Lecture 1: Optimal control with fractional elliptic PDEs constraints

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What is control theory?

- The word control has a double meaning. First controlling a system can be understood as testing or checking that its behavior is satisfactory. In a deeper sense, to control is also to act, to put things together in order to guarantee that the system behaves as desired.
- To fix ideas, assume we want to get a good behavior of a physical system governed by the state equation

$$A(y) = f(v). \tag{1.1}$$

- Here, y is the **state**, the unknown of the system that we are willing to control. It belongs to a vector space Y.
- On the other hand, v is the control. It belongs to the set of admissible controls $\mathcal{U}_{ad}$. This is the variable that we can choose freely in $\mathcal{U}_{ad}$ to act on the system.

What is control theory?

- Assume that $A : D(A) \subset Y \mapsto Y$ and $f : \mathcal{U}_{ad} \mapsto Y$ are two given (linear or nonlinear) mappings.
- The operator A determines the equation that must be satisfied by the state variable y, according to the laws of Physics. The function f indicates the way the control v acts on the system governing the state.
- Assume that, for each $v \in \mathcal{U}_{ad}$, (1.1) has one solution $y = y(v)$ on Y. Then, to control (1.1) is to find $v \in \mathcal{U}_{ad}$ such that y gets close to the prescribed state, a desired target.
- The "best" among all the existing controls achieving the desired goal is referred as the optimal control.

What is control theory?

- It is then reasonable to think that a fruitful way to choose a good control v is by minimizing a cost function of the form:

$$J_1(v) = \frac{1}{2}\|y(v) - y_d\|_Y^2 \quad \forall\, v \in \mathcal{U}_{ad} \tag{1.2}$$

or, more generally, for $\mu \geq 0$,

$$J_2(v) = \frac{1}{2}\|y(v) - y_d\|_Y^2 + \frac{\mu}{2}\|v\|_{\mathcal{U}}^2 \quad \forall\, v \in \mathcal{U}_{ad}. \tag{1.3}$$

- When minimizing J_2, we are minimizing the balance between the two terms. The first one requires to get close to the target y_d while the second one penalizes using too much costly control.
- When minimizing J_2 we are trying to drive the system to a state close to the target y_d without too much effort.

What is controllability?

1. There are two ways of specifying a "desired prescribed situation":
   - To fix a desired state y_d and require
   $$y(v) = y_d \tag{1.4}$$
   - or, at least
   $$y(v) \simeq y_d. \tag{1.5}$$

   This is the controllability viewpoint.
2. The main question is then the existence of an admissible control v so that the corresponding state $y(v)$ satisfies (1.4) or (1.5).
3. Once the existence of such a control v is established, it is meaningful to look for an optimal control, for instance, a control of minimal size.
4. As we shall see, this problem may be difficult (very difficult) to solve.
5. Beautiful Mathematics have been developed to solve these questions.

Control Problem Formulation: Let $Z_D := L^2(\mathbb{R}^N \setminus \Omega), \quad U_D := L^2(\Omega)$

- Given $\xi \geq 0$ a constant parameter, we consider the problem:

$$\min_{(u,z)\in(U_D,Z_D)} J(u) + \frac{\xi}{2}\|z\|^2_{Z_D}, \tag{2.1a}$$

subject to the constraints: $u \in U_D$ solves the elliptic PDE

$$\begin{cases} (-\Delta)^s u & = 0 \quad \text{in } \Omega, \\ u & = z \quad \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{2.1b}$$

and the control constraints

$$z \in Z_{ad,D} \subset Z_D \text{ a closed and convex subset.} \tag{2.1c}$$

- The precise conditions on the functional $J : U_D \to \mathbb{R}$ will be given later.
- $(-\Delta)^s$ is the fractional Laplacian that we will introduce soon.

What are the main steps?

1. Define the fractional Laplace operator $(-\Delta)^s$ $(0 < s < 1)$.
2. Introduce the function spaces needed to study the problem.
3. Introduce a suitable (and consistent) notion of solutions to the state equation (2.1b).
4. Show that (2.1b) has a unique suitable solution u.
5. Find sufficient conditions on the functional J such that the minimization problem (2.1) has a minimizer (a unique minimizer).
6. Derive the optimality conditions (optimal systems, optimal conditions).

The fractional Laplace operator: Using the Caffarelli-Silvestre Extension

Let $0 < s < 1$. For $u : \mathbb{R}^N \to \mathbb{R}$, we consider the harmonic extension $W : [0, \infty) \times \mathbb{R}^N \to \mathbb{R}$, that is, a solution of the elliptic Dirichlet problem

$$\begin{cases} W_{yy} + \frac{1-2s}{y} W_y + \Delta_x W = 0 & \text{in } (0,\infty) \times \mathbb{R}^N, \\ W(0,\cdot) = u & \text{in } \mathbb{R}^N. \end{cases} \tag{3.1}$$

Then the fractional Laplace operator can be defined as

$$(-\Delta)^s u(x) = -d_s \lim_{y \to 0^+} y^{1-2s} W_y(y,x), \quad x \in \mathbb{R}^N,$$

where d_s is an explicit constant depending only on s and given by

$$d_s := 2^{2s-1} \frac{\Gamma(s)}{\Gamma(1-s)}.$$

Of course the function u must belong to an appropriate space.

