Lecture 3: Controllability of fractional parabolic PDE

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1. Interior controllability properties
   - Null and exactly controllable
   - Approximately controllable

2. Exterior controllability properties
   - Null and exactly controllable
   - Approximately controllable

Main objectives

- Here we discuss the controllability properties of fractional heat equations.
- We will consider two types of control: interior and exterior controls.
- As we have already mentioned in Lecture 1, these problems may be difficult (very difficult) to solve.
- For each result, we shall give an equivalent characterization.
- It does not mean that the equivalent characterization is easy to prove. But most of the times it will be the only known alternative to solve the problem.
- Several topics in this lecture are still open problems and are excellent problems of research for PhD students and/or young researchers who are interested in this field of mathematics and its applications.

Formulation of our interior controllability problem

Let $\Omega \subset \mathbb{R}^N$ $(N \geq 1)$ be a bounded domain. We consider the following control problem:

$$\begin{cases} y_t + (-\Delta)^s y = f\chi_\omega & \text{in } \; Q := \Omega \times (0,T), \\ y = 0 & \text{in } \; \Sigma := (\mathbb{R}^N \setminus \Omega) \times (0,T), \\ y(\cdot,0) = y_0, & \text{in } \; \Omega. \end{cases} \tag{1.1}$$

- Here $(-\Delta)^s$ $(0 < s < 1)$ is the fractional Laplace operator.
- In (1.1), y is the state to be controlled.
- f is the control function which is localized in a nonempty open set $\omega \subset \Omega$, and χ_ω stands for the characteristic function of the set ω.

Definition of solutions to the state equation

- By a finite energy solution of (1.1) we mean a function

$$y \in L^2((0,T); H_0^s(\Omega)) \cap H^1((0,T); H^{-s}(\Omega))$$

such that $y(\cdot, 0) = y_0$ and the equality

$$\langle y_t, \phi \rangle_{H^{-s}(\Omega), H_0^s(\Omega)} + \mathcal{E}(y, \phi) = \langle f, \phi \rangle_{H^{-s}(\Omega), H_0^s(\Omega)} \tag{1.2}$$

holds for every $\phi \in H_0^s(\Omega)$ and a.e. $t \in (0,T)$, where for $u, v \in H_0^s(\Omega)$, we have set

$$\mathcal{E}(u, v) := \frac{C_{N,s}}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{N+2s}} \, dxdy.$$

- Notice that $L^2((0,T); H_0^s(\Omega)) \cap H^1((0,T); H^{-s}(\Omega)) \hookrightarrow C([0,T]; L^2(\Omega))$ so that the value $y(\cdot, 0)$ make sense.

Existence of solutions

Let $y_0 \in L^2(\Omega)$ and $f \in L^2((0,T); H^{-s}(\Omega))$.

1. Then (1.1) has a unique finite energy solution y.
2. In addition, if $y_0 = 0$ and $f \in L^2((0,T) \times \omega)$, then

$$y \in L^\infty((0,T); H_0^s(\Omega)) \cap H^1((0,T); L^2(\Omega)).$$

3. In any case, the unique solution y is given by (using the formula of variation of parameters)

$$y(x,t) = S(t)y_0(x) + \int_0^t S(t-\tau)f(x,\tau)\, d\tau$$

   where $S = (S(t))_{t \geq 0}$ is the strongly continuous semigroup generated by the operator $-(-\Delta)_D^s$ (the realization of $(-\Delta)^s$ with the zero Dirichlet exterior condition) that we have defined in Lecture 1.

The set of reachable states

The set of reachable states is given by

$$\mathcal{R}(y_0, T) := \{y(\cdot, T) : \ f \in L^2((0, T) \times \omega)\}$$

The three notions of controllability

1. (1.1) is said to be null controllable in time $T > 0$ iff $0 \in \mathcal{R}(y_0, T)$.
Equivalently $\exists\ f \in L^2((0,T) \times \omega)$ such that the solution y satisfies

$$y(\cdot, T) = 0 \text{ a.e. in } \Omega.$$

2. (1.1) is exactly controllable in time $T > 0$ iff $\mathcal{R}(y_0, T) = L^2(\Omega)$.
Equivalently $\forall\ y_d \in L^2(\Omega)$, $\exists f \ \in L^2((0,T) \times \omega)$ such that y satisfies

