theorem 5.15. *Take* $0 < \alpha \leqslant 1$ *and* $0 < \sigma \leqslant 1$ *such that* $\alpha + 2\sigma > 1$. *If* $u \in C_H^{0,\alpha}$ *then for each* $1 \leqslant |i| \leqslant n$ *we have* $A_i H^{-\sigma}u \in \mathcal{S}' \cap C(\mathbb{R}^n)$ *and*

$$A_i H^{-\sigma}u(x) = \int_{\mathbb{R}^n} (u(z) - u(x))A_i F_{-\sigma}(x,z)\ dz + u(x)A_i H^{-\sigma}1(x), \qquad x \in \mathbb{R}^n.$$

Proof. Let us first prove the result when $u = f \in \mathcal{S}$. It is enough to consider $1 \leqslant i \leqslant n$. We have

$$A_i H^{-\sigma}f(x) = A_i \int_{\mathbb{R}^n} (f(z) - f(x))F_{-\sigma}(x,z)\ dz + \partial_{x_i}f(x)H^{-\sigma}1(x) + f(x)A_i H^{-\sigma}1(x).$$

We want to put the A_i inside the integral. In order to do that, we apply a classical approximation argument given in the proof of Lemma 4.1 of [40], that we sketch here. By estimate (5.19) of Lemma 5.24 (see Section 5.5) and the fact that $\alpha + 2\sigma > 1$, the function

$$\begin{aligned} g(x) &= \int_{\mathbb{R}^n} \partial_{x_i}\left[(f(z) - f(x))F_{-\sigma}(x,z)\right]\ dz \\ &= \int_{\mathbb{R}^n} (f(z) - f(x))\partial_{x_i}F_{-\sigma}(x,z)\ dz - \partial_{x_i}f(x)H^{-\sigma}1(x), \end{aligned}$$

is well defined. Fix a function $\phi \in C^1(\mathbb{R})$ satisfying $0 \leqslant \phi \leqslant 1$, $\phi(t) = 0$ for $t \leqslant 1$, $\phi(t) = 1$ for $t \geqslant 2$, and $0 \leqslant \phi' \leqslant 2$. Define, for $0 < \varepsilon < 1/2$,

$$h_\varepsilon(x) = \int_{\mathbb{R}^n} (f(z) - f(x))F_{-\sigma}(x,z)\phi\left(\varepsilon^{-1}|x - z|\right)\ dz, \qquad x \in \mathbb{R}^n.$$

Then estimate (5.18) in Lemma 5.23 implies that, as $\varepsilon \to 0$, $h_\varepsilon(x)$ converges uniformly in $\mathbb{R}^n$ to

$$\int_{\mathbb{R}^n} (f(z) - f(x))F_{-\sigma}(x,z)\ dz, \qquad x \in \mathbb{R}^n.$$

Moreover, $h_\varepsilon \in C^1(\mathbb{R}^n)$ and, again by (5.18) and (5.19),

$$\begin{aligned}|g(x) - \partial_{x_i} h_\varepsilon(x)| &\leqslant \int_{\mathbb{R}^n} \left|\partial_{x_i}\left[(f(z)-f(x))F_{-\sigma}(x,z)\left(1-\phi\left(\varepsilon^{-1}|x-z|\right)\right)\right]\right|\ dz \\ &\leqslant C_f \int_{|x-z|<2\varepsilon} \left[F_{-\sigma}(x,z) + |x-z|\,|\nabla_x F_{-\sigma}(x,z)| + |x-z|\,F_{-\sigma}(x,z)\frac{1}{\varepsilon}\right]\ dz \\ &\leqslant C_f \cdot \Phi_{n,\sigma}(\varepsilon),\end{aligned}$$

where $\Phi_{n,\sigma}(\varepsilon) \to 0$ as $\varepsilon \to 0$, uniformly in $x \in \mathbb{R}^n$. Thus,

$$\partial_{x_i} \int_{\mathbb{R}^n} (f(z)-f(x))F_{-\sigma}(x,z)\ dz = \int_{\mathbb{R}^n} (f(z)-f(x))\partial_{x_i} F_{-\sigma}(x,z)\ dz - \partial_{x_i} f(x) H^{-\sigma} 1(x),$$

and the Theorem is valid when u is a Schwartz function. For the general case $u \in C_H^{0,\alpha}$ we argue as follows. If $n > 2\sigma - 1$ then, by (5.19), we can apply Proposition 5.6 with $\gamma = 1 - 2\sigma$ and $\varepsilon = \alpha + 2\sigma - 1$ and, if $n = 2\sigma - 1$ we can use Remark 5.7. □

5.3.3 The Hermite-Riesz transforms $\mathcal{R}_i$ and $\mathcal{R}_{ij}$

Using Hermite series expansions it is easy to check that the first and second order Hermite-Riesz transforms (2.33) and (2.34) are symmetric operators in $\mathcal{S}$ and that they map $\mathcal{S}$ into $\mathcal{S}$ continuously.

Taking $\sigma = 1/2$ in Theorem 5.15 we get:

Theorem 5.16. *If* $u \in C_H^{0,\alpha}$, $0 < \alpha \leqslant 1$, *then for all* $1 \leqslant |i| \leqslant n$, $\mathcal{R}_i u \in \mathcal{S}' \cap C(\mathbb{R}^n)$ *and*

$$\mathcal{R}_i u(x) = \int_{\mathbb{R}^n} (u(z)-u(x))A_i F_{-1/2}(x,z)\ dz + u(x) A_i H^{-1/2} 1(x), \qquad x \in \mathbb{R}^n.$$

By using the properties of the kernel of the second order Hermite-Riesz transform

$$R_{ij}(x,z) = A_i A_j F_{-1}(x,z)$$

given in Lemma 5.25 bellow, it is easy to get a pointwise description of $\mathcal{R}_{ij} f$, $f \in \mathcal{S}$. Hence we can use Proposition 5.6 with $\gamma = 0$ and $\varepsilon = \alpha$ to have the following Theorem.

Theorem 5.17. *If* $u \in C_H^{0,\alpha}$, $0 < \alpha \leqslant 1$, *then for all* $1 \leqslant |i|, |j| \leqslant n$ *we have* $\mathcal{R}_{ij} u \in \mathcal{S}' \cap C(\mathbb{R}^n)$ *and*

$$\mathcal{R}_{ij} u(x) = \int_{\mathbb{R}^n} (u(z)-u(x)) R_{ij}(x,z)\ dz + u(x) A_i A_j H^{-1} 1(x), \qquad x \in \mathbb{R}^n.$$

5.4 Proofs of the main results

5.4.1 Regularity properties of Hermite-Riesz transforms

Proof of Theorem 5.5. By Lemmas 5.23, 5.24, and 5.27 of Section 5.5 and Theorem 5.16, the result for $\mathcal{R}_i$ can be deduced applying Proposition 5.8 with $\gamma = 0$ and $\varepsilon = \alpha$.

Let us consider the operator $\mathcal{R}_{ij}$ for some $j \in \{1, \dots, n\}$. Then by Remark 5.11 we have

$$\begin{aligned}\mathcal{R}_{ij} &= A_i A_j H^{-1} = A_i \left(A_j H^{-1/2}\right) H^{-1/2} \\ &= A_i \left[(H+2)^{-1/2} A_j\right] H^{-1/2} = A_i (H+2)^{-1/2} \circ \mathcal{R}_j.\end{aligned}$$

Therefore, it is enough to prove that $A_i(H+2)^{-1/2}$ is a continuous operator on $C_H^{0,\alpha}$. When $f \in \mathcal{S}$ we can write

$$A_i(H+2)^{-1/2} f(x) = \int_{\mathbb{R}^n} (f(z) - f(x)) A_i F_{2,-1/2}(x,z)\ dz + f(x) A_i (H+2)^{-1/2} 1(x),$$

where

$$F_{2,-1/2}(x,z) = \frac{1}{\Gamma(1/2)} \int_0^1 \left(\frac{1-s}{1+s}\right) G_{t(s)}(x,z) d\mu_{-1/2}(s)$$

and

$$(H+2)^{-1/2} 1(x) = \int_{\mathbb{R}^n} F_{2,-1/2}(x,z) dz.$$

Following the proof of Lemmas 5.23 and 5.24, it can be checked that the kernel $A_i F_{2,-1/2}(x,z)$ and the function $(H+2)^{-1/2}1(x)$ share the same size and smoothness properties than the kernel $A_i F_{-1/2}(x,z)$ and the function $H^{-1/2}1(x)$ stated in the mentioned Lemmas. Thus, as a consequence of the results of Section 5.2, $A_i(H+2)^{-1/2} : C_H^{0,\alpha} \to C_H^{0,\alpha}$ continuously. Therefore $\mathcal{R}_{ij}$ is a bounded operator on $C_H^{0,\alpha}$ when $j \in \{1, \dots, n\}$.

