Lecture 2: Optimal control with fractional parabolic PDEs

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Recall the fractional Laplace operator

For a measurable function u and $\varepsilon > 0$ we let

$$(-\Delta)^s_\varepsilon u(x) = C_{N,s} \int_{\{y\in\mathbb{R}^N:\ |x-y|>\varepsilon\}} \frac{u(x)-u(y)}{|x-y|^{N+2s}}\, dy, \quad x \in \mathbb{R}^N.$$

The fractional Laplacian $(-\Delta)^s u$ of u is defined for $x \in \mathbb{R}^N$ by,

$$(-\Delta)^s u(x) = C_{N,s}\text{P.V.} \int_{\mathbb{R}^N} \frac{u(x)-u(y)}{|x-y|^{N+2s}}\, dy = \lim_{\varepsilon\downarrow 0} (-\Delta)^s_\varepsilon u(x)$$

provided that the limit exists for a.e. $x \in \mathbb{R}^N$. Here,

$$C_{N,s} := \frac{s2^{2s}\Gamma\left(\frac{N+2s}{2}\right)}{\pi^{\frac{N}{2}}\Gamma(1-s)} = \left(\int_{\mathbb{R}^N} \frac{1-\cos(\xi_1)}{|\xi|^{N+2s}}\, d\xi\right)^{-1},$$

and Γ denotes the usual Euler-Gamma function.

Fractional order Sobolev spaces: Let $\Omega \subset \mathbb{R}^N$ be a bounded domain

- For $0 < s < 1$ we let

$$H^s(\Omega) := \left\{ u \in L^2(\Omega) \; : \; \int_\Omega \int_\Omega \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} \, dxdy < \infty \right\},$$

and we endow it with the norm defined by

$$\|u\|_{H^s(\Omega)} := \left(\int_\Omega |u|^2 \, dx + \int_\Omega \int_\Omega \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} \, dxdy \right)^{\frac{1}{2}}.$$

- We also define $H_0^s(\Omega) := \{ u \in H^s(\mathbb{R}^N) \; : \; u = 0 \text{ in } \mathbb{R}^N \setminus \Omega \}$.
- We shall let $H^{-s}(\Omega)$ denote the dual of $H_0^s(\Omega)$ with respect to the pivot space $L^2(\Omega)$ so that we have the continuous and dense embeddings:

$$H_0^s(\Omega) \hookrightarrow L^2(\Omega) \hookrightarrow H^{-s}(\Omega).$$

Recall the integration by parts formula

- Let $u \in H^s(\mathbb{R}^N)$ be such that $(-\Delta)^s u \in L^2(\Omega)$. Then $\forall\ v \in H^s(\mathbb{R}^N)$,

$$\int_\Omega v(-\Delta)^s u\, dx = \frac{C_{N,s}}{2} \int_{\mathbb{R}^{2N}\setminus(\mathbb{R}^N\setminus\Omega)^2} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+2s}}\, dxdy - \int_{\mathbb{R}^N\setminus\Omega} v\mathcal{N}_s u\, dx.$$

$$\text{where } \mathcal{N}_s u(x) = C_{N,s} \int_\Omega \frac{u(x)-u(y)}{|x-y|^{N+2s}}\, dy, \qquad x \in \mathbb{R}^N \setminus \overline{\Omega}.$$

- Notice that if $u, v \in H_0^s(\Omega)$ then

$$\int_{\mathbb{R}^{2N}\setminus(\mathbb{R}^N\setminus\Omega)^2} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+2s}}\, dxdy = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+2s}}\, dxdy$$

Problem Formulation: Given $\xi \geq 0$ we consider the minimization problem:

$$\min_{(u,z)\in(\mathcal{U}_D,\mathcal{Z}_D)} \left(J(u) + \frac{\xi}{2}\|z\|^2_{Z_D}\right), \tag{2.1a}$$

subject to the constraints: Find $u \in \mathcal{U}_D$ solving

$$\begin{cases} \partial_t u + (-\Delta)^s u = 0 & \text{in } Q := (0,T)\times\Omega, \\ u = z & \text{in } \Sigma := [0,T)\times(\mathbb{R}^N\setminus\Omega), \\ u(0,\cdot) = 0 & \text{in } \Omega, \end{cases} \tag{2.1b}$$

and the control constraints

$$z \in \mathcal{Z}_{ad,D}, \tag{2.1c}$$

with $\mathcal{Z}_{ad,D} \subset \mathcal{Z}_D$ being a closed and convex subset. Here,

$$\mathcal{Z}_D := L^2(\Sigma),\ \ \mathcal{U}_D := L^2(Q).$$

The functional $J : \mathcal{U}_D \to \mathbb{R}$ is weakly lower-semicontinuous.

Remark: Boundary Control replaced with Exterior Control!

1. As we have already mentioned in the elliptic case (Lecture 1), here also, the exterior control plays the role for the fractional Laplacian that the boundary control does for the Laplace operator. Since the equation

$$(-\Delta)^s u = f \text{ in } \Omega, \quad u = g \text{ on } \partial\Omega.$$

is not well-posed, then the associated evolution equation cannot be well-posed.