$$y(\cdot, T) = y_d \text{ a.e. in } \Omega.$$

3. (1.1) is said to be approximately controllable in time $T > 0$ iff

$$\mathcal{R}(y_0, T) \text{ is dense in } L^2(\Omega).$$

Equivalent $\forall\ y_0, y_1 \in L^2(\Omega)$ and $\varepsilon > 0$, $\exists\ f \in L^2((0,T) \times \omega)$ such that

$$\|y(\cdot, T) - y_1\|_{L^2(\Omega)} < \varepsilon.$$

Remark

1. One of the most important property of the heat equation is its regularizing effect.
2. When $\Omega \setminus \omega \neq \emptyset$, the solutions of (1.1) belong to $C^\infty(\Omega \setminus \omega)$ at time $t = T$. Hence, the restriction of the elements of $\mathcal{R}(y_0, T)$ to $\Omega \setminus \omega$ are C^∞-functions.
3. Then, the trivial case $\omega = \Omega$ (i. e. when the control acts on the entire domain) being excepted, exact controllability may not hold.
4. In this sense, the notion of exact controllability is not very relevant for the heat equation. This is due to the strong time irreversibility of the heat equation.

Null and exact controllabilities are the same

The following assertions are equivalent.

1. The system (1.1) is null controllable in time $T > 0$.
2. The system (1.1) is exactly controllable in time $T > 0$.

Proof

1. It is clear that exactly controllable implies null controllable.
2. It is easy to see that if null controllability holds, then any initial data may be led to any final state of the form $S(T)y_0$ with $y_0 \in L^2(\Omega)$, i. e. to the range of the semigroup in time $t = T$.
3. Indeed, let $y_0, z_0 \in L^2(\Omega)$ and remark that

$$\mathcal{R}(y_0 - z_0, T) = \mathcal{R}(y_0, T) - S(T)z_0.$$

Since $0 \in \mathcal{R}(y_0 - z_0, T)$, it follows that $S(T)z_0 \in \mathcal{R}(y_0, T)$. □

First characterization of null/exact controllability

The following assertions are equivalent.

1. The system (1.1) is null controllable at time $T > 0$.
2. For every initial datum $y_0 \in L^2(-1,1)$, there exists a control function $f \in L^2((0,T) \times \omega)$ such that

$$\int_{-1}^{1} y_0(x)v(x,0)\,dx = \int_0^T \int_\omega f(x,t)v(x,t)\,dxdt, \tag{1.3}$$

where v is the unique solution of the associated dual system:

$$\begin{cases} -v_t(t,x) + (-\Delta)^s v(t,x) = 0 & \text{in } Q, \\ v = 0 & \text{on } \Sigma, \\ v(\cdot,T) = v_T & \text{in } \Omega. \end{cases} \tag{1.4}$$

Proof

Let $y_0 \in L^2(-1,1)$ and $f \in L^2((0,T)\times\omega)$. Taking v solution of the dual problem (1.4) as a test function in the definition of weak solutions to (1.1), using the integration by parts formula, we obtain that

$$0 = \int_0^T \langle y_t(\cdot,t) + (-\Delta)^s y(\cdot,t), v(\cdot,t)\rangle \, dt \tag{1.5}$$

$$= \int_\Omega \Big(y(x,T)v(x,T) - y(x,0)v(x,0) \Big)\, dx + \int_0^T \int_\omega y(x,t)f(x,t)\, dxdt.$$

1. If (1.3) holds, then by (1.5) $\int_{-1}^{1} y(x,T)v(x,T)\, dx = 0$ for every $v_T \in L^2(\Omega)$. Thus, $y(\cdot,T) = 0$ in Ω and (1.1) is null controllable.
2. Conversely, if (1.1) is null controllable, that is, if $y(\cdot,T)=0$ in Ω, then (1.3) follows from (1.5). □

Second characterization of null/exact controllability

The following assertions are equivalent.

1. The system (1.1) is null/exactly controllable in time $T > 0$.
2. The following observability inequality holds: There is a constant $C(T) > 0$ such that

$$\|v(0,\cdot)\|_{L^2(\Omega)}^2 \le C(T) \int_0^T \int_\omega |v|^2 \, dxdt, \tag{1.6}$$

where v is the unique solution of the dual system (1.4).

Remark

- Inequality (1.6) is called observation or observability inequality.
- It shows that the quantity

$$\int_0^T \int_\omega |v|^2 \, dxdt$$

(the observed one) which depends only on the restriction of v to the subset ω of Ω, uniquely determines the solution of the dual problem (1.4).