Note that

$$\mathcal{R}_{ij} = \partial^2_{x_i,x_j} H^{-1} + x_j \partial_{x_i} H^{-1} + x_i \partial_{x_j} H^{-1} + x_i x_j H^{-1} + \delta_{ij} H^{-1},$$

which at the level of kernels means that

$$R_{ij}(x,z) = \partial^2_{x_i,x_j} F_{-1}(x,z) + x_j \partial_{x_i} F_{-1}(x,z) + x_i \partial_{x_j} F_{-1}(x,z) + x_i x_j F_{-1}(x,z) + \delta_{ij} F_{-1}(x,z),$$

By the estimates given in Lemmas 5.23, 5.25 and 5.26, we can apply the statements of Section 5.2 to show that the operators $x_i \partial_{x_j} H^{-1}$, $x_i x_j H^{-1}$ and H^{-1} are bounded on $C_H^{0,\alpha}$. Hence, $\partial^2_{x_i,x_j} H^{-1}$ maps $C_H^{0,\alpha}$ into $C_H^{0,\alpha}$ continuously. Observe now that the operator $\mathcal{R}_{i,-j}$, for $j \in \{1, \dots, n\}$, can be written as

$$\mathcal{R}_{i,-j} = -\partial^2_{x_i,x_j} H^{-1} + x_j \partial_{x_i} H^{-1} - x_i \partial_{x_j} H^{-1} + x_i x_j H^{-1} + \delta_{ij} H^{-1}.$$

The observations above give the conclusion for $\mathcal{R}_{i,-j}$, $j \in \{1, \dots, n\}$. $\square$

For technical reasons we have to consider the *first order adjoint Hermite-Riesz transforms* that are defined by

$$\mathcal{R}_i^* f(x) = H^{-1/2} A_i f(x) = \int_{\mathbb{R}^n} F_{-1/2}(x,z)(A_i f)(z)\, dz, \qquad f \in \mathcal{S},\ x \in \mathbb{R}^n,\ 1 \leqslant |i| \leqslant n.$$

Theorem 5.18. *The operators $\mathcal{R}_i^*$, $1 \leqslant |i| \leqslant n$, are bounded operators on $C_H^{0,\alpha}$, $0 < \alpha < 1$.*

Proof. If $1 \leqslant i \leqslant n$ then, by Remark 5.11, $\mathcal{R}_{-i}^* = H^{-1/2} A_{-i} = A_{-i}(H+2)^{-1/2}$. This operator already appeared in the proof of Theorem 5.5 and there we showed that it is a bounded operator on $C_H^{0,\alpha}$.

On the other hand, $\mathcal{R}_{-i}^* = -H^{-1/2}\partial_{x_i} + H^{-1/2}x_i$. But by Lemmas 5.23, 5.24, and 5.26 of Section 5.5 and Proposition 5.8, we can see that the operator $f \mapsto H^{-1/2}x_i f$, initially defined on $\mathcal{S}$, maps $C_H^{0,\alpha}$ into itself continuously. Therefore, we obtain the same conclusion for the operator $f \mapsto \mathcal{R}_{-i}^* f - H^{-1/2}x_i f = -H^{-1/2}\partial_{x_i} f$. Consequently, $\mathcal{R}_i^* = H^{-1/2}A_i = H^{-1/2}\partial_{x_i} + H^{-1/2}x_i$ is a bounded operator on $C_H^{0,\alpha}$. □

5.4.2 Proof of Theorem A

We start with *(A1)*. By recalling the results in Lemmas 5.21 and 5.22, if we put $\gamma = 2\sigma < 1$ and $\varepsilon = \alpha - 2\sigma$ in Proposition 5.8, we get the conclusion.

Consider now *(A2)*. Using Remark 5.11 and Theorem 5.13 we have

$$H^\sigma u \in C_H^{1,\alpha-2\sigma} \Leftrightarrow A_i H^\sigma u,\ A_{-i}H^\sigma u \in C_H^{0,\alpha-2\sigma} \Leftrightarrow (H+2)^\sigma A_i u,\ (H-2)^\sigma A_{-i} u \in C_H^{0,\alpha-2\sigma}.$$

By Theorem 5.12 together with Lemmas 5.21 and 5.22, we can apply Proposition 5.8 with $\gamma = 2\sigma < 1$ and $\varepsilon = \alpha - 2\sigma$ in order to get $(H+2)^\sigma : C_H^{0,\alpha} \to C_H^{0,\alpha-2\sigma}$ continuously and then

$$\|(H+2)^\sigma A_i u\|_{C_H^{0,\alpha-2\sigma}} \leqslant C\,\|A_i u\|_{C_H^{0,\alpha}} \leqslant C\,\|u\|_{C_H^{1,\alpha}}.$$

An application of Theorem 5.13, Lemmas 5.21 and 5.22 and Proposition 5.8 gives us the estimate $\|(H-2)^\sigma A_{-i}u\|_{C_H^{0,\alpha-2\sigma}} \leqslant C\,\|u\|_{C_H^{1,\alpha}}$. Thus, $\|H^\sigma u\|_{C_H^{1,\alpha-2\sigma}} \leqslant C\,\|u\|_{C_H^{1,\alpha}}$.

Let us prove *(A3)*. We can write

$$H^\sigma = H^{\sigma-1/2} \circ H^{-1/2} \circ H = H^{\sigma-1/2} \circ \frac{1}{2}\sum_{i=1}^{n} \left(\mathcal{R}_{-i}^* A_{-i} + \mathcal{R}_i^* A_i\right),$$

where $\mathcal{R}_{\pm i}^*$ are the first order adjoint Hermite-Riesz transforms, that are bounded operators on $C_H^{0,\alpha}$ (Theorem 5.18). Consequently,

$$\frac{1}{2}\sum_{i=1}^{n} \left(\mathcal{R}_{-i}^* A_{-i} u + \mathcal{R}_i^* A_i u\right) =: v \in C_H^{0,\alpha}.$$

Now we distinguish two cases. If $\sigma - 1/2 > 0$ then $0 < \alpha - 2(\sigma - 1/2) < 1$ by hypothesis, so we can apply *(A1)* to obtain that $H^{\sigma-1/2}v \in C_H^{0,\alpha-2\sigma+1}$ and $\|H^\sigma u\|_{C_H^{0,\alpha-2\sigma+1}} \leqslant C\,\|u\|_{C_H^{1,\alpha}}$.

If $\sigma - 1/2 < 0$ then $0 < \alpha + 2(-\sigma + 1/2) < 1$ and we will get $H^{-(-\sigma+1/2)}v \in C_H^{0,\alpha-2\sigma+1}$ and $\|H^\sigma u\|_{C_H^{0,\alpha-2\sigma+1}} \leqslant C\|u\|_{C_H^{1,\alpha}}$ after we have proved Theorem B *(B1)*. If $\sigma = 1/2$ the result just follows from the boundedness of the first order adjoint Hermite-Riesz transforms on $C_H^{0,\alpha}$.