2. The well-posed elliptic PDE is given by

$$(-\Delta)^s u = f \text{ in } \Omega, \quad u = g \text{ on } \mathbb{R}^N \setminus \Omega.$$

This shows that the associated parabolic problem will be well-posed.

3. Therefore, we have to consider an exterior control problem as it is stated in the problem formulation.

The state equation

Here we consider the state equation: Find $u \in \mathcal{U}_D = L^2(Q)$ solving

$$\begin{cases} \partial_t u + (-\Delta)^s u = f & \text{in } Q, \\ u = z & \text{in } \Sigma, \\ u(\cdot, 0) = 0 & \text{in } \Omega. \end{cases}$$

We will consider three cases.

- First, we shall consider the case $z = 0$, introduce the right notion of solutions and prove their existence.
- Second, we shall consider the case where z is smooth, say, $z \in H^1((0,T); H^s(\mathbb{R}^N \setminus \Omega))$, introduce the right notion of solutions and prove their existence.
- Finally, we shall consider the case where z is not smooth, $z \in L^2(\Sigma)$, introduce the right notion of solutions and prove their existence.

The zero exterior data

- Let us consider first the following auxiliary problem:

$$\begin{cases} \partial_t w + (-\Delta)^s w & = f \quad \text{in } Q, \\ w & = 0 \quad \text{in } \Sigma, \\ w(\cdot, 0) & = 0 \quad \text{in } \Omega. \end{cases} \tag{3.1}$$

- Notice that (3.1) can be rewritten as the following Cauchy problem:

$$\begin{cases} \partial_t w + (-\Delta)^s_D w = f & \text{in } Q, \\ w(\cdot, 0) = 0 & \text{in } \Omega, \end{cases} \tag{3.2}$$

where we recall that $(-\Delta)^s_D$ is the realization of $(-\Delta)^s$ with the zero Dirichlet exterior condition. That is,

$$D((-\Delta)^s_D) = \{u \in H^s_0(\Omega) : (-\Delta)^s u \in L^2(\Omega)\}, (-\Delta)^s_D u = (-\Delta)^s u \text{ in } \Omega.$$

Remark 1

- We notice that $(-\Delta)_D^s$ is the selfadjoint operator on $L^2(\Omega)$ associated with the bilinear form $\mathcal{E} : H_0^s(\Omega) \times H_0^s(\Omega) \to \mathbb{R}$ given by

$$\mathcal{E}(u,v) := \frac{C_{N,s}}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+2s}}\, dxdy$$

in the sense that

$$\begin{cases} D((-\Delta)_D^s) = \{u \in H_0^s(\Omega) : \exists f \in L^2(\Omega) : \mathcal{E}(u,v) = \int_\Omega fv\, dx \; \forall v \in H_0^s(\Omega) \\ (-\Delta)_D^s u = f. \end{cases}$$

- The operator $(-\Delta)_D^s$ has a compact resolvent. This follows from the fact that the embedding $H_0^s(\Omega) \hookrightarrow L^2(\Omega)$ is compact.

Remark 2

- As a consequence $(-\Delta)^s_D$ has a discrete sprectrum formed with eigenvalues λ_n satisfying

$$0 < \lambda_1 \leq \lambda_2 \leq \cdots \lambda_n \leq \cdots , \quad \lim_{n\to\infty} \lambda_n = \infty.$$

- We shall denote by φ_n the associated normalized eignefunctions.
- The operator $-(-\Delta)^s_D$ generates a strongly continuous semigroup $(e^{-t(-\Delta)^s_D})_{t\geq 0}$ on $L^2(\Omega)$ given for every $u \in L^2(\Omega)$ by

$$(e^{-t(-\Delta)^s_D} u)(x) = \sum_{n=1}^{\infty} e^{-\lambda_n t}(u, \varphi_n)_{L^2(\Omega)}\varphi_n(x).$$

- In that case $u := e^{-t(-\Delta)^s_D} u_0$, $u_0 \in L^2(\Omega)$, is the unique strong solution

$$u_t + (-\Delta)^s_D u = 0 \ \text{ in } \ Q, \qquad u(\cdot, 0) = u_0 \ \text{ in } \ \Omega.$$

Remark 2

- Let $H^{-s}(\Omega)$ be the dual space of $H_0^s(\Omega)$ with respect to the pivot space $L^2(\Omega)$ so that $H_0^s(\Omega) \hookrightarrow L^2(\Omega) \hookrightarrow H^{-s}(\Omega)$.
- The operator $(-\Delta)_D^s$ can be also seen as a bounded operator from $H_0^s(\Omega)$ into $H^{-s}(\Omega)$ given for $u, v \in H_0^s(\Omega)$ by

$$\langle (-\Delta)_D^s u, v \rangle = \mathcal{E}(u, v).$$

- In that case the operator $-(-\Delta)_D^s$ also generates a strongly continuous semigroup $(S(t))_{t \geq 0}$ on $H^{-s}(\Omega)$.
- Also here, $u := S(t)u_0, \;\; u_0 \in H^{-s}(\Omega)$, is the unique strong solution of

$$u_t + (-\Delta)_D^s u = 0 \;\text{ in }\; Q, \qquad u(\cdot, 0) = u_0 \;\text{ in }\; \Omega.$$

- The two semigroups coincide on $L^2(\Omega)$.