- Inequality (1.6) is usually very difficult to prove even in the classical case $s = 1$. In the fractional case we still do NOT know how to prove such an inequality.

Comments

Recall that (1.1) is null controllable in time $T > 0$ if and only if

$$\|v(0,\cdot)\|^2_{L^2(\Omega)} \leq C(T)\int_0^T\int_\omega |v|^2\,dxdt, \tag{1.7}$$

- Once (1.7) is known to hold one can obtain the control with minimal L^2-norm among the admissible ones.
- To do that it is sufficient to minimize the functional

$$J(v_0) = \frac{1}{2}\int_0^T\int_\omega v^2\,dxdt + \int_\Omega v(0)y_0\,dx \tag{1.8}$$

on the Hilbert space

$$H = \left\{ v_0 :\ \text{the solution v of (1.4) satisfies } \int_0^T\int_\omega v^2\,dxdt < \infty \right\}.$$

Comments Cont.

- In fact, H is the completion of $L^2(\Omega)$ with respect to the norm

$$\left(\int_0^T \int_\omega v^2 \, dxdt\right)^{1/2}.$$

- This shows that H is much larger than $L^2(\Omega)$.
- Observe that J is convex and continuous on H.
- On the other hand, (1.7) guarantees the coercivity of J and the existence of minimizers.
- Due to the irreversibility of the system (1.1), (1.7) is not easy to prove.
- As I said before, we still do NOT know how to prove (1.7) in general.

Theorem (Biccari et al. 2020)

Let $N = 1$ and $\Omega = (-1, 1)$. Then the system (1.1) is null controllable for every $T > 0$ if and only if $1/2 < s < 1$.

Proof

- Let $\{\mu_k\}_{k\geq 1}$ be the eigenvalues of $(-\Delta)^s_D$ with eigenfunctions φ_k.
- For $k \in \mathbb{N}$, $v_k(x,t) = \varphi_k(x)e^{\mu_k(T-t)}$ is a solution of (1.4) with $v_T = \varphi_k$.
- Multiplying (1.1) with v_k and integrating over $Q = (0,T) \times (-1,1)$ we get that $u(x,T) = 0$ if and only if

$$\int_{-1}^{1}\int_{\omega} \varphi_k(x)e^{-\mu_k t}f(x,t)\,dxdt = -\int_{-1}^{1} u_0(x)\varphi_k(x)\,dx =: -u_k^0. \quad (1.9)$$

- Assume that there is a sequence q_k biorthogonal to $e^{-\mu_k t}$ on $(0,T)$, i.e.,

$$\int_0^T q_n(t)e^{-\mu_k t}\,dt = \delta_{n,k}.$$

Proof Cont.

- Then the following control function f satisfies (1.9):

$$f(x,t) := -\sum_{k\geq 1} \frac{u_k^0}{\|\varphi_k\|^2_{L^2(\omega)}} q_k(t)\varphi_k(x) \tag{1.10}$$

- By Müntz's theorem a biorthogonal sequence exists if and only if

$$\sum_{k\geq 1} \frac{1}{\mu_k} < +\infty. \tag{1.11}$$

- By a result of Fattorini and Russell, if we have the gap condition:

$$\text{There exists } \gamma > 0 \text{ such that } \mu_{k+1} - \mu_k \geq \gamma \text{ for all } k \geq 1, \tag{1.12}$$

then

$$\|q_k\|_{L^2(0,T)} \leq Ce^{\tau\mu_k},\ \forall\, k \in \mathbb{N},\ \tau > 0. \tag{1.13}$$

Proof Cont.

- We know that

$$\mu_k = \left(\frac{k\pi}{2} - \frac{(1-s)\pi}{4}\right)^{2s} + O\left(\frac{1}{k}\right).$$

- We can see that (1.11) and (1.12) are both satisfied if and only if $1/2 < s < 1$. If $0 < s \leq 1/2$, the series (1.11) diverges, since it behaves as a harmonic series.
- The convergence of the series in (1.10) is a consequence of (1.13) and the following lower bound: There is a constant $C > 0$ such that

$$\|\varphi_k\|_{L^2(\omega)} \geq C > 0, \quad \forall\, k \geq 1.$$

The proof is finished. □

What about the case $N \geq 2$? Let $\Omega \subset \mathbb{R}^N$ be a bounded smooth domain

1. Let $\omega \subset \Omega$ be a neighborhood of $\partial\Omega$ as follows:

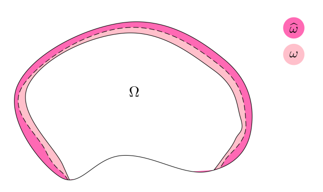

Figure: Domain Ω and control region ω.