The fractional Laplace operator: Using Singular Integrals

For a measurable function $u : \mathbb{R}^N \to \mathbb{R}$ and $\varepsilon > 0$ we let

$$(-\Delta)^s_\varepsilon u(x) = C_{N,s} \int_{\{y \in \mathbb{R}^N:\ |x-y|>\varepsilon\}} \frac{u(x)-u(y)}{|x-y|^{N+2s}}\, dy, \quad x \in \mathbb{R}^N.$$

The fractional Laplacian $(-\Delta)^s u$ of u is defined for $x \in \mathbb{R}^N$ by,

$$(-\Delta)^s u(x) = C_{N,s} \text{P.V.} \int_{\mathbb{R}^N} \frac{u(x)-u(y)}{|x-y|^{N+2s}}\, dy = \lim_{\varepsilon \downarrow 0} (-\Delta)^s_\varepsilon u(x)$$

provided that the limit exists for a.e. $x \in \mathbb{R}^N$. Here,

$$C_{N,s} := \frac{s 2^{2s} \Gamma\left(\frac{N+2s}{2}\right)}{\pi^{\frac{N}{2}} \Gamma(1-s)} = \left(\int_{\mathbb{R}^N} \frac{1-\cos(\xi_1)}{|\xi|^{N+2s}}\, d\xi \right)^{-1}.$$

- $(-\Delta)^s$ is a nonlocal operator. That is, $\text{supp}[(-\Delta)^s u] \not\subseteq \text{supp}[u]$.

Theorem (Let $0 < s < 1$ and u be smooth enough, says, $u \in \mathcal{S}(\mathbb{R}^N)$. Then)

$$(-\Delta)^s u(x) = -\frac{C_{N,s}}{2} \int_{\mathbb{R}^N} \frac{u(x+y) + u(x-y) - 2u(x)}{|y|^{N+2s}}\, dy. \tag{3.2}$$

Proof

- By letting $z = y - x$ we get

$$\begin{aligned}(-\Delta)^s u(x) &= -\, C_{N,s} \text{P.V.} \int_{\mathbb{R}^N} \frac{u(y) - u(x)}{|x-y|^{N+2s}}\, dy \\ &= -C_{N,s} \text{P.V.} \int_{\mathbb{R}^N} \frac{u(x+z) - u(x)}{|z|^{N+2s}}\, dz. \end{aligned} \tag{3.3}$$

- By letting $\tilde{z} = -z$ in (3.3) we get

$$\text{P.V.} \int_{\mathbb{R}^N} \frac{u(x+z) - u(x)}{|z|^{N+2s}}\, dz = \text{P.V.} \int_{\mathbb{R}^N} \frac{u(x-\tilde{z}) - u(x)}{|\tilde{z}|^{N+2s}}\, d\tilde{z}. \tag{3.4}$$

- Relabeling $\tilde{z}$ as z we get

$$
\begin{aligned}
&2\text{P.V.} \int_{\mathbb{R}^N} \frac{u(x+z)-u(x)}{|z|^{N+2s}}\, dz \\
=&\text{P.V.} \int_{\mathbb{R}^N} \frac{u(x+z)-u(x)}{|z|^{N+2s}}\, dz + \text{P.V.} \int_{\mathbb{R}^N} \frac{u(x-z)-u(x)}{|z|^{N+2s}}\, dz. \qquad (3.5)
\end{aligned}
$$

- If we rename z as y and summing up we get

$$
(-\Delta)^s u(x) = -\frac{C_{N,s}}{2}\text{P.V.} \int_{\mathbb{R}^N} \frac{u(x+y)+u(x-y)-2u(x)}{|y|^{N+2s}}\, dy. \qquad (3.6)
$$

- Since u is smooth, a second Taylor expansion gives

$$
\frac{u(x+y)+u(x-y)-2u(x)}{|y|^{N+2s}} \leq \frac{\|D^2 u\|_{L^\infty(\mathbb{R}^N)}}{|y|^{N+2s-2}}
$$

which is integrable near 0 for all $s \in (0,1)$. We can then remove the P.V. in (3.6) to get (3.2). □

The fractional Laplace operator: Using Fourier Analysis

- Let $0 < s < 1$. Using Fourier analysis, the fractional Laplace operator $(-\Delta)^s$ can be defined as the pseudo-differential operator with symbol $|\xi|^{2s}$. That is,

$$(-\Delta)^s u = C_{N,s}\mathcal{F}^{-1}\left(|\xi|^{2s}\mathcal{F}(u)(\xi)\right),$$

where

$$\mathcal{F}(u)(\xi) = \frac{1}{(2\pi)^{N/2}}\int_{\mathbb{R}^N} e^{-i\xi x}u(x)\,dx$$

is the Fourier transform, $\mathcal{F}^{-1}$ the inverse Fourier transform, and

$$C_{N,s} := \frac{s2^{2s}\Gamma\left(\frac{N+2s}{2}\right)}{\pi^{\frac{N}{2}}\Gamma(1-s)} = \left(\int_{\mathbb{R}^N}\frac{1-\cos(\xi_1)}{|\xi|^{N+2s}}\,d\xi\right)^{-1}.$$

- $(-\Delta)^s$ is the generator of the so called s-stable Lévy processes (Lévy flights in some of the physical literature).

Proof: We prove the result for smooth functions u.

- Let $u \in \mathcal{S}(\mathbb{R}^N)$. We denote by $\mathcal{L}$ the linear operator given by

$$\mathcal{L}u(x) := -\frac{C_{N,s}}{2}\int_{\mathbb{R}^N} \frac{u(x+y)+u(x-y)-2u(x)}{|y|^{N+2s}}\,dy.$$

- We are looking for the symbol of $\mathcal{L}$, that is $\hat{S} : \mathbb{R}^N \to \mathbb{R}$ such that

$$\mathcal{L}u = \mathcal{F}^{-1}(\hat{S}\mathcal{F}u). \tag{3.7}$$

- We want to prove that

$$\hat{S}(\xi) = |\xi|^{2s} \;\text{ where } \xi \text{ is the frequency variable.} \tag{3.8}$$