Notations

- Throughout the following for a Banach space $\mathbb{X}$ we shall let

$$H^1_{0,0}((0,T);\mathbb{X}) := \{u \in H^1((0,T);\mathbb{X}) : \; u(\cdot,0) = 0\}$$

  and

$$H^1_{0,T}((0,T);\mathbb{X}) := \{u \in H^1((0,T);\mathbb{X}) : \; u(\cdot,T) = 0\}.$$

- We shall let

$$\mathbb{U}_0 := L^2((0,T);H^s_0(\Omega)) \cap H^1_{0,0}((0,T);H^{-s}(\Omega))$$

  and

$$\mathbb{U} := L^2((0,T);H^s(\mathbb{R}^N)) \cap H^1_{0,0}((0,T);H^{-s}(\Omega)).$$

Definition of weak solutions: The zero exterior Dirichlet problem

- Let $f \in L^2((0,T); H^{-s}(\overline{\Omega}))$. A function $w \in \mathbb{U}_0$ is said to be a weak solution to (3.1) if the equality

$$\langle \partial_t w(\cdot,t), v\rangle + \frac{C_{N,s}}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{(w(x,t)-w(y,t))(v(x)-v(y))}{|x-y|^{N+2s}}\,dxdy = \langle f(t,\cdot), v\rangle,$$

  holds for every $v \in H_0^s(\Omega)$ and almost every $t \in (0,T)$. Here, $\langle\cdot,\cdot\rangle$ denotes the duality pairing between $H^{-s}(\Omega)$ and $H_0^s(\Omega)$.
- For $u, v \in H_0^s(\Omega)$, we shall always let

$$\mathcal{E}(v,w) := \frac{C_{N,s}}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+2s}}\,dxdy.$$

Theorem (Existence of Weak solutions to (3.1))

Let $f \in L^2((0,T); H^{-s}(\Omega))$. Then there exists a unique weak solution $w \in \mathbb{U}_0$ to (3.1) which is given by

$$w(x,t) = \int_0^t e^{-(t-\tau)(-\Delta)^s_D} f(x,\tau)\, d\tau,$$

where $(e^{-t(-\Delta)^s_D})_{t\geq 0}$ is the strongly continuous semigroup generated by $-(-\Delta)^s_D$. In addition, there is a constant $C > 0$ such that

$$\|w\|_{\mathbb{U}_0} \leq C\|f\|_{L^2((0,T);H^{-s}(\Omega))}. \tag{3.3}$$

Proof

This follows from semigroups theory and the formula of variation of parameters. □

Remark

We mention the following facts.

1. Recall that a weak solution of (3.1) belongs to $\mathbb{U}_0 := L^2((0,T); H_0^s(\Omega)) \cap H_{0,0}^1((0,T); H^{-s}(\Omega))$.
2. A well-known Lion's Theorem shows that $\mathbb{U}_0 \hookrightarrow C([0,T]; L^2(\Omega))$.
3. If $f \in L^2(Q)$, then using semigroups theory, we have that a weak solution to (3.1) enjoys the following regularity:

$$u \in C([0,T]; D((-\Delta)_D^s)) \cap H_{0,0}^1((0,T); L^2(\Omega)).$$

4. In that case the weak solution is even a strong solution. That is, the first equation is satisfied point-wise in Q.

Definition: Weak solutions: Higher regularity requirement on the datum z.

- Let $z \in H^1_{0,0}((0,T); H^s(\mathbb{R}^N \setminus \Omega))$ and $\tilde{z} \in H^1_{0,0}((0,T); H^s(\mathbb{R}^N))$ be such that $\tilde{z}|_\Sigma = z$.
- Then $u \in \mathbb{U} := L^2((0,T); H^s(\mathbb{R}^N)) \cap H^1_{0,0}((0,T); H^{-s}(\Omega))$ is said to be a weak solution to (2.1b) if $u - \tilde{z} \in \mathbb{U}_0$ and the equality

$$\langle \partial_t u(\cdot,t), v\rangle + \frac{C_{N,s}}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{(u(x,t)-u(y,t))(v(x)-v(y))}{|x-y|^{N+2s}}\,dxdy = 0,$$

holds for every $v \in H^s_0(\Omega)$ and almost every $t \in (0,T)$.

Theorem (Existence of weak solutions)

Let $z \in H^1_{0,0}((0,T); H^s(\mathbb{R}^N \setminus \Omega))$. *Then there exists a unique weak solution* $u \in \mathbb{U}$ *to* (2.1b). *In addition, there is a constant* $C > 0$ *such that*

$$\|u\|_{\mathbb{U}} \le C\|z\|_{H^1((0,T);H^s(\mathbb{R}^N\setminus\Omega))}. \tag{3.4}$$

Observation

1. Since Ω has a Lipschitz boundary, for $z \in H^1((0,T); H^s(\mathbb{R}^N \setminus \Omega))$, there is $\widetilde{z} \in H^1((0,T); H^s(\mathbb{R}^N))$ such that $\widetilde{z}|_\Sigma = z$ and

$$\|\tilde{z}\|_{H^1((0,T);H^s(\mathbb{R}^N))} \le C\|z\|_{H^1((0,T);H^s(\mathbb{R}^N\setminus\Omega))}.$$