By iteration of *(A1)*, *(A2)*, *(A3)* and using Remark 5.11 and Theorems 5.12 and 5.13 we can derive *(A4)*. The rather cumbersome details are left to the interested reader.

5.4.3 Proof of Theorem B

To prove *(B1)* note that if $\alpha + 2\sigma \leqslant 1$ then $0 < \sigma < 1/2$. Let us write

$$\begin{aligned} H^{-\sigma}u(x_1) - H^{-\sigma}u(x_2) &= \int_{\mathbb{R}^n} \big[u(z) - u(x_1)\big]\big[F_{-\sigma}(x_1, z) - F_{-\sigma}(x_2, z)\big]\ dz \\ &\quad + u(x_1)\big[H^{-\sigma}1(x_1) - H^{-\sigma}1(x_2)\big]. \end{aligned}$$

By Lemma 5.24 the second term above is bounded by $C[u]_{M^\alpha}|x_1 - x_2|^{\alpha+2\sigma}$. We split the remaining integral on $B = B(x_1, 2|x_1 - x_2|)$ and on B^c. We use Lemma 5.23 to get

$$\int_B |u(z) - u(x_1)|\, F_{-\sigma}(x_1, z)\ dz \leqslant C[u]_{C^{0,\alpha}} \int_B \frac{|x_1 - z|^\alpha}{|x_1 - z|^{n-2\sigma}}\ dz = C[u]_{C^{0,\alpha}}|x_1 - x_2|^{\alpha+2\sigma}.$$

Let $B' = B(x_2, 4|x_1 - x_2|)$. Then, by the triangle inequality,

$$\begin{aligned} \int_B |u(z) - u(x_1)|\, F_{-\sigma}(x_2, z)\ dz &\leqslant C[u]_{C^{0,\alpha}} \int_{B'} \frac{|x_1 - z|^\alpha}{|x_2 - z|^{n-2\sigma}}\ dz \\ &\leqslant C[u]_{C^{0,\alpha}}\left[\int_{B'} \frac{|x_1 - x_2|^\alpha}{|x_2 - z|^{n-2\sigma}}\ dz + \int_{B'} |x_2 - z|^{\alpha-n+2\sigma}\ dz\right] \\ &= C[u]_{C^{0,\alpha}}|x_1 - x_2|^{\alpha+2\sigma}. \end{aligned}$$

Denote by $\widetilde{B}$ the ball with center x_2 and radius $|x_1 - x_2|$. Note that, for $z \in \widetilde{B}^c$, $|z - x_1| < 2|z - x_2|$. Then Lemma 5.23 gives

$$\begin{aligned} \int_{B^c} |u(z) - u(x_1)|\,|F_{-\sigma}(x_1, z) - F_{-\sigma}(x_2, z)|\, dz &\leqslant C[u]_{C^{0,\alpha}} \int_{\widetilde{B}^c} \frac{|x_1 - x_2|\,|z - x_2|^\alpha}{|z - x_2|^{n+1-2\sigma}}\ e^{-\frac{|x-z|^2}{C}} dz \\ &\leqslant C[u]_{C^{0,\alpha}}|x_1 - x_2|^{\alpha+2\sigma}. \end{aligned}$$

Thus, $[H^{-\sigma}u]_{C^{0,\alpha+2\sigma}} \leqslant C\|u\|_{C_H^{0,\alpha}}$. For the decay, we put

$$H^{-\sigma}u(x) = \int_{\overline{B}} (u(z) - u(x))F_{-\sigma}(x, z)\ dz + \int_{\overline{B}^c} (u(z) - u(x))F_{-\sigma}(x, z)\ dz + u(x)H^{-\sigma}1(x),$$

where $\overline{B} = B\left(x, \frac{1}{1+|x|}\right)$. We have

$$\int_{\overline{B}} |u(z) - u(x)|\, F_{-\sigma}(x, z)\ dz \leqslant C[u]_{C^{0,\alpha}} \int_{\overline{B}} \frac{|z - x|^\alpha}{|x - z|^{n-2\sigma}}\ dz \leqslant [u]_{C^{0,\alpha}} \frac{C}{(1 + |x|)^{\alpha+2\sigma}},$$

and

$$\left|\int_{\overline{B}^c}(u(z)-u(x))F_{-\sigma}(x,z)\ dz + u(x)H^{-\sigma}1(x)\right| \leqslant \int_{\overline{B}^c}|u(z)|\,F_{-\sigma}(x,z)\ dz \\ +2\,|u(x)|\,\left|H^{-\sigma}1(x)\right|.$$

To estimate the very last integral we can proceed as we did for the integral part of the operator T in the proof of Proposition 5.8, by splitting the integral in annulus and we skip the details. By Lemma 5.24, $|u(x)H^{-\sigma}1(x)| \leqslant C[u]_{M^\alpha}(1+|x|)^{-(\alpha+2\sigma)}$. This concludes the proof of Theorem B *(B1)*.

In order to prove *(B2)* we observe that by using the boundedness of the first order Hermite-Riesz transforms on $C_H^{0,\alpha}$ we get

$$\|A_i H^{-\sigma}u\|_{C_H^{0,\alpha+2\sigma-1}} = \|\mathcal{R}_i H^{-\sigma+1/2}u\|_{C_H^{0,\alpha+2\sigma-1}} \leqslant C\|H^{-\sigma+1/2}u\|_{C_H^{0,\alpha+2\sigma-1}} \leqslant C\,\|u\|_{C_H^{0,\alpha}},$$

where in the last inequality we applied Theorem A *(A1)* if $-\sigma+1/2>0$ and the case *(B1)* just proved above if $-\sigma+1/2<0$. The case $\sigma=1/2$ is contained in Theorem 5.5.

Under the hypotheses of *(B3)* we have to prove that $A_iA_jH^{-\sigma}u$ belongs to $C_H^{0,\alpha+2\sigma-2}$. But $A_iA_jH^{-\sigma}u=\mathcal{R}_{ij}H^{1-\sigma}u$. Therefore, Theorem A *(A1)* and Theorem 5.5 give the result.

5.5 Computational Lemmas

Lemma 5.19. *For each positive number* a *let*

$$\psi^a_{s,z}(x)=e^{-a\left[s|x+z|^2+\frac{1}{s}|x-z|^2\right]},\qquad x,z\in\mathbb{R}^n,\ \ s\in(0,1).$$

Then,

$$\psi^a_{s,z}(x)\leqslant e^{-\frac{a}{4}|x||x-z|}e^{-\frac{a}{4}\frac{|x-z|^2}{s}}. \tag{5.10}$$

Proof. We have

$$\psi^a_{s,z}(x)\leqslant e^{-\frac{a}{2}\frac{|x-z|^2}{s}}e^{-\frac{a}{2}\left[s|x+z|^2+\frac{1}{s}|x-z|^2\right]}\leqslant e^{-\frac{a}{2}\frac{|x-z|^2}{s}}e^{-\frac{a}{2}|x-z||x+z|}.$$

The first inequality above is obvious. For the second one we argue as follows: if $|x+z|\leqslant|x-z|$ then it is clearly valid; when $|x-z|<|x+z|$ we minimize the function $\theta(s)=\frac{a}{2}\left[s\,|x+z|^2+\frac{1}{s}\,|x-z|^2\right]$, $s\in(0,1)$, to get $\theta(s)\geqslant\frac{a}{2}\,|x-z|\,|x+z|$. To obtain the desired estimate let us first assume that $x\cdot z>0$. Then $|x+z|\geqslant|x|$ and

$$e^{-\frac{a}{2}\frac{|x-z|^2}{s}}e^{-\frac{a}{2}|x-z||x+z|}\leqslant e^{-\frac{a}{4}\frac{|x-z|^2}{s}}e^{-\frac{a}{4}|x||x-z|}.$$