Proof

- Notice that

$$\frac{u(x+y)+u(x-y)-2u(x)}{|y|^{N+2s}} \leq 4\Bigg(\chi_{B_1(y)}|y|^{2-N-2s}\sup_{B_1(x)}|D^2u|$$
$$+\chi_{\mathbb{R}^N\setminus B_1(y)}|y|^{-N-2s}|u(x+y)+u(x-y)-2u(x)|\Bigg) \in L^1(\mathbb{R}^{2N})$$

- Using Fubini-Tonelli theorem, we get

$$\begin{aligned}\hat{S}(\xi)(\mathcal{F}u)(\xi) &== -\frac{C_{N,s}}{2}\int_{\mathbb{R}^N}\frac{\mathcal{F}\Big(u(x+y)+u(x-y)-2u(x)\Big)}{|y|^{N+2s}}\,dy\\ &= -\frac{C_{N,s}}{2}\int_{\mathbb{R}^N}\frac{e^{i\xi\cdot y}+e^{-i\xi\cdot y}-2}{|y|^{N+2s}}\,dy(\mathcal{F}u)(\xi)\\ &= C_{N,s}\int_{\mathbb{R}^N}\frac{1-\cos(\xi\cdot y)}{|y|^{N+2s}}\,dy(\mathcal{F}u)(\xi). \qquad (3.9)\end{aligned}$$

Proof Cont

- In order to obtain (3.8), it suffices to show that

$$\int_{\mathbb{R}^N} \frac{1-\cos(\xi\cdot y)}{|y|^{N+2s}}\,dy = (C_{N,s})^{-1}\,|\xi|^{2s}. \tag{3.10}$$

- Observe that, if $\xi = (\xi_1, \dots, \xi_N) \in \mathbb{R}^N$, then near $\xi = 0$ we have

$$\frac{1-\cos(\xi_1)}{|\xi|^{N+2s}} \leq \frac{|\xi_1|^2}{|\xi|^{N+2s}} \leq \frac{1}{|\xi|^{N-2+2s}}.$$

- Thus

$$\int_{\mathbb{R}^N} \frac{1-\cos(\xi_1)}{|\xi|^{N+2s}}\,d\xi \;\text{ is finite and positive.} \tag{3.11}$$

Proof Cont

- Consider the function $I : \mathbb{R}^N \to \mathbb{R}$ given by

$$I(\xi) = \int_{\mathbb{R}^N} \frac{1 - \cos(\xi \cdot y)}{|y|^{N+2s}} \, dy.$$

- We claim that I is rotationally invariant, that is

$$I(\xi) = I(|\xi| e_1) \tag{3.12}$$

  where e_1 denotes the first direction vector of $\mathbb{R}^N$.
- Indeed, if $N = 1$, then $I(-\xi) = I(\xi)$ and we can deduce (3.12).
- If $N \geq 2$, we consider a rotation R for which $R(|\xi| e_1) = \xi$ and we denote by R^T its transpose.

Proof Cont

- Then the substitution $\tilde{y} = R^T y$ gives the following:

$$I(\xi) = \int_{\mathbb{R}^N} \frac{1-\cos((R(|\xi|e_1))\cdot y)}{|y|^{N+2s}}\,dy = \int_{\mathbb{R}^N} \frac{1-\cos((|\xi|e_1)\cdot(R^T y))}{|y|^{N+2s}}\,dy$$
$$= \int_{\mathbb{R}^N} \frac{1-\cos((|\xi|e_1)\cdot\tilde{y})}{|y|^{N+2s}}\,d\tilde{y} = I(|\xi|e_1). \quad \text{That shows (3.12).}$$

- As a consequence of (3.11) and (3.12), the substitution $\zeta = |\xi|y$ gives

$$I(\xi) = I(|\xi|e_1) = \int_{\mathbb{R}^N} \frac{1-\cos(|\xi|y_1)}{|y|^{N+2s}}\,dy$$
$$= \frac{1}{|\xi|^N}\int_{\mathbb{R}^N} \frac{1-\cos(\zeta_1)}{|\zeta/|\xi||^{N+2s}}\,d\zeta = C_{N,s}^{-1}|\xi|^{2s}.$$

- We have shown (3.10). □

Fractional order Sobolev spaces: Let $\Omega \subset \mathbb{R}^N$ ($N \geq 1$) be a bounded Lipschitz domain and $0 < s < 1$

- We let
$$H^s(\Omega) := \left\{ u \in L^2(\Omega) \ : \ \int_\Omega \int_\Omega \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\, dxdy < \infty \right\},$$
and we endow it with the norm defined by
$$\|u\|_{H^s(\Omega)} := \left(\int_\Omega |u|^2\, dx + \int_\Omega \int_\Omega \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\, dxdy \right)^{\frac{1}{2}}.$$
- We also define $H_0^s(\Omega) := \left\{ u \in H^s(\mathbb{R}^N) \ : \ u = 0 \text{ in } \mathbb{R}^N \setminus \Omega \right\}$.
- We let $H^{-s}(\Omega)$ denote the dual space of $H_0^s(\Omega)$ so that we have the continuous embeddings: $H_0^s(\Omega) \hookrightarrow L^2(\Omega) \hookrightarrow H^{-s}(\Omega)$.
- Let $H_c^s(\Omega) = \overline{\mathcal{D}(\Omega)}^{H^s(\Omega)}$.

Properties of Fractional order Sobolev spaces

- For every $0 < s < 1$ we have that

$$H_0^s(\Omega) = \left\{ u \in H_c^s(\Omega) : \frac{u}{\delta^s} \in L^2(\Omega) \right\},$$

where $\delta(x) := \mathrm{dist}(x, \partial\Omega)$, $x \in \Omega$.

- If $s \neq \frac{1}{2}$, then $H_c^s(\Omega) = H_0^s(\Omega)$.
- If $s = \frac{1}{2}$, then $H_0^{\frac{1}{2}}(\Omega)$ is a proper subspace of $H_c^s(\Omega)$.
- $H_c^s(\Omega) = H^s(\Omega)$ for every $0 < s \leq \frac{1}{2}$.