2. Then (1.1) is null/exactly controllable for every $T > 0 \Leftrightarrow 1/2 < s < 1$.
3. This is proved by using the associated wave equation, the Lebeau-Robiano strategy, and transmutation techniques.
4. For an arbitrary nonempty $\omega \subset \Omega$, we still do NOT know if (1.1) is null/exactly controllable.

Remark 1

1. Null controllability implies approximate controllability.
   - Indeed, we have shown that, whenever null controllability holds,
   $$S(T)[L^2(\Omega)] \subset \mathcal{R}(y_0, T)$$
   for all $y_0 \in L^2(\Omega)$.
   - Taking into account that all the eigenfunctions of the fractional Laplacian with zero Dirichlet exterior condition belong to $S(T)[L^2(\Omega)]$ we can deduce that the set of reachable states is dense and, consequently, that approximate controllability holds.
2. The converse is NOT in general true. That is, approximate controllability does not always implies null/exact controllability.

Remark 2

For the approximate controllability of (1.1) it is enough to consider the initial datum $y_0 = 0$.

1. Indeed, the linearity of the system implies that

$$\mathcal{R}(y_0, T) = \mathcal{R}(0, T) + S(T)y_0.$$

2. Since $S(T)y_0 \in L^2(\Omega)$ we can deduce that

$$\mathcal{R}(y_0, T) \text{ is dense in } \; L^2(\Omega)$$

if and only if

$$\mathcal{R}(0, T) \text{ is dense in } \; L^2(\Omega).$$

□

Remark 3

Approximate controllability together with uniform estimates on the approximate controls functions may led to null/exact controllability properties.

1. More precisely, given y_1, we have that $y_1 \in \mathcal{R}(y_0, T)$ if and only if there exists a sequence $(f^\varepsilon)_{\varepsilon>0}$ of controls such that

$$\|y(\cdot, T) - y_1\|_{L^2(\Omega)} < \varepsilon$$

and $(f^\varepsilon)_{\varepsilon>0}$ is bounded in $L^2((0,T)\times\omega)$.

2. Indeed, in this case any weak limit in $L^2((0,T)\times\omega)$ of the sequence $(f^\varepsilon)_{\varepsilon>0}$ of controls gives an exact control which makes that

$$y(T,\cdot) = y_1.$$

□

Characterization of the approximate controllability property

The following assertions are equivalent.

1. The system (1.1) is approximately controllable in time $T > 0$.
2. The dual system (1.4) has the unique continuation property: That is, let $\omega \subset \Omega$ be an arbitrary nonempty open set and v the unique solution of the adjoint system (1.4), that is,

$$\begin{cases} -v_t(t,x) + (-\Delta)^s v(t,x) = 0 & \text{in } \; Q, \\ v = 0 & \text{on } \; \Sigma, \\ v(\cdot, T) = v_T & \text{in } \; \Omega. \end{cases}$$

If

$$v = 0 \;\text{ in } (0,T) \times \omega, \;\; \text{then} \;\; v = 0 \;\text{ in } \; Q = (0,T) \times \Omega.$$

Ideas use for the proof

- The proof of this result requires a fine analysis and analytic continuation properties of solutions of the adjoint system (1.4).
- It also uses the following unique continuation property of the eigenvalues problem: Let $\lambda > 0$ and $\varphi \in H_0^s(\Omega)$ satisfy

$$(-\Delta)^s \varphi = \lambda \varphi \ \text{ in } \ \Omega \tag{1.14}$$

  in the weak sense.
- Let $\omega \subset \Omega$ be an arbitrary nonempty open set. If $\varphi = 0$ in ω, then $\varphi = 0$ in Ω.
- This latter result has been first proved by W. (SICON 2019).
- After this result, several unique continuation properties have been obtained in the literature.

Theorem (W. SICON 2019)

Let $\omega \subset \Omega$ be an arbitrary nonempty open set. Then the system (1.1) is always approximately controllable for any $T > 0$ and $f \in L^2((0,T)\times\omega)$.