If $x\cdot z\leqslant0$ then $|x-z|\geqslant|x|$ and

$$e^{-\frac{a}{2}\frac{|x-z|^2}{s}}e^{-\frac{a}{2}|x-z||x+z|}\leqslant e^{-\frac{a}{4}\frac{|x-z|^2}{s}}e^{-\frac{a}{4}\frac{|x||x-z|}{s}}\leqslant e^{-\frac{a}{4}\frac{|x-z|^2}{s}}e^{-\frac{a}{4}|x||x-z|}.$$

Thus, (5.10) follows. □

Proof of Lemma 2.6, Chapter 2. Follows directly from Lemma 5.19. □

Lemma 5.20. *Let $\eta, \rho \in \mathbb{R}$. Then for all $x, z \in \mathbb{R}^n$,*

$$\int_0^1 \left(\frac{1-s}{s}\right)^{n/2} \frac{1}{s^\eta} \; e^{-C[s|x+z|^2 + \frac{1}{s}|x-z|^2]} \; d\mu_\rho(s) \leqslant C e^{-\frac{|x||x-z|}{C}} e^{-\frac{|x-z|^2}{C}} \cdot I_{\eta,\rho}(x,z),$$

where

$$I_{\eta,\rho}(x,z) = \begin{cases} \dfrac{1}{|x-z|^{n+2\eta+2\rho}}, & \textit{if} \;\; n/2+\eta+\rho > 0, \\ 1 + \log\left(\frac{C}{|x-z|^2}\right) \chi_{\{\frac{C}{|x-z|^2} > 1\}}(x-z), & \textit{if} \;\; n/2+\eta+\rho = 0, \\ 1, & \textit{if} \;\; n/2+\eta+\rho < 0. \end{cases}$$

Proof. By (5.10),

$$\begin{aligned} &\int_0^1 \left(\frac{1-s}{s}\right)^{n/2} \frac{1}{s^\eta} \; e^{-C[s|x+z|^2 + \frac{1}{s}|x-z|^2]} \; d\mu_\rho(s) \\ &\leqslant C e^{-\frac{|x||x-z|}{C}} \left(\int_0^{1/2} \frac{1}{s^{n/2+\eta+\rho}} \; e^{-\frac{|x-z|^2}{Cs}} \; \frac{ds}{s} + e^{-\frac{|x-z|^2}{C}} \int_{1/2}^1 (1-s)^{n/2} \; d\mu_\rho(s) \right) \\ &= C e^{-\frac{|x||x-z|}{C}} \left(\frac{C}{|x-z|^{n+2\eta+2\rho}} \int_{\frac{|x-z|^2}{C}}^\infty r^{n/2+\eta+\rho} e^{-2r} \; \frac{dr}{r} + C e^{-\frac{|x-z|^2}{C}} \right) \\ &\leqslant C e^{-\frac{|x||x-z|}{C}} \left(\frac{e^{-\frac{|x-z|^2}{C}}}{|x-z|^{n+2\eta+2\rho}} \int_{\frac{|x-z|^2}{C}}^\infty r^{n/2+\eta+\rho} e^{-r} \; \frac{dr}{r} + e^{-\frac{|x-z|^2}{C}} \right) =: C e^{-\frac{|x||x-z|}{C}} \cdot I. \end{aligned}$$

If $n/2+\eta+\rho > 0$,

$$I \leqslant \frac{e^{-\frac{|x-z|^2}{C}}}{|x-z|^{n+2\eta+2\rho}} \int_0^\infty r^{n/2+\eta+\rho} e^{-r} \; \frac{dr}{r} + e^{-\frac{|x-z|^2}{C}} \leqslant \frac{C}{|x-z|^{n+2\eta+2\rho}} \; e^{-\frac{|x-z|^2}{C}}.$$

Suppose now that $n/2+\eta+\rho \leqslant 0$. Consider two cases: if $\frac{|x-z|^2}{C} \geqslant 1$ then,

$$\begin{aligned} I &\leqslant e^{-\frac{|x-z|^2}{C}} \left[\frac{1}{|x-z|^{n+2\eta+2\rho}} \int_1^\infty r^{n/2+\eta+\rho} e^{-r} \frac{dr}{r} + 1 \right] \\ &= C e^{-\frac{|x-z|^2}{C}} \left[\frac{C}{|x-z|^{n+2\eta+2\rho}} + 1 \right] \leqslant C, \end{aligned}$$

and when $\frac{|x-z|^2}{C} < 1$ we have

$$\begin{aligned} I &\leqslant e^{-\frac{|x-z|^2}{C}} \left[\frac{1}{|x-z|^{n+2\eta+2\rho}} \left(\int_{\frac{|x-z|^2}{C}}^1 r^{n/2+\eta+\rho} \; \frac{dr}{r} + C \right) + 1 \right] \\ &\leqslant C e^{-\frac{|x-z|^2}{C}} \cdot \begin{cases} 1 + \log\left(\frac{C}{|x-z|^2}\right), & \text{if} \;\; n/2+\eta+\rho = 0, \\ 1, & \text{if} \;\; n/2+\eta+\rho < 0. \end{cases} \end{aligned}$$

□

Lemma 5.21. *Denote by* $\mathcal{F}$ *any of the kernels* $F_\sigma(x,z)$ *(defined in* (4.6)*) or* $F_{\pm 2k,\sigma}(x,z)$ *(given in Theorems 5.12 and 5.13). Then,*

$$|\mathcal{F}(x,z)| \leqslant \frac{C}{|x-z|^{n+2\sigma}}\ e^{-\frac{|x||x-z|}{C}} e^{-\frac{|x-z|^2}{C}}, \tag{5.11}$$

for all $x, z \in \mathbb{R}^n$ *and*

$$|\mathcal{F}(x_1,z) - \mathcal{F}(x_2,z)| \leqslant \frac{C\,|x_1-x_2|}{|x_2-z|^{n+1+2\sigma}}\ e^{-\frac{|z||x_2-z|}{C}} e^{-\frac{|x_2-z|^2}{C}}, \tag{5.12}$$

for all $x_1, x_2 \in \mathbb{R}^n$ *such that* $|x_1 - z| > 2\,|x_1 - x_2|$.

Proof. Let us first consider $\mathcal{F} = F_\sigma$. The estimate in (5.11) for $\mathcal{F} = F_\sigma$ is already stated in Lemma 4.8 of Chapter 4. Nevertheless, we can prove it here quickly by using, in (5.4), Lemma 5.20 with $\eta = 0$ and $\rho = \sigma$. To get (5.12) we observe that, by the Mean Value Theorem,

$$\left|G_{t(s)}(x_1,z) - G_{t(s)}(x_2,z)\right| \leqslant C\,|x_1-x_2| \left(\frac{1-s}{s}\right)^{n/2} \frac{1}{s^{1/2}}\ e^{-\frac{1}{8}\left[s|\xi+z|^2 + \frac{1}{s}|\xi-z|^2\right]}, \tag{5.13}$$

for some $\xi = (1-\lambda)x_1 + \lambda x_2$, $\lambda \in [0,1]$. Then, by Lemma 5.20 with $\eta = 1/2$ and $\rho = \sigma$,

$$\begin{aligned}
|F_\sigma(x_1,z) - F_\sigma(x_2,z)| &\leqslant |x_1-x_2| \sup_{\{\xi=(1-\lambda)x_1+\lambda x_2:\lambda\in[0,1]\}} |\nabla_x F_\sigma(\xi,z)| \\
&\leqslant C\,|x_1-x_2| \sup_{\xi} \int_0^1 \left(\frac{1-s}{s}\right)^{n/2} \frac{1}{s^{1/2}}\ e^{-\frac{1}{8}\left[s|\xi+z|^2+\frac{1}{s}|\xi-z|^2\right]}\ d\mu_\sigma(s) \\
&\leqslant C\,|x_1-x_2| \sup_{\xi} \frac{1}{|\xi-z|^{n+1+2\sigma}}\ e^{-\frac{|z||\xi-z|}{C}} e^{-\frac{|\xi-z|^2}{C}} \\
&\leqslant C \frac{|x_1-x_2|}{|x_2-z|^{n+1+2\sigma}}\ e^{-\frac{|z||x_2-z|}{C}} e^{-\frac{|x_2-z|^2}{C}},
\end{aligned}$$

