Regularity theory for a family of Integral-differential operators

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Passive transport and Brownian effect

Passive transport in a flow (dust or pollutant in a river for instance) can be modeled as

$$\partial_t \theta + u \cdot \nabla \theta = \Delta \theta,$$

where θ is the density of the particles, u the velocity of the flow, and $\Delta\theta$ models the brownian effects.

Anomalous diffusion and fractional Laplacians

In case of a turbulent flow, there is a lot of "swirls" which eject the particles making them to "jump".

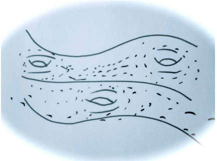

Figure: Passive transport in a turbulent flow

A classical way to model "anomalous" diffusion is using fractional Laplacians:

$$\partial_t \theta + v \cdot \nabla \theta - \Delta^{s/2} \theta = 0.$$

Probabilistic interpretation and integral form

This corresponds to a Levy process (vs Brownian process) involving jumps of size y.
Actually, for $0 < s < 1$

$$\Delta^{s/2}\theta(x) = C_{s,N} \int_{\mathbb{R}^N} \frac{\theta(y) - \theta(x)}{|x-y|^{s+N}} dy.$$

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They can be generalized to non homogenous operators of the form

$$\int_{\mathbb{R}^N} (\theta(y) - \theta(x)) K(x,y)\, dy,$$

with

$$\frac{1}{\Lambda |x-y|^{s+N}} \leq K(x,y) \leq \Lambda \frac{1}{|x-y|^{s+N}}.$$

2D projection and the Half Laplacian

The half Laplacian $\Delta^{1/2}$ corresponds to the "Dirichlet to Neumann" map.

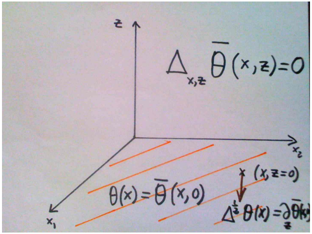


Figure: 2D projection

It is used for modeling in different situation (Dislocation dynamics, Surface Quasi-Geostrophic equation...)

Surface Quasi Geostrophic equation

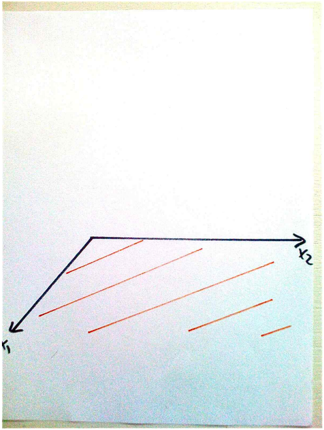
We consider the potential temperature function $\theta : \mathbb{R}^2 \to \mathbb{R}$ at the surface of the earth.

$$\partial_t \theta + u \cdot \nabla \theta = \Delta^{1/2} \theta,$$
$$u = R^{\perp} \theta.$$

$R^{\perp}\theta = (R_2\theta, -R_1\theta)$
where:

$$\widehat{R_i\theta} = \frac{\xi_i}{|\xi|} \hat{\theta}.$$

Note $\operatorname{div} u = 0$ (incompressibility).



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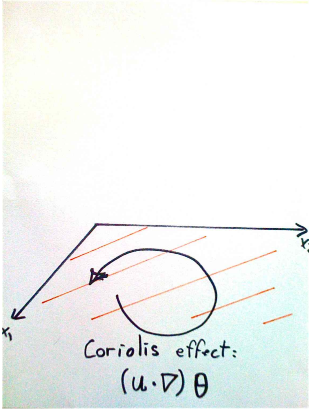

Figure: SQG equation

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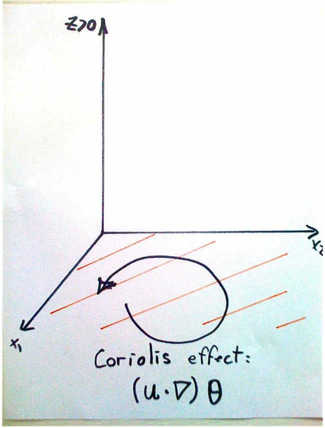