Proof

- From Hahn-Banach Theorem, $\mathcal{R}(0,T)$ is dense in $L^2(\Omega) \Leftrightarrow v_T \in L^2(\Omega)$ is such that $\int_\Omega y(T)v_T\,dx = 0$, for a solution y of (1.1), then $v_T = 0$.
- Let v be the solution of the dual system (1.4) with final datum v_T.
- Multiplying (1.1) by v, (1.4) by y and integrating, we get:

$$
\begin{aligned}
&\int_0^T\int_\omega fv\,dxdt = \int_Q (y_t + (-\Delta)^s y)\,v\,dxdt \\
&= \int_Q (-v_t + (-\Delta)^s v)\,y\,dxdt + \int_\Omega \Big(y(T)v(T) - y(0)v(0)\Big)\,dx \\
&+ \int_\Sigma \left(v\mathcal{N}_s y - y\mathcal{N}_s v\right)\,dxdt = \int_\Omega y(T)v_T\,dx.
\end{aligned}
$$

Proof Cont

- Hence, $\int_\Omega y(T)v_T\ dx = 0$ if and only if $\int_0^T \int_\omega fv\ dxdt = 0$.
- If
$$\int_0^T \int_\omega fv\ dxdt = 0$$
for all $f \in L^2((0,T)\times\omega)$, then
$$v \equiv 0 \text{ in } (0,T)\times\omega.$$
- It follows from the unique continuation property for solutions of the dual system (1.4) that $v = 0$ in $Q := (0,T)\times\Omega$.
- Since the solution v of the dual system (1.4) is unique, it follows that $v_T = 0$. □

Variational approach to approximate controllability

- Here we give another proof of the approximate controllability result. The proof has the advantage of being constructive and it allows to compute explicitly the approximate controls.
- Let us fix the control time $T > 0$ and the initial datum $y_0 = 0$. Let $y_1 \in L^2(\Omega)$ be the final target and $\varepsilon > 0$ be given. Recall that we are looking for a control f such that the solution y of (1.1) satisfies

$$\|y(T) - y_1\|_{L^2(\Omega)} \leq \varepsilon. \tag{1.15}$$

- We define the functional $J_\varepsilon : L^2(\Omega) \to \mathbb{R}$ by

$$J_\varepsilon(v_T) = \frac{1}{2}\int_0^T \int_\omega v^2 \, dxdt + \varepsilon \|v_T\|_{L^2(\Omega)} - \int_\Omega y_1 v_T \, dx \tag{1.16}$$

where v is the solution of the adjoint equation (1.4) with final datum v_T.

The minimum of J_ε gives a control to our problem

If $\hat{v}_T$ is a minimum point of J_ε in $L^2(\Omega)$ and $\hat{v}$ is the solution of the adjoint equation (1.4) with final datum $\hat{v}_T$, then $f = \hat{v}|_\omega$ is a control for (1.1), that is, (1.15) is satisfied.

Proof

- Suppose J_ε attains its minimum value at $\hat{v}_T \in L^2(\Omega)$. Then for any $v_0 \in L^2(\Omega)$ and $h \in \mathbb{R}$ we have $J_\varepsilon(\hat{v}_T) \leq J_\varepsilon(\hat{v}_T + hv_0)$.
- On the other hand

$$
\begin{aligned}
&J_\varepsilon(\hat{v}_T + hv_0)\\
=&\frac{1}{2}\int_0^T\int_\omega |\hat{v} + hv|^2\,dxdt + \varepsilon\|\hat{v}_T + hv_0\|_{L^2(\Omega)} - \int_\Omega y_1(\hat{v}_T + hv_0)\,dx\\
=&\frac{1}{2}\int_0^T\int_\omega |\hat{v}|^2\,dxdt + \frac{h^2}{2}\int_0^T\int_\omega |v|^2\,dxdt + h\int_0^T\int_\omega |\hat{v}v|^2\,dxdt\\
&+\varepsilon\|\hat{v}_T + hv_0\|_{L^2(\Omega)} - \int_\Omega y_1(\hat{v}_T + hv_0)\,dx.
\end{aligned}
$$

Proof Cont.

- Thus,

$$0 \leq \varepsilon\Big(\|\hat{v} + hv_0\|_{L^2(\Omega)} - \|\hat{v}\|_{L^2(\Omega)}\Big) + \frac{h^2}{2}\int_0^T\int_\omega |v|^2\,dxdt + h\Big(\int_0^T\int_\omega |\hat{v}v|^2\,dxdt - \int_\Omega y_1 v_0\,dx\Big).$$

- Since

$$\|\hat{v} + hv_0\|_{L^2(\Omega)} - \|\hat{v}\|_{L^2(\Omega)} \leq |h|\|v_0\|_{L^2(\Omega)}$$

we obtain that for all $h \in \mathbb{R}$ and $v_0 \in L^2(\Omega)$,

$$0 \leq \varepsilon|h|\|v_0\|_{L^2(\Omega)} + \frac{h^2}{2}\int_0^T\int_\omega |v|^2\,dxdt + h\int_0^T\int_\omega |\hat{v}v|^2\,dxdt - h\int_\Omega y_1 v_0\,dx.$$

Proof Cont.