where in the last inequality we used that $|\xi - z| \geqslant \frac{1}{2}\,|x_2 - z|$, since $|x_1 - z| > 2\,|x_1 - x_2|$. In a similar way we can prove both estimates for $\mathcal{F} = F_{2k,\sigma}$ because $0 \leqslant F_{2k,\sigma}(x,z) \leqslant F_\sigma(x,z)$. Note that, by (5.6)-(5.7), up to a multiplicative constant we have

$$\begin{aligned}
\left|F_{-2k,\sigma}(x,z)\right| &= \Bigg| \int_0^{1/2} \left(\frac{1+s}{1-s}\right)^k G_{t(s)}(x,z)\ d\mu_\sigma(s) \\
&\qquad + \int_{1/2}^1 \left(\frac{1+s}{1-s}\right)^k \left[G_{t(s)}(x,z) - \phi_{2k}(x)\right] d\mu_\sigma(s) \Bigg| \\
&\leqslant C \left[F_\sigma(x,z) + e^{-\frac{|x||x-z|}{C}} e^{-\frac{|x-z|^2}{C}} \int_{1/2}^1 \left(\frac{1-s}{1+s}\right)^{n/2} d\mu_\sigma(s) \right].
\end{aligned}$$

Therefore, (5.11) is valid for $F_{-2k,\sigma}$. By (5.13),

$$\int_0^{1/2}\left(\frac{1+s}{1-s}\right)^k \left|G_{t(s)}(x_1,z)-G_{t(s)}(x_2,z)\right| d\mu_\sigma(s)$$
$$\leqslant C\,|x_1-x_2|\sup_{\xi}\int_0^1\left(\frac{1-s}{s}\right)^{n/2}\frac{1}{s^{1/2}}\; e^{-\frac{1}{8}[s|\xi+z|^2+\frac{1}{s}|\xi-z|^2]}\; d\mu_\sigma(s). \tag{5.14}$$

Recall the definition of $M_r(x,z)$ given in (5.5). It can be checked that

$$\left|\frac{d^k}{dr^k}\nabla_x M_r(x,z)\right| \leqslant Ce^{-\frac{|x||x-z|}{C}}e^{-\frac{|x-z|^2}{C}},\qquad r\in(0,1/3).$$

Thus, by Taylor's formula,

$$\left|\nabla_x\left[G_{t(s)}(x,z)-\phi_{2k}(x,z,s)\right]\right| \leqslant C\left(\frac{1-s}{1+s}\right)^{k+n/2} e^{-\frac{|x||x-z|}{C}}e^{-\frac{|x-z|^2}{C}},\qquad s\in(1/2,1), \tag{5.15}$$

and consequently, when $|x_1-z|>2\,|x_1-x_2|$,

$$\int_{1/2}^{1}\left(\frac{1+s}{1-s}\right)^k \left|(G_{t(s)}(x_1,z)-\phi_{2k}(x_1,z,s))-(G_{t(s)}(x_2,z)-\phi_{2k}(x_2,z,s))\right| d\mu_\sigma(s)$$
$$\leqslant C\,|x_1-x_2|\sup_{\xi}\int_{1/2}^1\left(\frac{1+s}{1-s}\right)^k\left|\nabla_x\left[G_{t(s)}(\xi,z)-\phi_{2k}(\xi,z,s)\right]\right|\; d\mu_\sigma(s)$$
$$\leqslant C\,|x_1-x_2|\sup_{\xi} e^{-\frac{|z||\xi-z|}{C}}e^{-\frac{|\xi-z|^2}{C}} \leqslant C\,|x_1-x_2|\,e^{-\frac{|z||x_2-z|}{C}}e^{-\frac{|x_2-z|^2}{C}}. \tag{5.16}$$

Pasting estimates (5.14) and (5.16), (5.12) follows for $\mathcal{F}=F_{-2k,\sigma}$. □

Lemma 5.22. *Denote by* $\mathcal{B}$ *any of the functions* B_σ *or* $B_{\pm 2k,\sigma}$ *defined in* (4.6) *and in Theorems 5.12 and 5.13. Then* $\mathcal{B}\in C^\infty(\mathbb{R}^n)$ *and for all* $x\in\mathbb{R}^n$,

$$|\mathcal{B}(x)|\leqslant C\left(1+|x|^{2\sigma}\right),\qquad \textit{and}\qquad |\nabla\mathcal{B}(x)|\leqslant C\begin{cases}|x|, & \textit{if } |x|\leqslant 1,\\ |x|^{2\sigma-1}, & \textit{if } |x|>1.\end{cases} \tag{5.17}$$

Proof. The first inequality in (5.17) for the case $\mathcal{B}=B_\sigma$ is contained in Lemma 4.8 of Chapter 4. Identity (2.29) gives

$$B_\sigma(x)=\frac{1}{\Gamma(-\sigma)}\int_0^1\left[\left(\frac{1-s^2}{1+s^2}\right)^{n/2} e^{-\frac{s}{1+s^2}|x|^2}-1\right]\; d\mu_\sigma(s).$$

We differentiate under the integral sign to see that $B_\sigma\in C^\infty(\mathbb{R}^n)$ and

$$|\nabla B_\sigma(x)| = \left|\frac{2x}{\Gamma(-\sigma)}\int_0^1\frac{s}{1+s^2}\left(\frac{1-s^2}{1+s^2}\right)^{n/2} e^{-\frac{s}{1+s^2}|x|^2}\; d\mu_\sigma(s)\right|$$
$$\leqslant C\,|x|\int_0^1 se^{-\frac{s}{2}|x|^2}d\mu_\sigma(s) =: \widetilde{I}(x).$$

By (4.9),

$$\widetilde{I}(x) \leqslant C\,|x| \left[\int_0^{1/2} e^{-\frac{s}{2}|x|^2}\,\frac{ds}{s^\sigma} + e^{-\frac{|x|^2}{C}} \int_{1/2}^1 \,d\mu_\sigma(s)\right] = C\,|x|^{2\sigma-1} \int_0^{\frac{|x|^2}{4}} e^{-r}\,\frac{dr}{r^\sigma} + C\,|x|\,e^{-\frac{|x|^2}{C}}.$$

If $|x| \leqslant 1$,

$$\int_0^{\frac{|x|^2}{4}} e^{-r}\,\frac{dr}{r^\sigma} \leqslant \int_0^{\frac{|x|^2}{4}} \frac{dr}{r^\sigma} = C\,|x|^{2-2\sigma},$$

and if $|x| > 1$,

$$\int_0^{\frac{|x|^2}{4}} e^{-r}\,\frac{dr}{r^\sigma} \leqslant \int_0^{1/4} \frac{dr}{r^\sigma} + \int_{1/4}^{\frac{|x|^2}{4}} e^{-r}\,dr = C - e^{-\frac{|x|^2}{C}} \leqslant C.$$

Hence, (5.17) for $\mathcal{B} = B_\sigma$ is proved.

We can write

$$\begin{aligned} B_{2k,\sigma}(x) = {}& \frac{1}{\Gamma(-\sigma)} \int_0^1 \left[\left(\frac{1-s}{1+s}\right)^k \left(\frac{1-s^2}{1+s^2}\right)^{n/2} - 1\right] e^{-\frac{s}{1+s^2}|x|^2}\,d\mu_\sigma(s) \\ & + \frac{1}{\Gamma(-\sigma)} \int_0^1 \left(e^{-\frac{s}{1+s^2}|x|^2} - 1\right)\,d\mu_\sigma(s) =: I + II. \end{aligned}$$

The bounds for I and II can be deduced exactly as in the proof of Lemma 4.8 of Chapter 4. On the other hand, observe that $\left|\nabla B_{2k,\sigma}(x)\right| \leqslant \widetilde{I}(x)$. Thus, (5.17) follows with $\mathcal{B} = B_{2k,\sigma}$.