Sobolev embeddings: $\Omega \subset \mathbb{R}^N$ is a bounded Lipschitz domain

1. It is well-known that the following continuous embeddings hold:

$$H^s(\Omega), H_c^s(\Omega), H_0^s(\Omega) \hookrightarrow \begin{cases} L^{\frac{2N}{N-2s}}(\Omega) & \text{if } N > 2s, \\ L^p(\Omega) \quad \forall\, p \in [1,\infty) & \text{if } N = 2s, \\ C^{0,1-\frac{N}{2s}}(\overline{\Omega}) & \text{if } N < 2s. \end{cases}$$

2. The following embeddings are compact:

$$H^s(\Omega), H_c^s(\Omega), H_0^s(\Omega) \hookrightarrow L^2(\Omega).$$

3. The properties (1) and (2) hold for $H_c^s(\Omega), H_0^s(\Omega)$ without any regularity assumption on Ω.
4. These embeddings can be used to study the regularity of solutions of elliptic and parabolic PDEs associated with $(-\Delta)^s$.

The nonlocal normal derivative

- For a measurable function $u : \mathbb{R}^N \to \mathbb{R}$, we define

$$\mathcal{N}_s u(x) = C_{N,s} \int_\Omega \frac{u(x) - u(y)}{|x - y|^{N+2s}}\, dy, \quad x \in \mathbb{R}^N \setminus \overline{\Omega}.$$

- Then $\mathcal{N}_s$ is a nonlocal operator and it is well-defined for $u \in H^s(\mathbb{R}^N)$.
- More precisely, we have that $\mathcal{N}_s$ maps $H^s(\mathbb{R}^N)$ continuously into $H^s_{\mathrm{loc}}(\mathbb{R}^N \setminus \Omega)$.
- If $u \in V := \{u \in H^s_0(\Omega) : (-\Delta)^s u \in L^2(\Omega)\}$, then $\mathcal{N}_s u \in L^2(\mathbb{R}^N \setminus \Omega)$ and there is a constant $C > 0$ such that

$$\|\mathcal{N}_s u\|_{L^2(\mathbb{R}^N \setminus \Omega)} \leq C \|u\|_{H^s_0(\Omega)}.$$

Integration by parts formula for $(-\Delta)^s$.

- Let $u \in \mathbb{H} := \{u \in H^s(\mathbb{R}^N) : (-\Delta)^s u \in L^2(\Omega)\}$. Then $\forall\, v \in H^s(\mathbb{R}^N)$,

$$\int_\Omega v(-\Delta)^s u\, dx = \frac{C_{N,s}}{2} \int_{\mathbb{R}^{2N}\setminus(\mathbb{R}^N\setminus\Omega)^2} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+2s}}\, dxdy - \int_{\mathbb{R}^N\setminus\Omega} v\mathcal{N}_s u\, dx$$

  where $\mathbb{R}^{2N} \setminus (\mathbb{R}^N \setminus \Omega)^2 = (\Omega \times \Omega) \cup (\Omega \times (\mathbb{R}^N \setminus \Omega)) \cup ((\mathbb{R}^N \setminus \Omega) \times \Omega)$.

- Notice that if $u, v \in H_0^s(\Omega)$ then

$$\int_{\mathbb{R}^{2N}\setminus(\mathbb{R}^N\setminus\Omega)^2} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+2s}}\, dxdy = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+2s}}\, dxdy$$

Proof

We have that

$$
\begin{aligned}
&\frac{C_{N,s}}{2}\int_{\mathbb{R}^{2N}\setminus(\mathbb{R}^N\setminus\Omega)^2}\frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+2s}}\,dxdy\\
=&C_{N,s}\int_{\Omega}\int_{\mathbb{R}^N}v(x)\frac{u(x)-u(y)}{|x-y|^{N+2s}}\,dydxC_{N,s}\int_{\mathbb{R}^N\setminus\Omega}\int_{\Omega}v(x)\frac{u(x)-u(y)}{|x-y|^{N+2s}}\,dydx\\
=&C_{N,s}\int_{\Omega}v(x)\int_{\mathbb{R}^N}\frac{u(x)-u(y)}{|x-y|^{N+2s}}\,dydx+C_{N,s}\int_{\mathbb{R}^N\setminus\Omega}v(x)\int_{\Omega}\frac{u(x)-u(y)}{|x-y|^{N+2s}}\,dydx\\
=&\int_{\Omega}v(-\Delta)^s u\,dx+\int_{\mathbb{R}^N\setminus\Omega}v\mathcal{N}_s u\,dx.
\end{aligned}
$$

□

Recall our control problem: Let $Z_D := L^2(\mathbb{R}^N \setminus \Omega)$, $\quad U_D := L^2(\Omega)$ where $\Omega \subset \mathbb{R}^N$ is a bounded Lipschitz domain

Given $\xi \geq 0$ a constant parameter, we consider the minimization problem:

$$\min_{(u,z)\in(U_D,Z_D)} J(u) + \frac{\xi}{2}\|z\|^2_{Z_D}, \tag{5.1a}$$

subject to the constraints: Find $u \in U_D$ solving

$$\begin{cases} (-\Delta)^s u & = 0 \quad \text{in } \Omega, \\ u & = z \quad \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{5.1b}$$

and the control constraints

$$z \in Z_{ad,D} \subset Z_D \text{ a closed and convex subset.} \tag{5.1c}$$

Our notion of weak solutions to the state equation

We begin by considering a more general form of (5.1b), that is,

$$\begin{cases} (-\Delta)^s u = f & \text{in } \Omega, \\ u = z & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases} \tag{5.2}$$

Definition: Weak solutions

Let $f \in H^{-s}(\Omega)$, $z \in H^s(\mathbb{R}^N \setminus \Omega)$ and $\widetilde{z} \in H^s(\mathbb{R}^N)$ be such that $\widetilde{z}|_{\mathbb{R}^N\setminus\Omega} = z$. A function $u \in H^s(\mathbb{R}^N)$ is said to be a weak solution of (5.2) if $u - \widetilde{z} \in H_0^s(\Omega)$ and the equality

$$\mathcal{E}(u, v) := \frac{C_{N,s}}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{N+2s}} \, dxdy = \langle f, v \rangle,$$

holds for every $v \in H_0^s(\Omega)$. Here $\langle \cdot, \cdot \rangle$ denotes the duality pairing between $H^{-s}(\Omega)$ and $H_0^s(\Omega)$.