- Dividing by $h > 0$ and passing to the limit as $h \to 0$ we obtain

$$0 \leq \varepsilon \|v_0\|_{L^2(\Omega)} + \int_0^T \int_\omega |\hat{v} v|^2 \, dxdt - \int_\Omega y_1 v_0 \, dx. \tag{1.17}$$

- The same calculation with $h < 0$ gives that

$$\left| \int_0^T \int_\omega |\hat{v} v|^2 \, dxdt - \int_\Omega y_1 v_0 \, dx \right| \leq \varepsilon \|v_0\|_{L^2(\Omega)}. \tag{1.18}$$

- On the other hand, taking $f = \hat{v}|_\omega$ in (1.1), multiplying (1.1) by v solution of the adjoint equation (1.4) and integrating by parts, we get

$$\int_0^T \int_\omega \hat{v} v \, dxdt = \int_\Omega y(T) v_0 \, dx. \tag{1.19}$$

Proof Cont.

- From the last two relations it follows that

$$\left|\int_\Omega \Big(y(T) - y_1\Big) v_0 \, dx\right| \le \varepsilon \|v_0\|_{L^2(\Omega)}, \quad \forall v_0 \in L^2(\Omega). \tag{1.20}$$

- This is equivalent to

$$\|y(T) - y_1\|_{L^2(\Omega)} \le \varepsilon.$$

- We have shown the approximate controllability. □

J_ε attains its minimum in $L^2(\Omega)$

There exists $\hat{v}_T \in L^2(\Omega)$ such that

$$J_\varepsilon(\hat{v}_T) = \min_{v_T \in L^2(\Omega)} J_\varepsilon(v_T). \tag{1.21}$$

Proof

- It is easy to see that J_ε is convex and continuous. Thus, the existence of a minimum is ensured if J_ε is coercive, i.e.

$$J_\varepsilon(v_T) \to \infty \text{ when } \|v_T\|_{L^2(\Omega)} \to \infty. \tag{1.22}$$

- We shall prove that

$$\liminf_{\|v_T\|_{L^2(\Omega)} \to \infty} \frac{J_\varepsilon(v_T)}{\|v_T\|_{L^2(\Omega)}} \geq \varepsilon. \tag{1.23}$$

- Evidently, (1.23) implies (1.22) and the proof will be finished.

Proof Cont.

- Let $(v_{T,j}) \subset L^2(\Omega)$ be a sequence of final datum for the adjoint equation (1.4) with $\|v_{T,j}\|_{L^2(\Omega)} \to \infty$. We normalize then

$$\tilde{v}_{T,j} = \frac{v_{T,j}}{\|v_{T,j}\|_{L^2(\Omega)}} \text{ so that } \|\tilde{v}_{T,j}\|_{L^2(\Omega)} = 1.$$

- Let $\tilde{v}_j$ be the solution of (1.4) with final datum $\tilde{v}_{T,j}$. Then

$$\frac{J_\varepsilon(v_{T,j})}{\|v_{T,j}\|_{L^2(\Omega)}} = \frac{1}{2}\|v_{T,j}\|_{L^2(\Omega)} \int_0^T \int_\omega |\tilde{v}_j|^2 \, dxdt + \varepsilon - \int_\Omega y_1 \tilde{v}_{T,j} \, dx.$$

- The following two cases occur:

Proof Cont.

1. $\liminf_{j\to\infty} \int_0^T \int_\omega |\tilde{v}_j|^2 \, dxdt > 0$. In this case we immediately obtain

$$\frac{J_\varepsilon(v_{T,j})}{\|v_{T,j}\|_{L^2(\Omega)}} \to \infty.$$

2. $\liminf_{j\to\infty} \int_0^T \int_\omega |\tilde{v}_j|^2 \, dxdt = 0$. Then $\tilde{v}_{T,j}$ is bounded in $L^2(\Omega)$.
   - By extracting a subsequence, we have that $\tilde{v}_{T,j} \rightharpoonup v_0$ in $L^2(\Omega)$ and $\tilde{v}_j \rightharpoonup v$ in $L^2((0,T); H_0^s(\Omega)) \cap H^1((0,T); H^{-s}(\Omega))$, where v is the the solution of (1.4) with final datum v_0 at $t = T$.
   - By the lower semi-continuity we have that

$$\int_0^T \int_\omega |v|^2 \, dxdt \leq \liminf_{j\to\infty} \int_0^T \int_\omega |\tilde{v}_j|^2 \, dxdt = 0.$$

   Therefore $v = 0$ in $(0,T) \times \omega$.