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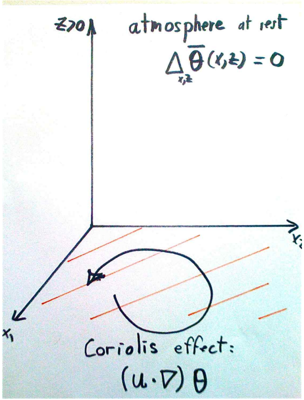

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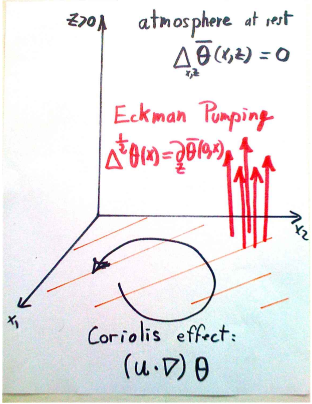

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Random waiting time and fractional in time derivatives

Example of anomalous diffusion due to random waiting time:
Assume that the eddies "trap" the particles for a random (long enough) time.
It gives rise to a memory effect (non-Markovian equations): to propagate the flow at a given time, you need to know the position of the particles and the eddies, but also how long the particles have already been trapped in the eddies.

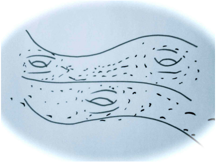

Figure: Passive transport in a turbulent flow

The Caputo fractional in time derivative

This leads to the Caputo fractional in time derivative ($0 < \alpha < 1$):

$$ {}_0D_t^\alpha \theta(t) = C_\alpha \int_0^t \frac{\theta'(s)}{(t-s)^\alpha}\, ds. $$

It can be written in the following integral form:

$$ {}_0D_t^\alpha \theta(t) = C_\alpha \int_{-\infty}^t \frac{\theta(s)-\theta(t)}{(t-s)^{\alpha+1}}\, ds, $$

where $u(x) = u_0$ (initial value) for $x < 0$.

Results

The solutions of the following problems, with initial values bounded in L^2, are C^∞ for $t > 0$.

- (Caffarelli-V., 2010, Annals of Math.) The quasi-Geostrophic equation.
- (Caffarelli -Chan -V., J. of the AMS)

$$\partial_t \theta(t,x) - \int_{\mathbb{R}^N} \phi'(\theta(t,y) - \theta(t,x)) K(y-x) dy = 0,$$

  Where ϕ is strictly convex and K is comparable with a fractional Laplacian:
- (Caffarelli- Allen- V., 2015):

$$_0D_t^\alpha \theta - \int_{\mathbb{R}^N} (\theta(t,y) - \theta(t,x)) K(t,x,y) dy = 0,$$

De Giorgi method

Theorem

(De giorgi 57) Let ϕ be strictly convex. Then any local minimizers of

$$V(u) = \int_\Omega \phi(\nabla u)\, dx$$

is C^∞ strictly inside Ω.

- u is solution of the associated Euler-Lagrange equation.
- let $\theta = \partial_e u$. It is solution to

$$\mathrm{div}(A(x)\nabla\theta) = 0,$$

  where $A(x)$ (function of u) is bounded elliptic but not a priori regular.
- The method developed by De Giorgi shows that $\theta \in C^\alpha$.

Applications to our problems

We need the following tools:

- ENERGY: We need to have an energy dissipation.
- The estimate of the "levels" of energy has to be localizable.
- C^α is the threshold to obtain higher regularity (via more standard mehods).
- In all these applications, we can concentrate on linear problems (as De Giorgi).
  (But this is not crucial, see Caffarelli-Vaszquez (porous media), or Chan-V. (Hamilton-Jacobi)).

Main steps

- L^2 (bounded energy) to L^∞ (uniformly bounded).
- L^∞ (uniformly bounded) to C^α (modulus of continuity).

Main difficulty

The main difficulty: For nonlocal problems, how to localize the energy ?

- Via the extension operator (Caffarelli-V.). But leads to a very degenerated energy inequality.
  Does not work if the operator is not homogenous.

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- Via a "weak" localization (Caffarelli-Chan-V.): Far easier...
- What about the fractional derivatives in time ?
  It works pretty much the same way. It can be seen as a kind of nonlocal elliptic problem in space time.

Thank you

THANK YOU !!