Observation

1. Notice that since Ω has a Lipschitz continuous boundary, for $z \in H^s(\mathbb{R}^N \setminus \Omega)$, there is $\widetilde{z} \in H^s(\mathbb{R}^N)$ such that $\widetilde{z}|_{(\mathbb{R}^N\setminus\Omega)} = z$ and
$$\|\tilde{z}\|_{H^s(\mathbb{R}^N)} \leq C\|z\|_{H^s(\mathbb{R}^N\setminus\Omega)}.$$
2. Since $\tilde{z} \in H^s(\mathbb{R}^N)$, we have that $(-\Delta)^s\tilde{z} \in H^{-s}(\mathbb{R}^N)$ and there is a constant $C > 0$ such that
$$\|(-\Delta)^s\tilde{z}\|_{H^{-s}(\mathbb{R}^N)} \leq C\|\tilde{z}\|_{H^s(\mathbb{R}^N)} \leq C\|z\|_{H^s(\mathbb{R}^N\setminus\Omega)}. \quad (5.3)$$
This follows from the fact that $(-\Delta)^s$ can be seen as a bounded operator from $H^s(\mathbb{R}^N)$ into its dual $H^{-s}(\mathbb{R}^N)$.

Theorem (Existence of weak solutions)

Let $f \in H^{-s}(\Omega)$ and $z \in H^s(\mathbb{R}^N \setminus \Omega)$. Then there exists a unique weak solution u of the state equation (5.2). In addition there is a constant $C > 0$ such that

$$\|u\|_{H^s(\mathbb{R}^N)} \leq C \left(\|f\|_{H^{-s}(\Omega)} + \|z\|_{H^s(\mathbb{R}^N \setminus \Omega)} \right). \tag{5.4}$$

Proof

- Let $\tilde{u} = u - \tilde{z}$. Then $\tilde{u} \in H_0^s(\Omega)$ and is a weak solution of

$$(-\Delta)^s \tilde{u} = f - (-\Delta)^s \tilde{z} \quad \text{in } \Omega$$

  in the sense that for every $v \in H_0^s(\Omega)$, we have

$$\mathcal{E}(\tilde{u}, v) = \langle f - (-\Delta)^s \tilde{z}, v \rangle. \tag{5.5}$$

- $\mathcal{E}$ is continuous and coercive and the right hand side of (5.5) is a continuous linear functional on $H_0^s(\Omega)$.
- By Lax-Miligram Lemma $\exists! \tilde{u} \in H_0^s(\Omega)$ satisfying (5.5). In addition,

$$\|\tilde{u}\|_{H_0^s(\Omega)} \leq \|f - (-\Delta)^s \tilde{z}\|_{H^{-s}(\Omega)} \leq C \left(\|f\|_{H^{-s}(\Omega)} + \|z\|_{H^s(\mathbb{R}^N \setminus \Omega)} \right).$$

Proof Cont

- We have shown that $u := \tilde{u} + \tilde{z}$ solve (5.2) in the weak sense.
- Using the previous estimate, we get that

$$\begin{aligned}\|u\|_{H^s(\mathbb{R}^N)} = \|\tilde{u} + \tilde{z}\|_{H^s(\mathbb{R}^N)} &\le \|\tilde{u}\|_{H_0^s(\Omega)} + \|\tilde{z}\|_{H^s(\mathbb{R}^N)} \\ &\le \|f - (-\Delta)^s \tilde{z}\|_{H^{-s}(\Omega)} + \|\tilde{z}\|_{H^s(\mathbb{R}^N)} \\ &\le C\left(\|f\|_{H^{-s}(\Omega)} + \|z\|_{H^s(\mathbb{R}^N\setminus\Omega)}\right),\end{aligned}$$

where we have also used (5.3). We have shown (5.4). □

Remark

1. Such a result is usually sufficient in most situations.
2. But it is not really useful in the current context of the optimal control problem (5.1) since we are interested in taking the control space $Z_D = L^2(\mathbb{R}^N \setminus \Omega)$.
3. We need solutions to the fractional Dirichlet problem (5.2) when the control $z \in L^2(\mathbb{R}^N \setminus \Omega)$.
4. We have to introduce a weaker notion of solutions for (5.2).

Definition: Very-weak solutions to the state equation

Let $z \in L^2(\mathbb{R}^N \setminus \Omega)$ and $f \in H^{-s}(\Omega)$. A function $u \in L^2(\mathbb{R}^N)$ is said to be a very-weak solution to (5.2) (or a solution by transposition) if $u = z$ a.e. in $\mathbb{R}^N \setminus \Omega$ and the identity

$$\int_\Omega u(-\Delta)^s v\, dx = \langle f, v\rangle - \int_{\mathbb{R}^N\setminus\Omega} z\mathcal{N}_s v\, dx, \tag{5.6}$$

holds for every $v \in \mathcal{D}(\Omega)$.