2. Since $\tilde{z} \in H^1((0,T); H^s(\mathbb{R}^N))$, then $(-\Delta)^s\tilde{z} \in H^1((0,T); H^{-s}(\mathbb{R}^N))$ and there is a constant $C > 0$ such that

$$\begin{aligned}\|(-\Delta)^s\tilde{z}\|_{H^1((0,T);H^{-s}(\mathbb{R}^N))} &\le C\|\tilde{z}\|_{H^1((0,T);H^s(\mathbb{R}^N))}\\ &\le C\|z\|_{H^1((0,T);H^s(\mathbb{R}^N\setminus\Omega))}.\end{aligned} \tag{3.5}$$

Proof

- First assume that z depends only on x and consider the s-Harmonic extension $\tilde{z} \in H^s(\mathbb{R}^N)$ of $z \in H^s(\mathbb{R}^N \setminus \Omega)$ that solves

$$\begin{cases} (-\Delta)^s \tilde{z} = 0 & \text{in } \Omega, \\ \tilde{z} = z & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{3.6}$$

  in a weak sense.

- That is, given $z \in H^s(\mathbb{R}^N \setminus \Omega)$, there exists a function $\tilde{z} \in H^s(\mathbb{R}^N)$ such that $\tilde{z}|_{\mathbb{R}^N \setminus \Omega} = z$ and $\tilde{z}$ solves (3.6) in the sense that

$$\frac{C_{N,s}}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(\tilde{z}(x) - \tilde{z}(y))(v(x) - v(y))}{|x-y|^{N+2s}} \, dxdy = 0 \quad \text{for all } v \in H_0^s(\Omega),$$

  and there is a constant $C > 0$ such that

$$\|\tilde{z}\|_{H^s(\mathbb{R}^N)} \le C \|z\|_{H^s(\mathbb{R}^N \setminus \Omega)}. \tag{3.7}$$

Proof Cont

- The existence of a weak solution to (3.6) and the continuous dependence on the datum z has been proved in Lecture 1.
- Now if $z \in H^1_{0,0}((0,T);H^s(\mathbb{R}^N \setminus \Omega))$, then it follows from the above arguments that $\tilde{z} \in H^1_{0,0}((0,T);H^s(\mathbb{R}^N))$.
- Next, we show the existence of a unique solution to (2.1b) by using a lifting argument. Let $w := u - \tilde{z}$. Then $w|_\Sigma = 0$ and

$$\begin{cases} \partial_t w + (-\Delta)^s w = -\partial_t \tilde{z} & \text{in } Q, \\ w = 0 & \text{in } \Sigma, \\ w(\cdot,0) = 0 & \text{in } \Omega. \end{cases} \tag{3.8}$$

- Since $\partial_t z \in L^2((0,T);H^s(\mathbb{R}^N \setminus \Omega))$, then $\partial_t \tilde{z} \in L^2((0,T);H^s(\mathbb{R}^N))$.
- Hence, there exists a unique $w \in \mathbb{U}_0$ solving (3.8).

Proof Cont

- Thus, the unique solution $u \in \mathbb{U}$ of our initial system is given by $u = w + \tilde{z}$.
- It remains to show the estimate (3.4).
- Firstly, since $w = 0$ in Σ, it follows from (3.3) that there is a constant $C > 0$ such that

$$\|w\|_{\mathbb{U}} = \|w\|_{\mathbb{U}_0} \leq C\|\partial_t \tilde{z}\|_{L^2((0,T);H^s(\mathbb{R}^N))}. \tag{3.9}$$

- Secondly, it follows from (3.7) that there is a constant $C > 0$ such that

$$\|\tilde{z}\|_{L^2((0,T);H^s(\mathbb{R}^N))} \leq C\|z\|_{L^2((0,T);H^s(\mathbb{R}^N\setminus\Omega))}. \tag{3.10}$$

Proof Cont

- Thirdly, using (3.9) and (3.10) we get that there is a constant $C > 0$ such that

$$\begin{aligned}\|u\|_{\mathbb{U}} &= \|w+\tilde{z}\|_{\mathbb{U}} \leq \|w\|_{\mathbb{U}} + \|\tilde{z}\|_{\mathbb{U}} \\ &\leq C\Big(\|\partial_t \tilde{z}\|_{L^2((0,T);H^s(\mathbb{R}^N))} + \|z\|_{L^2((0,T);H^s(\mathbb{R}^N\setminus\Omega))} \\ &\quad + \|\tilde{z}\|_{H^1((0,T);H^s(\mathbb{R}^N))}\Big) \end{aligned} \tag{3.11}$$

- Since $\tilde{z} \in H^1((0,T);H^s(\mathbb{R}^N))$, it follows from (3.7) that

$$\|\tilde{z}\|_{H^1((0,T);H^s(\mathbb{R}^N))} \leq C\|z\|_{H^1((0,T);H^s(\mathbb{R}^N\setminus\Omega))}. \tag{3.12}$$