When $\mathcal{B} = B_{-2k,\sigma}$,

$$\begin{aligned} B_{-2k,\sigma}(x) = {}& \frac{1}{\Gamma(-\sigma)} \int_0^{1/2} \left[\left(\frac{1+s}{1-s}\right)^k \left(\frac{1-s^2}{1+s^2}\right)^{n/2} e^{-\frac{s}{1+s^2}|x|^2} - 1\right]\,d\mu_\sigma(s) \\ & + \frac{1}{\Gamma(-\sigma)} \int_{1/2}^1 \left[\left(\frac{1+s}{1-s}\right)^k \int_{\mathbb{R}^n} \left(G_{t(s)}(x,z) - \phi_{2k}(x,z,s)\right)\,dz - 1\right]\,d\mu_\sigma(s) \\ & =: III + IV. \end{aligned}$$

If up to the constant $1/\Gamma(-\sigma)$ we write III as

$$\int_0^{1/2} \left[\left(\frac{1+s}{1-s}\right)^k \left(\frac{1-s^2}{1+s^2}\right)^{n/2} - 1\right]\,d\mu_\sigma(s) + \int_0^{1/2} \left(e^{-\frac{s}{1+s^2}|x|^2} - 1\right)\,d\mu_\sigma(s),$$

then we can handle these two terms as we did for I and II above to get $|III| \leqslant C\left(1 + |x|^{2\sigma}\right)$. By (5.6)-(5.7), $|IV| \leqslant C$. For the gradient of $B_{-2k,\sigma}$ similar estimates to those used for ∇B_σ can be applied for the term $\nabla_x III$. Finally, (5.15) implies that $|\nabla_x IV| \leqslant C$. The proof is complete. □

The following Lemma contains a small refinement of the estimate for the kernel $F_{-\sigma}(x,z)$ given in [17, Proposition 2].

Lemma 5.23. *Take* $\sigma \in (0,1]$. *Then for all* $x, z \in \mathbb{R}^n$,

$$0 \leqslant F_{-\sigma}(x,z) \leqslant C \begin{cases} \frac{1}{|x-z|^{n-2\sigma}} \; e^{-\frac{|x||x-z|}{C}} e^{-\frac{|x-z|^2}{C}}, & \text{if } n > 2\sigma, \\ e^{-\frac{|x||x-z|}{C}} e^{-\frac{|x-z|^2}{C}} \left[1 + \log\left(\frac{C}{|x-z|^2}\right) \chi_{\{\frac{C}{|x-z|^2}>1\}}(x-z)\right], & \text{if } n = 2\sigma, \\ e^{-\frac{|x||x-z|}{C}} e^{-\frac{|x-z|^2}{C}}, & \text{if } n < 2\sigma. \end{cases} \tag{5.18}$$

If $F(x,z)$ *denotes any of the kernels* $\nabla_x F_{-\sigma}(x,z)$, $x_i F_{-\sigma}(x,z)$ *or* $z_i F_{-\sigma}(x,z)$ *then,*

$$|F(x,z)| \leqslant C \begin{cases} \frac{1}{|x-z|^{n+1-2\sigma}} \; e^{-\frac{|x||x-z|}{C}} e^{-\frac{|x-z|^2}{C}}, & \text{if } n > 2\sigma - 1, \\ e^{-\frac{|x||x-z|}{C}} e^{-\frac{|x-z|^2}{C}} \left[1 + \log\left(\frac{C}{|x-z|^2}\right) \chi_{\{\frac{C}{|x-z|^2}>1\}}(x-z)\right], & \text{if } n = 2\sigma - 1. \end{cases} \tag{5.19}$$

Moreover, when $|x_1 - z| \geqslant 2\,|x_1 - x_2|$,

$$\begin{aligned} &|F_{-\sigma}(x_1,z) - F_{-\sigma}(x_2,z)| \\ &\leqslant C\,|x_1 - x_2| \begin{cases} \frac{1}{|x_2-z|^{n+1-2\sigma}} \; e^{-\frac{|z||x_2-z|}{C}} e^{-\frac{|x_2-z|^2}{C}}, & \text{if } \sigma \neq 1, \\ e^{-\frac{|z||x_2-z|}{C}} e^{-\frac{|x_2-z|^2}{C}} \left[1 + \log\left(\frac{C}{|x-z|^2}\right) \chi_{\{\frac{C}{|x-z|^2}>1\}}(x-z)\right], & \text{if } \sigma = 1, \end{cases} \end{aligned} \tag{5.20}$$

and

$$|F(x_1,z) - F(x_2,z)| \leqslant C \frac{|x_1 - x_2|}{|x_2 - z|^{n+2-2\sigma}} \; e^{-\frac{|z||x_2-z|}{C}} e^{-\frac{|x_2-z|^2}{C}}. \tag{5.21}$$

Proof. By (2.27),

$$\begin{aligned} F_{-\sigma}(x,z) &= \frac{1}{\Gamma(\sigma)} \int_0^1 G_{t(s)}(x,z)\; d\mu_{-\sigma}(s) \\ &= C \int_0^1 \left(\frac{1-s^2}{s}\right)^{n/2} e^{-\frac{1}{4}[s|x+z|^2 + \frac{1}{s}|x-z|^2]}\; d\mu_{-\sigma}(s). \end{aligned} \tag{5.22}$$

Then apply Lemma 5.20 with $\eta = 0$ and $\rho = -\sigma$ to get (5.18). Differentiation with respect to x inside the integral in (5.22) gives

$$|\nabla_x F_{-\sigma}(x,z)| \leqslant C \int_0^1 \left(\frac{1-s}{s}\right)^{n/2} \frac{1}{s^{1/2}} \; e^{-\frac{1}{8}[s|x+z|^2 + \frac{1}{s}|x-z|^2]}\; d\mu_{-\sigma}(s), \tag{5.23}$$

and then Lemma 5.20 with $\eta = 1/2$ and $\rho = -\sigma$ implies (5.19) when $F(x,z) = \nabla_x F_{-\sigma}(x,z)$. Take $x, z \in \mathbb{R}^n$. If $x \cdot z \geqslant 0$ then $|x| \leqslant |x+z|$ and in this situation $|x|\, F_{-\sigma}(x,z)$ is bounded by the RHS of (5.23). If $x \cdot z < 0$ we have $|x| \leqslant |x-z|$ and in this case

$$|x|\, F_{-\sigma}(x,z) \leqslant C\,|x-z|\, e^{-\frac{1}{8}|x-z|^2} \int_0^1 \left(\frac{1-s}{s}\right)^{n/2} e^{-\frac{1}{8}[s|x+z|^2 + \frac{1}{s}|x-z|^2]}\; d\mu_{-\sigma}(s).$$

Therefore, by Lemma 5.20, we obtain (5.19) for $F(x,z) = x_i F_{-\sigma}(x,z)$. The same reasoning applies to $F(x,z) = z_i F_{-\sigma}(x,z)$ since $|z| \leqslant |z-x| + |x|$. To derive (5.20) we follow the proof of (5.12) in Lemma 5.21 with $-\sigma$ in the place of σ, and we use Lemma 5.20. Estimate (5.21) for $F(x,z) = \nabla_x F_{-\sigma}(x,z)$ can be deduced by using the Mean Value Theorem and Lemma 5.20, since

$$\partial^2_{x_i,x_j} F_{-\sigma}(x,z) = \frac{1}{\Gamma(\sigma)} \int_0^1 \left(\frac{1-s^2}{4\pi s}\right)^{n/2} e^{-\frac{1}{4}[s|x+z|^2 + \frac{1}{s}|x-z|^2]} \times$$
$$\times \left[\left(-\frac{s}{2}(x_i+z_i) - \frac{1}{2s}(x_i - z_i)\right)\left(-\frac{s}{2}(x_j+z_j) - \frac{1}{2s}(x_j - z_j)\right) - \delta_{ij}\left(\frac{s}{2} + \frac{1}{2s}\right)\right] \, d\mu_{-\sigma}(s),$$

gives that

$$\left|D^2_x F_{-\sigma}(x,z)\right| \leqslant C \int_0^1 \left(\frac{1-s}{s}\right)^{n/2} \frac{1}{s} \, e^{-C[s|x+z|^2 + \frac{1}{s}|x-z|^2]} \, d\mu_{-\sigma}(s). \tag{5.24}$$