Remark

1. Notice that if $v \in \mathcal{D}(\Omega)$, then $(-\Delta)^s v \in L^2(\Omega)$, $\mathcal{N}_s v \in L^2(\mathbb{R}^N \setminus \Omega)$ and there is a constant $C > 0$ such that

$$\|\mathcal{N}_s v\|_{L^2(\mathbb{R}^N\setminus\Omega)} \le C\|v\|_{H_0^s(\Omega)}.$$

2. We may replace $\mathcal{D}(\Omega)$ with $V := \{u \in H_0^s(\Omega) : (-\Delta)^s u \in L^2(\Omega)\}$.

Theorem (Existence of very-weak solutions (Antil, Khatri & W. 2019))

Let $f \in H^{-s}(\overline{\Omega})$ and $z \in L^2(\mathbb{R}^N \setminus \Omega)$. Then there exists a unique very-weak solution $u \in L^2(\mathbb{R}^N)$ to (5.2), and there is a constant $C > 0$ such that

$$\|u\|_{L^2(\Omega)} \leq C \left(\|f\|_{H^{-s}(\Omega)} + \|z\|_{L^2(\mathbb{R}^N \setminus \Omega)} \right). \tag{5.7}$$

In addition, if $z \in H^s(\mathbb{R}^N \setminus \Omega)$, then the following assertions hold.

1. *Every weak solution of (5.2) is also a very-weak solution.*
2. *Every very-weak solution of (5.2) that belongs to $H^s(\mathbb{R}^N)$ is also a weak solution.*

Theorem (Lions)

Let $(F, \|\cdot\|_F)$ be a Hilbert space. Let Φ be a subspace of F endowed with a pre-Hilbert scalar product $(((\cdot,\cdot)))$ and associated norm $|||\cdot|||$. Moreover, let $E : F \times \Phi \to \mathbb{C}$ be a sesquilinear form. Assume that the following hold:

1. *The embedding $\Phi \hookrightarrow F$ is continuous, that is, there is a constant $C > 0$ such that*
$$\|\varphi\|_F \leq C_1 |||\varphi||| \quad \forall\, \varphi \in \Phi. \tag{5.8}$$
2. *For all $\varphi \in \Phi$, the mapping $u \mapsto E(u,\varphi)$ is continuous on F.*
3. *There is a constant $C_2 > 0$ such that*
$$|E(\varphi,\varphi)| \geq C_2 |||\varphi|||^2 \quad \text{for all} \quad \varphi \in \Phi. \tag{5.9}$$

If $\varphi \mapsto L(\varphi)$ is a continuous linear functional on Φ, then there exists a function $u \in F$ (which is unique if Φ is a Hilbert space) verifying

$$E(u,\varphi) = L(\varphi) \quad \text{for all} \quad \varphi \in \Phi.$$

Proof of the existence theorem: We verify the conditions of Lions' Theorem

- We endow $\Phi := \mathcal{D}(\Omega)$ with the norm $|||u||| := \|u\|_{H_0^s(\Omega)}$.
- Clearly, the embedding $\mathcal{D}(\Omega) \hookrightarrow L^2(\Omega)$ is continuous.
- Let $\mathbb{E} : L^2(\Omega) \times \mathcal{D}(\Omega) \to \mathbb{R}$ and $L : \mathcal{D}(\Omega) \to \mathbb{R}$ be given by

$$\mathbb{E}(u, v) := \int_\Omega u(-\Delta)^s v \, dx \text{ and } \; Lv := \langle f, v \rangle - \int_{\mathbb{R}^N \setminus \Omega} z\mathcal{N}_s v \, dx.$$

- Let $v \in \mathcal{D}(\Omega)$. Claim: The map $u \mapsto \mathbb{E}(u, v)$ is continuous on $L^2(\Omega)$.

$$|\mathbb{E}(u, v)| = \left| \int_\Omega u(-\Delta)^s v \, dx \right| \leq \|u\|_{L^2(\Omega)} \|(-\Delta)^s v\|_{L^2(\Omega)}.$$

- Claim: $\mathbb{E}$ is coercive on $\mathcal{D}(\Omega)$. There is a constant $C > 0$ such that

$$\mathbb{E}(v, v) = \int_\Omega v(-\Delta)^s v \, dx = \mathcal{E}(v, v) \geq C\|v\|^2_{H_0^s(\Omega)} \; \forall v \in \mathcal{D}(\Omega).$$

Proof: Cont

- Using the previous estimates we get that for every $v \in \mathcal{D}(\Omega)$,

$$\begin{aligned}|L(v)| \leq & \|f\|_{H^{-s}(\Omega)}\|v\|_{H_0^s(\Omega)} + \|z\|_{L^2(\mathbb{R}^N\setminus\Omega)}\|\mathcal{N}_s v\|_{L^2(\mathbb{R}^N\setminus\Omega)} \\ \leq & C\left(\|f\|_{H^{-s}(\Omega)} + \|z\|_{L^2(\mathbb{R}^N\setminus\Omega)}\right)\|v\|_{H_0^s(\Omega)}.\end{aligned}$$

- By Lions' Theorem, (5.2) has a very weak solution $u \in L^2(\mathbb{R}^N)$.
- Assume (5.2) has two very weak solutions u_1, u_2. Let $u := u_1 - u_2$. Then

$$\int_\Omega u(-\Delta)^s v\, dx = 0 \ \text{ for all } v \in \mathcal{D}(\Omega).$$

- $\forall w \in L^2(\Omega)$, $\exists! v \in V$ satisfying $(-\Delta)^s v = w$ in Ω. Thus. $\displaystyle\int_\Omega uw\, dx = 0$ for all $w \in L^2(\Omega)$. Hence, $u = 0$ in Ω, i.e. $u_1 = u_2$ in Ω.
- The estimate (5.7) is easy to verify. $\square$

Remark

1. In view of the existence theorem of very weak solution, the following (solution-map) control-to-state map

$$S : Z_D \to U_D, \quad z \mapsto Sz = u,$$

is well-defined, linear, and continuous.