Proof Cont

- Note that $\partial_t \tilde{z}$ is a solution of the Dirichlet problem (3.6) with z replaced with $\partial_t z$. Thus, $\partial_t \tilde{z} \in L^2((0,T); H^s(\mathbb{R}^N))$. Hence, using (3.7), we get that

$$\|\partial_t \tilde{z}\|_{L^2((0,T);H^s(\mathbb{R}^N))} \le C\|\partial_t z\|_{L^2((0,T);H^s(\mathbb{R}^N\setminus\Omega))}. \tag{3.13}$$

- Combining (3.12) and (3.13) we get from (3.11) that

$$\|u\|_{\mathbb{U}} \le C\left(\|z\|_{L^2((0,T);H^s(\mathbb{R}^N\setminus\Omega))} + \|\partial_t z\|_{L^2((0,T);H^s(\mathbb{R}^N\setminus\Omega))}\right).$$

- We have shown (3.4) and the proof is finished. □

Remark 1

- Let $(\varphi_n)_{n\in\mathbb{N}}$ be the orthonormal basis of eigenfunctions of $(-\Delta)_D^s$ associated with the eigenvalues $(\lambda_n)_{n\in\mathbb{N}}$. The unique weak solution u of (2.1b) is given by

$$u(x,t) = -\sum_{n=1}^{\infty}\left(\int_0^t \left(z(\cdot,t-\tau),\mathcal{N}_s\varphi_n\right)_{L^2(\mathbb{R}^N\setminus\Omega)} e^{-\lambda_n\tau}\,d\tau\right)\varphi_n(x).$$

- This formula is important when one wants to analyze the controllability properties of our system that we will do in Lecture 3.

Remark 2

1. Even though such a result is typically sufficient in most situations, nevertheless it is not directly useful in the current context of the optimal control problem (2.1) since we are interested in taking the space $\mathcal{Z}_D = L^2(\Sigma)$.
2. Thus, we need existence of solutions (in some sense) to the fractional Dirichlet problem (2.1b) when $z \in L^2(\Sigma)$.
3. We need to introduce another weaker notion of solutions for (2.1b) as in the elliptic case done in Lecture 1.

Our goal

Our next goal is to reduce the regularity requirement on the datum z in both space and time. We shall call the resulting solution u a very-weak solution.

Definition: Very-weak solution

1. Let $z \in L^2(\Sigma)$. A function $u \in L^2((0,T) \times \mathbb{R}^N)$ is said to be a very-weak solution to (2.1b) if the identity

$$\int_Q u\left(-\partial_t v + (-\Delta)^s v\right)\, dxdt = -\int_\Sigma z\mathcal{N}_s v\, dxdt, \tag{3.14}$$

holds for every $v \in L^2((0,T); V) \cap H^1_{0,T}((0,T); L^2(\Omega))$, where we recall that $V := \{v \in H^s_0(\Omega) :\ (-\Delta)^s v \in L^2(\Omega)\} = D((-\Delta)^s_D)$.

2. Recall that $\mathcal{N}_s v$ is the nonlocal normal derivative of v given by

$$\mathcal{N}_s v(x) = C_{N,s} \int_\Omega \frac{v(x) - v(y)}{|x-y|^{N+2s}}\, dy,\ x \in \mathbb{R}^N \setminus \overline{\Omega}.$$

Theorem (Recall Lions' Existence Theorem)

Let $(F, \|\cdot\|_F)$ be a Hilbert space. Let Φ be a subspace of F endowed with a pre-Hilbert scalar product $(((\cdot,\cdot)))$ and associated norm $|||\cdot|||$. Moreover, let $E : F \times \Phi \to \mathbb{C}$ be a sesquilinear form. Assume that the following hold:

1. *The embedding $\Phi \hookrightarrow F$ is continuous, that is, there is $C > 0$ such that*
$$\|\varphi\|_F \leq C|||\varphi||| \quad \forall\, \varphi \text{ in } \Phi. \tag{3.15}$$
2. *For all $\varphi \in \Phi$, the mapping $u \mapsto E(u,\varphi)$ is continuous on F.*
3. *There is a constant $C > 0$ such that*
$$|E(\varphi,\varphi)| \geq C|||\varphi|||^2 \quad \text{for all} \quad \varphi \in \Phi. \tag{3.16}$$

If $\varphi \mapsto L(\varphi)$ is a continuous linear functional on Φ, then there is $u \in F$ (unique if Φ is a Hilbert space) verifying

$$E(u,\varphi) = L(\varphi) \quad \text{for all} \quad \varphi \in \Phi.$$

Theorem (Existence of very-weak solutions (Antil, Verma, & W. 2020))

Let $z \in L^2(\Sigma)$. Then there exists a unique very-weak solution u to (2.1b) *that fulfills*

$$\|u\|_{L^2(Q)} \leq C\|z\|_{L^2(\Sigma)}, \tag{3.17}$$

for a constant $C > 0$. In addition, if $z \in H^1_{0,0}((0,T); H^s(\mathbb{R}^N \setminus \Omega))$, then the following assertions hold.