Proof Cont.

- It follows from the unique continuation property for solutions of the adjoint equation (1.4) that $v = 0$ in $(0, T) \times \Omega$. Consequently $v_0 = 0$.
- Therefore, $\tilde{v}_{T,j} \rightharpoonup 0$ weakly in $L^2(\Omega)$ and consequently,

$$\int_\Omega y_1 \tilde{v}_{T,j}\, dx \to 0 \ \text{ as well.}$$

- Hence,

$$\liminf_{j\to\infty} \frac{J_\varepsilon(v_{T,j})}{\|v_{T,j}\|_{L^2(\Omega)}} \geq \liminf_{j\to\infty} \left(\varepsilon - \int_\Omega y_1 \tilde{v}_{T,j}\, dx\right) = \varepsilon.$$

- We have shown (1.23). The proof is finished. □

Remark

- The second approach of the proof of the approximate controllability does not only guarantee the existence of a control but also provides a method to obtain the control by minimizing a convex, continuous, and coercive functional in $L^2(\Omega)$.
- In the proof of the coercivity, the relevance of $\varepsilon\|v_T\|_{L^2(\Omega)}$ is clear.
- Indeed, the covercivity of J_ε depends heavily on this term. This is not only for technical reasons.
- The existence of a minimum of J_ε with $\varepsilon = 0$ implies the existence of a control which makes $y(T) = y_1$. But even in the classical case $s = 1$, this is not true unless y_1 is very regular in $\Omega \setminus \omega$. Even in the case $s = 1$, for general $y_1 \in L^2(\Omega)$, the term $\varepsilon\|v_T\|_{L^2(\Omega)}$ is needed.
- Notice that both proofs use the unique continuation property for solutions of the adjoint equation. This is very important in controllability problems.

Our exterior control problem:We consider the following controllabilty problem:

$$\begin{cases} u_t(x,t) + (-\Delta)^s u(x,t) = 0 & \text{in } \; Q := (0,T)\times\Omega, \\ u = g\chi_{\mathcal{O}} & \text{in } \; \Sigma := (0,T)\times(\mathbb{R}^N\setminus\Omega), \\ u(\cdot,0) = u_0 & \text{in } \; \Omega. \end{cases} \tag{2.1}$$

Here, u is the state to be controlled and g is the control function which is localized in a non-empty open set $\mathcal{O}\subset(\mathbb{R}^N\setminus\Omega)$.

Existence of solutions: According to Lecture 2, we have the following existence result of the system (2.1)

For every $g\in L^2((0,T)\times\mathcal{O})$, the system (2.1) has a unique very-weak solution (or solution by transposition) $u\in L^2(\mathbb{R}^N)$. That is,

$$\int_Q u\left(-\partial_t v + (-\Delta)^s v\right)\, dxdt = -\int_\Sigma g\mathcal{N}_s v \, dxdt, \tag{2.2}$$

holds for every $v\in L^2((0,T);V)\cap H^1_{0,T}((0,T);L^2(\Omega))$, where we recall that $V := \{v\in H_0^s(\Omega):\ (-\Delta)^s v\in L^2(\Omega)\}$.

Definition

The three notions of controllability are defined similarly, where here the set of reachable states is given by

$$\mathcal{R}(u_0, T) = \{u(\cdot, T) : \ g \in L^2((0, T) \times \mathcal{O})\}.$$

Remark

We observe that using the theory of evolution equations, we can show that any weak solution belongs to $C((0, T]; L^2(\Omega))$ so that the value $u(\cdot, T)$ makes sense in the formula of $\mathcal{R}(u_0, T)$.

As in the case of the interior control, we also have the following result

The following assertions are equivalent.

1. The system (2.1) is null controllable in time $T > 0$.
2. The system (2.1) is exactly controllable in time $T > 0$.

First characterization of null/exact controllability

The following assertions are equivalent.