2. We also notice that for $z \in Z_D$, we have that $u := Sz \in L^2(\mathbb{R}^N)$.

3. As a result we can write the *reduced fractional Dirichlet exterior control problem* as follows:

$$\min_{z \in Z_{ad,D}} \mathcal{J}(z) := J(Sz) + \frac{\xi}{2}\|z\|^2_{Z_D}, \tag{5.10}$$

where $\xi \geq 0$.

Theorem (Well-posedness of the control problem (Antil, Khatri &W., 2019))

- *Let $Z_{ad,D}$ be a closed and convex subset of Z_D.*
- *Let either $\xi > 0$ or $Z_{ad,D}$ be bounded.*
- *Let $J : U_D \to \mathbb{R}$ be weakly lower-semicontinuous. That is, if $u_n \rightharpoonup u$ weakly in $U_D := L^2(\Omega)$ as $n \to \infty$, then*

$$J(u) \leq \liminf_{n\to\infty} J(u_n).$$

1. *Then there exists a solution $\bar{z}$ to (5.1) or equivalently to (5.10).*
2. *If either J is convex and $\xi > 0$ or J is strictly convex and $\xi \geq 0$, then $\bar{z}$ is unique.*

Proof

- We notice that for $\mathcal{J} : Z_{ad,D} \to \mathbb{R}$, we can construct a minimizing sequence $\{z_n\}_{n\in\mathbb{N}}$ such that

$$\inf_{z\in Z_{ad,D}} \mathcal{J}(z) = \lim_{n\to\infty} \mathcal{J}(z_n).$$

- If $\xi > 0$ or $Z_{ad,D} \subset Z_D$ is bounded, then $\{z_n\}_{n\in\mathbb{N}}$ is a bounded sequence in Z_D which is a Hilbert space.
- Due to the reflexivity of Z_D, we have that (up to a subsequence if necessary) $z_n \rightharpoonup \bar{z}$ (weak convergence) in Z_D as $n \to \infty$.
- Since $Z_{ad,D}$ is closed in Z_D, we have that $\bar{z} \in Z_{ad,D}$.
- Since $S : Z_{ad,D} \to U_D$ is linear and continuous, we have that it is weakly continuous. This implies that $Sz_n \rightharpoonup S\bar{z}$ in U_D as $n \to \infty$.
- We have to show that $(S\bar{z}, \bar{z}) = (\bar{u}, \bar{z})$ fulfills the state equation.

Proof: Cont

- More precisely, we need to study the identity

$$\int_\Omega u_n(-\Delta)^s v\,dx = -\int_{\mathbb{R}^N\setminus\Omega} z_n \mathcal{N}_s v\,dx, \quad \forall\, v \in V. \tag{5.11}$$

- Since $u_n \rightharpoonup S\bar{z} =: \bar{u}$ in U_D as $n \to \infty$ and $z_n \rightharpoonup \bar{z}$ in $Z_{ad,D}$ as $n \to \infty$, we can immediately take the limit in (5.11) to obtain that

$$\int_\Omega \bar{u}(-\Delta)^s v\,dx = -\int_{\mathbb{R}^N\setminus\Omega} \bar{z} \mathcal{N}_s v\,dx, \quad \forall\, v \in V.$$

- Thus $(\bar{u}, \bar{z}) \in U_D \times Z_{ad,D}$ fulfills the state equation.
- That $\bar{z}$ is the minimizer of (5.10) follows from the weakly lower-semicontinuity of $\mathcal{J}$. Indeed,

$$\mathcal{J}(\bar{z}) \le \liminf_{n\to\infty} \mathcal{J}(z_n) \le \lim_{n\to\infty} \mathcal{J}(z_n) = \inf_{z\in Z_{ad,D}} \mathcal{J}(z) \le \mathcal{J}(\bar{z}).$$

Proof: Cont

The uniqueness of $\bar{z}$ follows from the strict convexity of $\mathcal{J}$.

- Indeed, assume $\mathcal{J}$ has two minimizers $\bar{z}_1$ and $\bar{z}_2$.
- Then for all $z \in Z_{ad,D}$ we have that

$$\mathcal{J}(\bar{z}_1) = \mathcal{J}(\bar{z}_2) \le \mathcal{J}(z).$$

- Taking $z = \dfrac{\bar{z}_1 + \bar{z}_2}{2}$ and using the strict convexity of $\mathcal{J}$ we get

$$\mathcal{J}(\bar{z}_1) = \mathcal{J}(\bar{z}_2) \le \mathcal{J}(z) < \frac{1}{2}\mathcal{J}(\bar{z}_1) + \frac{1}{2}\mathcal{J}(\bar{z}_2) = \mathcal{J}(\bar{z}_1),$$

  a contradiction. We have shown uniqueness and the proof is finished. $\square$

Our next goal

We want to derive the first order necessary optimality conditions for our control problem. We begin by identifying the adjoint operator S^*.