1. *Every weak solution of* (2.1b) *is also a very-weak solution.*
2. *Every very-weak solution of* (2.1b) *that belongs to $\mathbb{U}$ is also a weak solution.*

Proof: We verify the conditions of Lions' Theorem

- We endow $\Phi := \mathcal{D}(Q)$ with the pre-Hilbert norm

$$|||u|||^2 := \|u\|^2_{L^2((0,T);H_0^s(\Omega))}.$$

- Clearly, the embedding $\Phi \hookrightarrow L^2(Q)$ is continuous.
- Let $\mathbb{E} : L^2(Q) \times \Phi \to \mathbb{R}$ and $L : \Phi \to \mathbb{R}$ be given by

$$\mathbb{E}(u,v) := \int_Q u\Big(-\partial_t v + (-\Delta)^s v\Big)\, dxdt \text{ and } Lv := -\int_\Sigma z\mathcal{N}_s v\, dxdt.$$

- Let $v \in \Phi$ be fixed. Claim: The map $u \mapsto \mathbb{E}(u,v)$ is continuous on $L^2(Q)$. Indeed, for every $u \in L^2(Q)$, we have

$$\begin{aligned}|\mathbb{E}(u,v)| &= \left|\int_Q u\left(-\partial_t v + (-\Delta)^s v\right)\, dxdt\right| \\ &\le \|u\|_{L^2(Q)}\| - \partial_t v + (-\Delta)^s v\|_{L^2(Q)}.\end{aligned}$$

Proof: Cont

- Claim: $\mathbb{E}$ is coercive on Φ, i.e. there is a constant $C > 0$ such that

$$\mathbb{E}(v,v) = \int_Q v\left(-\partial_t v + (-\Delta)^s v\right)\, dxdt \geq C|||v|||^2, \ \ \forall v \in \Phi.$$

- Indeed, let $v \in \Phi$. Integrating by parts, we get that

$$E(v,v) \geq C\|v\|^2_{L^2((0,T);H_0^s(\Omega))}.$$

- Using the previous estimates we get that for every $v \in \Phi$,

$$|L(v)| \leq \|z\|_{L^2(\Sigma)}\|\mathcal{N}_s v\|_{L^2(\Sigma)} \leq C\|z\|_{L^2(\Sigma)}\|v\|_{L^2((0,T);H_0^s(\Omega))}.$$

- By Lions' Theorem, the system (2.1b) has a very weak solution $u \in L^2((0,T)\times\mathbb{R}^N)$ (recall that $u = z$ in $\Sigma = (0,T)\times(\mathbb{R}^N\setminus\Omega)$).

Proof: Cont: Uniqueness

- Assume that the system has two solutions u_1 and u_2 with the same exterior datum z. Let $u := u_1 - u_2$. Then by definition,

$$\int_Q u\left(-\partial_t v + (-\Delta)^s v\right)\, dxdt = 0 \tag{3.18}$$

  for every $v \in L^2((0,T);V) \cap H^1_{0,T}((0,T);L^2(\Omega))$.

- Notice that for every $w \in L^2(Q)$, there is a unique $v \in L^2((0,T);V) \cap H^1_{0,T}((0,T);L^2(\Omega))$ satisfying

$$-\partial_t v + (-\Delta)^s v = w \quad \text{in} \quad Q.$$

- Thus we can deduce that $\displaystyle\int_Q uw\, dxdt = 0$ for every $w \in L^2(Q)$.
- By the fundamental lemma of the calculus of variation we can conclude that $u = 0$ in Q. We have shown uniqueness.

Proof: Cont: The estimate (3.17)

- Notice that $\mathcal{N}_s v \in L^2(\Sigma)$. We define the mapping

$$\mathcal{M} : L^2(Q) \to L^2(\Sigma), \quad \zeta \mapsto \mathcal{M}\zeta := -\mathcal{N}_s v.$$

Then $\mathcal{M}$ is linear and continuous: There is a $C > 0$ such that

$$\|\mathcal{M}\zeta\|_{L^2(\Sigma)} = \|\mathcal{N}_s v\|_{L^2(\Sigma)} \le C\|\zeta\|_{L^2(Q)}.$$

- Finally, we notice that there is a constant $C > 0$ such that

$$\left|\int_Q u\zeta \, dxdt\right| = \left|\int_\Sigma z\mathcal{M}\zeta \, dxdt\right| \le \|z\|_{L^2((\Sigma)}\|\mathcal{N}_s v\|_{L^2(\Sigma)}$$
$$\le C\|z\|_{L^2(\Sigma)}\|\zeta\|_{L^2(Q)}.$$

Dividing both sides of the preceding estimate by $\|\zeta\|_{L^2(Q)}$ and taking the supremum over $\zeta \in L^2(Q)$, we obtain (3.17). □

Recall that $\mathcal{Z}_D := L^2(\Sigma), \quad \mathcal{U}_D := L^2(Q)$

1. In view of the existence theorem of very weak solution, the following (solution-map) control-to-state map

$$S : \mathcal{Z}_D \to \mathcal{U}_D, \quad z \mapsto Sz =: u,$$

   is well-defined, linear and continuous.