Similar ideas can also be used to prove (5.21) when $F(x,z)$ is either $x_i F_{-\sigma}(x,z)$ or $z_i F_{-\sigma}(x,z)$. We skip the details. □

Lemma 5.24. *The function* $H^{-\sigma}1$ *belongs to the space* $C^\infty(\mathbb{R}^n)$ *and*

$$\left|H^{-\sigma}1(x)\right| \leqslant \frac{C}{(1+|x|)^{2\sigma}}, \qquad \textit{and} \qquad \left|\nabla H^{-\sigma}1(x)\right| \leqslant \frac{C}{(1+|x|)^{1+2\sigma}}.$$

Proof. Observe that (2.27) applied to (5.9) gives

$$H^{-\sigma}1(x) = \frac{1}{\Gamma(\sigma)} \int_0^1 e^{-t(s)H}1(x) \, d\mu_{-\sigma}(s) = C \int_0^1 \left(\frac{1-s^2}{1+s^2}\right)^{n/2} e^{-\frac{s}{1+s^2}|x|^2} \, d\mu_{-\sigma}(s). \tag{5.25}$$

Since

$$\left|\nabla_x e^{-t(s)H}1(x)\right| = 2\,|x| \, \frac{s}{1+s^2} \left(\frac{1-s^2}{1+s^2}\right)^{n/2} e^{-\frac{s}{1+s^2}|x|^2} \leqslant C s^{1/2} e^{-\frac{s}{C}|x|^2}, \tag{5.26}$$

differentiation inside the integral sign in (5.25) is justified. By repeating this argument we obtain $H^{-\sigma}1 \in C^\infty(\mathbb{R}^n)$. To study the size of $H^{-\sigma}1$ note that we can restrict to the case $|x| > 1$ because $H^{-\sigma}1$ is a continuous function. By (4.9),

$$\begin{aligned}\int_0^{1/2} \left(\frac{1-s^2}{1+s^2}\right)^{n/2} e^{-\frac{s}{1+s^2}|x|^2} d\mu_{-\sigma}(s) &\leqslant C \int_0^{1/2} e^{-\frac{s}{C}|x|^2} \frac{ds}{s^{1-\sigma}} \\ &= C\,|x|^{-2\sigma} \int_0^{\frac{|x|^2}{2C}} e^{-r} \frac{dr}{r^{1-\sigma}} \leqslant C\,|x|^{-2\sigma},\end{aligned}$$

and

$$\int_{1/2}^1 \left(\frac{1-s^2}{1+s^2}\right)^{n/2} e^{-\frac{s}{1+s^2}|x|^2} d\mu_{-\sigma}(s) \leqslant Ce^{-C|x|^2} \int_{1/2}^1 \frac{(1-s)^{n/2-1}}{(-\log(1-s))^{1-\sigma}} \, ds = Ce^{-C|x|^2}.$$

Plugging these two estimates into (5.25) we get the bound for $H^{-\sigma}1$. For the growth of the gradient, (5.26) and similar estimates as above can be used to obtain the result. □

Lemma 5.25. *For* $1 \leqslant |i|, |j| \leqslant n$, *denote by* $R(x,z)$ *any of the following kernels:* $\partial^2_{x_i,x_j} F_{-1}(x,z)$, $x_i \partial_{x_j} F_{-1}(x,z)$ *or* $x_i x_j F_{-1}(x,z)$. *Then*

$$|R(x,z)| \leqslant \frac{C}{|x-z|^n} \, e^{-\frac{|x||x-z|}{C}} e^{-\frac{|x-z|^2}{C}}, \tag{5.27}$$

and when $|x_1 - z| \geqslant 2\,|x_1 - x_2|$,

$$|R(x_1,z) - R(x_2,z)| \leqslant C \frac{|x_1 - x_2|}{|x_2 - z|^{n+1}} \, e^{-\frac{|z||x_2-z|}{C}} e^{-\frac{|x_2-z|^2}{C}}. \tag{5.28}$$

As a consequence, the kernels $R_{ij}(x,z) = A_i A_j F_{-1}(x,z)$ *of the second order Hermite-Riesz transforms also satisfy these size and smoothness estimates.*

Proof. We put $\sigma = 1$ in (5.24) and we use Lemma 5.20 with $\eta = 1$ and $\rho = -1$ to obtain the desired estimate for $D^2_x F_{-1}$. From (5.23),

$$\left| x_i \partial_{x_j} F_{-1}(x,z) \right| \leqslant C\,|x| \int_0^1 \left(\frac{1-s}{s} \right)^{n/2} \frac{1}{s^{1/2}} \, e^{-\frac{1}{8}\left[s|x+z|^2 + \frac{1}{s}|x-z|^2\right]} \, d\mu_{-1}(s), \tag{5.29}$$

If $|x| \leqslant 2$ then Lemma 5.20 with $\eta = 1/2$ and $\rho = -1$ applied to (5.29) gives

$$\left| x_i \partial_{x_j} F_{-1}(x,z) \right| \leqslant \frac{C}{|x-z|^{n-1}} \, e^{-\frac{|x||x-z|}{C}} e^{-\frac{|x-z|^2}{C}}.$$

Assume that $|x| > 2$ in (5.29). Consider first the case $|x| < 2\,|x-z|$. Then by Lemma 5.20,

$$\begin{aligned} \left| x_i \partial_{x_j} F_{-1}(x,z) \right| &\leqslant C \int_0^1 \left(\frac{1-s}{s} \right)^{n/2} e^{-C\left[s|x+z|^2 + \frac{1}{s}|x-z|^2\right]} \, d\mu_{-1}(s) \\ &\leqslant \frac{C}{|x-z|^{n-2}} \, e^{-\frac{|x||x-z|}{C}} e^{-\frac{|x-z|^2}{C}}. \end{aligned}$$

In the other case, namely $|x| \geqslant 2\,|x-z|$, we use the fact that $|x| > 2$ to see that $|x+z|^2 = 2\,|x|^2 - |x-z|^2 + 2\,|z|^2 > |x|^2$. Hence,

$$\begin{aligned} \left| x_j \partial_{x_j} F_{-1}(x,z) \right| &\leqslant C \int_0^1 \left(\frac{1-s}{s} \right)^{n/2} \frac{1}{s^{1/2}} \, e^{-C\left[s|x+z|^2 + \frac{1}{s}|x-z|^2\right]} \, d\mu_{-1}(s) \\ &\leqslant \frac{C}{|x-z|^{n}} \, e^{-\frac{|x||x-z|}{C}} e^{-\frac{|x-z|^2}{C}}. \end{aligned}$$

Collecting terms we have (5.27) for $R(x,z) = x_j \partial_{x_j} F_{-1}(x,z)$. Finally, to obtain (5.27) with $R(x,z) = x_i x_j F_{-1}(x,z)$ note that by (5.22),

$$\left| x_i x_j F_{-1}(x,z) \right| \leqslant C\,|x|^2 \int_0^1 \left(\frac{1-s}{s} \right)^{n/2} e^{-\frac{1}{8}\left[s|x+z|^2 + \frac{1}{s}|x-z|^2\right]} \, d\mu_{-1}(s),$$

and consider the cases $|x| \leqslant 2$ and $|x| > 2$ as before. In the second situation assume first that $|x| \leqslant 2|x-z|$ and then that $|x| \geqslant 2|x-z|$ (which implies $|x| \leqslant |x+z|$) and use the method of the proof given for $x_j \partial_{x_i} F_{-1}$ above.