Lemma

For the state equation (5.1b) *the adjoint operator* $S^* : U_D \to Z_D$ *of the control-to-state map*

$$S : Z_D \to U_D, \;\; z \mapsto Sz = u,$$

is given by

$$S^* w = -\mathcal{N}_s p \in Z_D,$$

where $w \in U_D$ *and* $p \in H_0^s(\Omega)$ *is the weak solution to the adjoint problem*

$$\begin{cases} (-\Delta)^s p &= w \quad \text{in } \Omega, \\ p &= 0 \quad \text{in } \mathbb{R}^N \setminus \Omega. \end{cases} \tag{5.12}$$

Proof

- According to the definition of S^*, for every $w \in U_D$ and $z \in Z_D$,

$$(w, Sz)_{L^2(\Omega)} = (S^* w, z)_{L^2(\mathbb{R}^N \setminus \Omega)}.$$

- Next, testing the adjoint equation (5.12) with Sz and using the fact that Sz is a very-weak solution of (5.2) with $f = 0$, we arrive at

$$\begin{aligned}(w, Sz)_{L^2(\Omega)} =& (Sz, (-\Delta)^s p)_{L^2(\Omega)} \\ =& -(z, \mathcal{N}_s p)_{L^2(\mathbb{R}^N \setminus \Omega)} \\ =& (z, S^* w)_{L^2(\mathbb{R}^N \setminus \Omega)}.\end{aligned}$$

This yields the asserted result. $\square$

Theorem (First order necessary optimality conditions (Antil, Khatri & W. 2019): Assume that $\xi > 0$.)

Let $\mathcal{Z}$ be open such that $Z_{ad,D} \subset \mathcal{Z}$. Let J be continuously Fréchet differentiable with $J'(u) \in U_D$. If $\bar{z}$ is a minimizer of (5.10) over $Z_{ad,D}$, then

$$\int_{L^2(\mathbb{R}^N\setminus\Omega)} \left(-\mathcal{N}_s\bar{p} + \xi\bar{z}\right)(z-\bar{z})\ dx \geq 0, \quad \forall z \in Z_{ad,D}, \tag{5.13}$$

where $\bar{p} \in H_0^s(\Omega)$ solves the adjoint equation

$$\begin{cases} (-\Delta)^s\bar{p} = J'(\bar{u}) & \text{in } \Omega, \\ \bar{p} = 0 & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases} \tag{5.14}$$

Equivalently we can write (5.13) as $\bar{z} = \mathcal{P}_{Z_{ad,D}}\left(\frac{1}{\xi}\mathcal{N}_s\bar{p}\right)$, where $\mathcal{P}_{Z_{ad,D}}$ is the projection onto the set $Z_{ad,D}$. If J is convex, then (5.13) is also a sufficient condition.

Proof

- The proof is a direct application of the differentiability properties of J and the chain rule together with the previous stated results.
- Indeed, given $h \in Z_{ad,D}$, the directional derivative of $\mathcal{J}$ is given by

$$\begin{aligned}\mathcal{J}'(\bar{z})h =& (J'(S\bar{z}), Sh)_{L^2(\Omega)} + \xi(\bar{z}, h)_{L^2(\mathbb{R}^N\setminus\Omega)}\\ =& (S^* J'(S\bar{z}), h)_{L^2(\mathbb{R}^N\setminus\Omega)} + \xi(\bar{z}, h)_{L^2(\mathbb{R}^N\setminus\Omega)}.\end{aligned}$$

- In the first step we have used that $J'(S\bar{z}) \in L^2(\Omega)$. In the second step we have used that S^* is well-defined.
- From our previous results we have that $S^* J'(S\bar{z}) = \mathcal{N}_s\bar{p} \in L^2(\mathbb{R}^N \setminus \Omega)$ where $\bar{p}$ solves (5.14). Combining these facts we get (5.13). □

Remark 1

- We recall a rather surprising result for the adjoint equation (5.12). The standard maximal elliptic regularity that is known to hold for the classical Laplacian on smooth open sets does not hold in the case of the fractional Laplacian i.e., the solution p of the adjoint equation does not always belong to $H^{2s}(\Omega)$.
- Notice that $w \in L^2(\Omega)$ and $p = [(-\Delta)_D^s]^{-1} w$. Assume that Ω is a smooth bounded open set. If $0 < s < \frac{1}{2}$, then $D((-\Delta)_D^s) \subseteq H_0^{2s}(\Omega)$. Hence, $p \in H^{2s}(\Omega)$ in that case.
- If $\frac{1}{2} \leq s < 1$, then $D((-\Delta)_D^s) \not\subset H^{2s}(\Omega)$. Thus, in that case p does not always belong to $H^{2s}(\Omega)$.
- Hence, we do not have maximal elliptic regularity.
- We only have local maximal elliptic regularity results.

Remark 2

- Since $\mathcal{P}_{Z_{ad,D}}$ is a contraction (Lipschitz) we can conclude that the optimal control $\bar{z}$ has the same regularity as $\mathcal{N}_s\bar{p}$, i.e., they are in $L^2(\mathbb{R}^N \setminus \Omega)$ globally and in $H^s_{\rm loc}(\mathbb{R}^N \setminus \Omega)$ locally.
- The maximal global regularity for the associated state $\bar{u}$ is only $L^2(\mathbb{R}^N)$.
- As it is well-known, in the case of the classical Laplacian, one can use a boot-strap argument to improve the regularity of $\bar{u}$ globally. However, this is not the case for the fractional exterior value problems.
- We recall that our operator $(-\Delta)^s_D$ is different from the spectral Dirichlet fractional Laplacian $(-\Delta_D)^s$ (the fractional powers of the realization $-\Delta_D$ of the Laplace operator with the zero Dirichlet boundary condition).
- For the latter operator, it is known that a maximal elliptic regularity can be achieved. Notice that these fractional powers $(-\Delta_D)^s$ have the same eigenfunctions as $-\Delta_D$.

Example

Let $u_d \in L^2(\Omega)$ be a given fixed target.

1. The functional $J : L^2(\Omega) \to \mathbb{R}$ given by

$$J(u) := \frac{1}{2} \int_\Omega |u - u_d|^2 \, dx$$

   satisfied all our hypothesis.

2. In that case, for $h \in L^2(\Omega)$, we have that

$$J'(u)h = \lim_{\lambda \to 0} \frac{J(u + \lambda h) - J(u)}{\lambda} = \int_\Omega (u - u_d) h \, dx.$$

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