2. Notice that for $z \in \mathcal{Z}_D$, we have that $u := Sz \in L^2((0,T) \times \mathbb{R}^N)$.
3. Let $J : \mathcal{U}_D \to \mathbb{R}$ and consider the reduced functional

$$\mathcal{J} : \mathcal{Z}_D \to \mathbb{R}, \ \ z \mapsto \mathcal{J}(z) := \left(J(Sz) + \frac{\xi}{2} \|z\|^2_{\mathcal{Z}_D} \right).$$

4. We can rewrite the reduced optimal control problem as follows:

$$\min_{z \in \mathcal{Z}_{ad,D}} \mathcal{J}(z). \tag{4.1}$$

Theorem (Existence of optimal solutions (Antil, Verma & W., 2020))

- *Let $\mathcal{Z}_{ad,D}$ be a closed and convex subset of $\mathcal{Z}_D$. Let either $\xi > 0$ with $J \geq 0$ or $\mathcal{Z}_{ad,D}$ bounded and $J : \mathcal{U}_D \to \mathbb{R}$ weakly lower-semicontinuous. Then there exists a solution $\bar{z}$ to (4.1).*
- *If either J is convex and $\xi > 0$ or J is strictly convex and $\xi \geq 0$, then $\bar{z}$ is unique.*

Proof

- For the functional $\mathcal{J}:\mathcal{Z}_{ad,D}\to\mathbb{R}$, it is possible to construct a minimizing sequence $\{z_n\}_{n\in\mathbb{N}}$ such that

$$\inf_{z\in\mathcal{Z}_{ad,D}}\mathcal{J}(z)=\lim_{n\to\infty}\mathcal{J}(z_n).$$

- If $\xi>0$ with $J\geq 0$ or $\mathcal{Z}_{ad,D}\subset\mathcal{Z}_D$ is bounded, then $\{z_n\}_{n\in\mathbb{N}}$ is a bounded sequence in $\mathcal{Z}_D$ which is a Hilbert space.
- As a result, we have that (up to a subsequence if necessary) $z_n\rightharpoonup\bar{z}$ (weak convergence) in $\mathcal{Z}_D$ as $n\to\infty$.
- Since $\mathcal{Z}_{ad,D}$ is closed and convex, hence, is weakly closed, we have that $\bar{z}\in\mathcal{Z}_{ad,D}$.
- It remains to show that $(S\bar{z},\bar{z})=(\bar{u},\bar{z})$ satisfies the state equation and that $\bar{z}$ is a minimizer to (4.1).

Proof Cont.

- Notice that (u_n, z_n) satisfies the identity

$$\int_Q u_n \left(-\partial_t v + (-\Delta)^s v\right)\, dxdt = -\int_\Sigma z_n \mathcal{N}_s v\, dxdt \tag{4.2}$$

for all $v \in L^2((0,T);V) \cap H^1_{0,T}((0,T);L^2(\Omega))$.

- Since $u_n := Sz_n \rightharpoonup S\bar{z} =: \bar{u}$ in $\mathcal{U}_D$ as $n \to \infty$, and $z_n \rightharpoonup \bar{z}$ in $\mathcal{Z}_{ad,D}$ as $n \to \infty$, we can immediately take the limit in (4.2) as $n \to \infty$, and conclude that $(\bar{u}, \bar{z}) \in \mathcal{U}_D \times \mathcal{Z}_{ad,D}$ fulfills the state equation. That is,

$$\int_Q \bar{u} \left(-\partial_t v + (-\Delta)^s v\right)\, dxdt = -\int_\Sigma \bar{z} \mathcal{N}_s v\, dxdt$$

for all $v \in L^2((0,T);V) \cap H^1_{0,T}((0,T);L^2(\Omega))$.

Proof Cont.

- Next, that $\bar{z}$ is the minimizer of (4.1) follows from the fact that $\mathcal{J}$ is weakly lower semicontinuous. Indeed, $\mathcal{J}$ is the sum of two weakly lower semicontinuous functions (recall the norm is continuous and convex).
- Finally, the uniqueness of $\bar{z}$ follows from the stated assumptions on J and ξ which leads to the strict convexity of the functional $\mathcal{J}$. □

Our next goal is to derive the optimality conditions

In order to derive the first order necessary optimality conditions, we need an expression of the adjoint operator S^*, where we recall that S is the control-to-state map given by

$$S : \mathcal{Z}_D \to \mathcal{U}_D, \quad z \mapsto Sz =: u.$$

The operator S^*

The adjoint operator $S^* =: \mathcal{U}_D \to \mathcal{Z}_D$ for the state equation (2.1b) is given by

$$S^* w = -\mathcal{N}_s p \in \mathcal{Z}_D,$$

where $w \in \mathcal{U}_D$, $p \in L^2((0,T); H_0^s(\Omega)) \cap H^1_{0,T}((0,T); H^{-s}(\Omega))$ is the weak solution to the following adjoint problem:

$$\begin{cases} -\partial_t p + (-\Delta)^s p & = w \quad \text{in } Q, \\ p & = 0 \quad \text{in } \Sigma, \\ p(T,\cdot) & = 0 \quad \text{in } \Omega, \end{cases} \tag{4.3}$$

and $\mathcal{N}_s p$ is the nonlocal normal derivative of p given for $(x,t) \in \Sigma$ by

$$\mathcal{N}_s p(x,t) = \int_\Omega \frac{p(x,t) - p(y,t)}{|x-y|^{N+2s}}\, dy, \quad x \in \mathbb{R}^N \setminus \overline{\Omega}.$$