To prove (5.28) we can use the Mean Value Theorem and Lemma 5.19 (see the proof of (5.12) and (5.21)). We omit the details. □

Lemma 5.26. *Denote by* $K(x,z)$ *any of the functions* $|x|^{2\sigma} F_{-\sigma}(x,z)$, $|z|^{2\sigma} F_{-\sigma}(x,z)$, $0 < \sigma \leqslant 1$, *or the kernel* $x_i \partial_{x_j} F_{-1}(x,z)$. *Then*

$$\sup_x \int_{\mathbb{R}^n} |K(x,z)| \, dz \leqslant C.$$

Proof. Consider the function $|x|^{2\sigma} F_{-\sigma}(x,z)$. If $|x| \leqslant 2$ then, by (5.18),

$$|x|^{2\sigma} \int_{\mathbb{R}^n} F_{-\sigma}(x,z) \, dz \leqslant C.$$

If $|x| > 2$ we consider two regions of integration: $|x| < |x-z|$ and $|x| \geqslant |x-z|$. In the first region, by Lemma 5.20,

$$|x|^{2\sigma} F_{-\sigma}(x,z) \leqslant C \int_0^1 \left(\frac{1-s}{s}\right)^{n/2} \frac{1}{s^{-\sigma}} \, e^{-C[s|x+z|^2 + \frac{1}{s}|x-z|^2]} \, d\mu_{-\sigma}(s) \leqslant C\Phi(x-z),$$

with $\Phi \in L^1(\mathbb{R}^n)$. To study the second region of integration, namely $|x| \geqslant |x-z|$, we use the fact that $|x+z| > |x|$ and we split the integral defining $F_{-\sigma}$ into two intervals: $(0,1/2)$ and $(1/2,1)$. To estimate the part of the integral over the interval $(0,1/2)$ we note that, by using (4.9) and three different changes of variables, we have

$$\begin{aligned}
\int_{|x|\geqslant|x-z|} \int_0^{1/2} G_{t(s)}(x,z) \, d\mu_{-\sigma}(s) \, dz &\leqslant C \int_{|x|\geqslant|x-z|} \int_0^{1/2} \frac{1}{s^{n/2}} \, e^{-\frac{1}{4}[s|x|^2 + \frac{1}{s}|x-z|^2]} \, \frac{ds}{s^{1-\sigma}} \, dz \\
&= C\,|x|^{n-2\sigma} \int_{|x|\geqslant|x-z|} \int_0^{\frac{|x|^2}{2}} \frac{e^{-\frac{1}{4}[r + \frac{1}{r}|x|^2|x-z|^2]}}{r^{n/2-\sigma+1}} \, dr \, dz \\
&= C\,|x|^{-2\sigma} \int_{|x|^2\geqslant|w|} \int_0^{\frac{|x|^2}{2}} \frac{1}{r^{n/2}} \, e^{-\frac{1}{4}[r + \frac{1}{r}|w|^2]} \, \frac{dr}{r^{1-\sigma}} \, dw \\
&= C\,|x|^{-2\sigma} \int_0^{|x|^2} \int_0^{\frac{|x|^2}{2}} \frac{1}{r^{n/2}} \, e^{-\frac{1}{4}\left[r + \frac{\rho^2}{r}\right]} \, \frac{dr}{r^{1-\sigma}} \, \rho^{n-1} \, d\rho \\
&\leqslant C\,|x|^{-2\sigma} \int_0^\infty \frac{e^{-\frac{r}{4}}}{r^{n/2-\sigma}} \left[\int_0^\infty e^{-\frac{\rho^2}{4r}} \rho^n \, \frac{d\rho}{\rho}\right] \frac{dr}{r} \\
&= C\,|x|^{-2\sigma} \left[\int_0^\infty e^{-\frac{r}{4}} r^\sigma \, \frac{dr}{r}\right] \left[\int_0^\infty e^{-t} t^{n/2} \, \frac{dt}{t}\right] \\
&= C\,|x|^{-2\sigma}.
\end{aligned}$$

The integral over the interval $(1/2,1)$ is bounded by

$$|x|^{2\sigma}\int_{1/2}^{1}(1-s)^{n/2}e^{-C|x|^2}e^{-\frac{|x-z|^2}{C}}\ d\mu_{-\sigma}(s)\leqslant Ce^{-\frac{|x-z|^2}{C}}\in L^1(\mathbb{R}^n).$$

Hence we get the conclusion for $K(x,z)=|x|^{2\sigma}F_{-\sigma}(x,z)$. To prove the result for the function $|z|^{2\sigma}F_{-\sigma}(x,z)$ observe that $|z|^{2\sigma}\leqslant C\left(|z-x|^{2\sigma}+|x|^{2\sigma}\right)$, so we can apply the estimates above. When $F(x,z)=x_i\partial_{x_j}F_{-1}(x,z)$ we can argue as we did for $|x|^{2\sigma}F_{-\sigma}(x,z)$ above, because of (5.29). ☐

Lemma 5.27. *For all* $1\leqslant|i|\leqslant n$ *and* $0<r_1<r_2\leqslant\infty$,

$$\sup_x\left|\int_{r_1<|x-z|\leqslant r_2}A_iF_{-1/2}(x,z)\ dz\right|\leqslant C,$$

where $C>0$ *is independent of* r_1 *and* r_2.

Proof. By estimate (5.19) given in Lemma 5.23 it is enough to consider $r_2<1$. From Lemma 5.26 with $\sigma=1/2$ we have that

$$\int_{\mathbb{R}^n}x_iF_{-1/2}(x,z)\ dz\leqslant C.$$

We can write

$$\int_{r_1<|x-z|<r_2}\partial_{x_i}F_{-1/2}(x,z)\ dz=\int_{r_1<|x-z|<r_2}I(x,z)\ dz+\int_{r_1<|x-z|<r_2}II(x,z)\ dz,$$

where

$$I(x,z):=\frac{1}{\Gamma(1/2)}\int_0^1\left(\frac{1-s^2}{4\pi s}\right)^{n/2}e^{-\frac14\left[s|x+z|^2+\frac1s|x-z|^2\right]}\left(-\frac{s}{2}(x_i+z_i)\right)\ d\mu_{-1/2}(s).$$

Lemma 5.20 shows that

$$|I(x,z)|\leqslant C\int_0^1\left(\frac{1-s}{s}\right)^{n/2}\frac{1}{s^{-1/2}}\ e^{-\frac14\left[s|x+z|^2+\frac1s|x-z|^2\right]}\ d\mu_{-1/2}(s)\leqslant\Phi(x-z),$$

for some integrable function Φ. To deal with $II(x,z)$ we consider the integral

$$\widetilde{II}(x,z)=\frac{1}{\Gamma(1/2)}\int_0^1\left(\frac{1-s^2}{4\pi s}\right)^{n/2}e^{-\frac14\left[s|2x|^2+\frac1s|x-z|^2\right]}\frac{-(x_i-z_i)}{2s}\ d\mu_{-1/2}(s),$$

which verifies

$$\left|\int_{r_1<|x-z|<r_2}\widetilde{II}(x,z)\ dz\right|=0.$$

Therefore, by applying the Mean Value Theorem and some argument parallel to the one used in the proof of Lemma 5.20,

$$\begin{aligned}\left|\mathrm{II}(x,z)-\widetilde{\mathrm{II}}(x,z)\right| &\leqslant C\int_0^1\left(\frac{1-s}{s}\right)^{n/2}\frac{1}{s^{1/2}}\,e^{-\frac{|x-z|^2}{Cs}}\left|e^{-\frac{1}{4}s|x+z|^2}-e^{-\frac{1}{4}s|2x|^2}\right|\,d\mu_{-1/2}(s)\\ &\leqslant C\int_0^1\left(\frac{1-s}{s}\right)^{n/2}e^{-\frac{|x-z|^2}{Cs}}\,d\mu_{-1/2}(s)\leqslant\Psi(x-z),\end{aligned}$$

for some $\Psi\in L^1(\mathbb{R}^n)$. $\square$

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