Proof

- Since S is linear and bounded, it follows that S^* is well-defined.
- Now for every $w \in \mathcal{U}_D$ and $z \in \mathcal{Z}_D$, we have that

$$(w, Sz)_{L^2(Q)} = (S^* w, z)_{L^2(\Sigma)}.$$

- We notice that using semigroups theory we have that $p \in L^2((0,T); V) \cap H^1_{0,T}((0,T); L^2(\Omega))$. Thus, $\partial_t p$, $(-\Delta)^s p \in L^2(Q)$.
- Next, testing the equation (4.3) with Sz which solves the state equation in the very-weak sense we get that

$$\begin{aligned}(w, Sz)_{L^2(Q)} &= (-\partial_t p + (-\Delta)^s p, Sz)_{L^2(Q)}\\ &= -(z, \mathcal{N}_s p)_{L^2((\Sigma)}\\ &= (z, S^* w)_{L^2(\Sigma)},\end{aligned}$$

where we have used the definition of very-weak solutions. □

Theorem (The optimality conditions (Antil, Verma & W. (2020))

Assume that $\xi > 0$. Let $\mathcal{Z} \subset \mathcal{Z}_D$ be open such that $\mathcal{Z}_{ad,D} \subset \mathcal{Z}$. Let $u \mapsto J(u) : \mathcal{U}_D \to \mathbb{R}$ be continuously Fréchet differentiable with $J'(u) \in \mathcal{U}_D$. If $\bar{z}$ is a minimizer of (4.1) over $\mathcal{Z}_{ad,D}$, then the first order necessary optimality conditions are given by

$$(-\mathcal{N}_s\bar{p} + \xi\bar{z}, z - \bar{z})_{L^2(\Sigma)} \geq 0, \quad \forall z \in \mathcal{Z}_{ad,D}, \tag{4.4}$$

where $\bar{p} \in L^2((0,T); H_0^s(\Omega)) \cap H^1_{0,T}((0,T); H^{-s}(\Omega))$ solves

$$\begin{cases} -\partial_t\bar{p} + (-\Delta)^s\bar{p} = J'(\bar{u}) & \text{in } Q, \\ \bar{p} = 0 & \text{in } \Sigma, \\ \bar{p}(T,\cdot) = 0 & \text{in } \Omega, \end{cases} \tag{4.5}$$

with $\bar{u} := S\bar{z}$. Finally, (4.4) is equivalent to

$$\bar{z} = \mathcal{P}_{Z_{ad},D}\left(\xi^{-1}\mathcal{N}_s\bar{p}\right). \tag{4.6}$$

Proof

- The statements are a direct consequence of the differentiability properties of J and the chain rule, combined with the formula of S^*.
- We have introduced the open set $\mathcal{Z}$ to properly define the derivative of $\mathcal{J}$. Let $h \in \mathcal{Z}_D$ be given. Then the directional derivative of $\mathcal{J}$ is given by

$$\begin{aligned} \mathcal{J}'(\bar{z})h &= (J'(S\bar{z}), Sh)_{L^2(Q)} + \xi(\bar{z}, h)_{L^2(\Sigma))} \\ &= (S^* J'(S\bar{z}) + \xi\bar{z}, h)_{L^2(\Sigma)}, \end{aligned} \tag{4.7}$$

  where we have used that $J'(S\bar{z}) \in L^2(Q)$.
- Next from the formula of S^*, we have that

$$S^* J'(S\bar{z}) = -\mathcal{N}_s \bar{p},$$

  where $\bar{p}$ solves the adjoint equation (4.5).

Proof Cont

- Recall that $\bar{p} \in L^2((0,T); H_0^s(\Omega)) \cap H^1_{0,T}((0,T); H^{-s}(\Omega))$ solving (4.5) also has the following regularity: $\partial_t \bar{p} \in L^2(Q)$.
- This implies that $(-\Delta)^s \bar{p} \in L^2(Q)$.
- This implies that $\mathcal{N}_s \bar{p} \in L^2(\Sigma)$.
- Substituting this expression of $S^* J'(S\bar{z})$ in (4.7), we obtain for every $h \in L^2(\Sigma)$,

$$\mathcal{J}'(\bar{z})h = (-\mathcal{N}_s \bar{p} + \xi \bar{z}, h)_{L^2(\Sigma)}.$$

- From this, we can deduce (4.4).
- Finally, (4.6) follows as in the elliptic case.

□

Example

Let $u_d \in L^2(Q)$ be a given fixed target.

1. The functional $J : L^2(Q) \to \mathbb{R}$ given by

$$J(u) := \frac{1}{2} \int_Q |u - u_d|^2 \, dxdt$$

satisfies all our hypothesis.

2. In that case, for $h \in L^2(Q)$, we have that

$$J'(u)h = \lim_{\lambda \to 0} \frac{J(u + \lambda h) - J(u)}{\lambda} = \int_Q (u - u_d) h \, dxdt.$$

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THANKS!