1. The system (2.1) is null controllable in time $T > 0$.
2. For every $u_0 \in L^2(\Omega)$, there exists a control function g such that

$$\int_\Omega u_0(x)w(x,0)\,dx = \int_0^T \int_{\mathcal{O}} g(x,t)\mathcal{N}_s w(x,t)\,dxdt,$$

where w is the unique solution of the associated dual system:

$$\begin{cases} -w_t(x,t) + (-\Delta)^s w(x,t) = 0 & \text{in } (0,T)\times\Omega, \\ w = 0 & \text{on } (0,T)\times(\mathbb{R}^N\setminus\Omega), \\ w(T,\cdot) = w_T & \text{in } \Omega, \end{cases}$$

for $w_T \in L^2(\Omega)$, and $\mathcal{N}_s$ is the nonlocal normal derivative given by

$$\mathcal{N}_s w(x) = C_{N,s}\int_\Omega \frac{w(x)-w(y)}{|x-y|^{N+2s}}\,dy, \quad x \in (\mathbb{R}^N\setminus\overline{\Omega}).$$

Second characterization of the null/exact controllability

The following assertions are equivalent.

1. The system (2.1) is null or exactly controllable in time $T > 0$.
2. The following observability inequality holds:

$$\|w(0,\cdot)\|^2_{L^2(\Omega)} \le C \int_0^T \int_{\mathcal{O}} |\mathcal{N}_s w|^2 \, dxdt, \tag{2.3}$$

where w is the unique solution of the associated dual system:

$$\begin{cases} -w_t(x,t) + (-\Delta)^s w(x,t) = 0 & \text{in } (0,T)\times\Omega, \\ w = 0 & \text{on } (0,T)\times(\mathbb{R}^N\setminus\Omega), \\ w(\cdot,T) = w_T & \text{in } \Omega. \end{cases} \tag{2.4}$$

Theorem (W. & Zamorano (2020))

Let $N = 1$ *and* $\Omega = (-1, 1)$*. Then the system* (2.1) *is null controllable for every* $T > 0$ *if and only if* $1/2 < s < 1$*.*

Proof

The proof uses the same ideas as the case of the interior control.

- Biorthogonal sequences.
- Gap conditions.
- Müntz theorem.
- Fattorini and Russel theorem.

Remark

For the exterior control, if $N \geq 2$, we still do NOT know if the system is null controllable or not. This is still an open problem.

Remark

Here also, for the approximate controllability of the system (2.1) it is also enough to consider the case where the initial datum $u_0 = 0$.

Characterization of approximate controllability: Warma (SICON 2019)

The following assertions are equivalent.

1. The system (2.1) is approximately controllable in time $T > 0$.
2. The dual system (2.3) has the unique continuation property: That is, let $\mathcal{O} \subset (\mathbb{R}^N \setminus \Omega)$ be an arbitrary nonempty open set and w the unique solution of (2.4).

$$\text{If } \mathcal{N}_s w = 0 \ \text{ in } \ (0,T) \times \mathcal{O}, \text{ then } \ w = 0 \ \text{ in } \ (0,T) \times \Omega.$$

Theorem (W. (SICON 2019))

Let $\mathcal{O} \subset (\mathbb{R}^N \setminus \Omega)$ be an arbitrary nonempty open set. Then the system (2.1) is always approximately controllable for any $T > 0$ and $g \in L^2((0,T) \times \mathcal{O})$.

Proof: Let $g \in L^2((0,T) \times \mathcal{O})$.

- Let u be the solution of (2.1) and v the unique solution of (dual) (2.3) with $v_T \in L^2(\Omega)$. Using the definition of very weak solutions we get that

$$\begin{aligned} 0 &= \int_Q u\left(-v_t + (-\Delta)^s v\right)\, dxdt \\ &= \int_\Omega u(T)v_T\, dx - \int_\Sigma u\mathcal{N}_s v\, dx\, dt. \end{aligned} \tag{2.5}$$

- It follows from (2.5) that

$$0 = \int_\Omega u(T)v_T\, dx - \int_0^T \int_{\mathcal{O}} g\mathcal{N}_s v\, dx\, dt. \tag{2.6}$$

Proof Cont.

- To prove that the set $\{(u(\cdot, T) : \; g \in L^2((0, T) \times \mathcal{O})\}$ is dense in $L^2(\Omega)$, we have to show that if $v_T \in L^2(\Omega)$ is such that

$$\int_\Omega u(x, T) v_T(x) \, dx = 0, \tag{2.7}$$

for any $g \in L^2((0, T) \times \mathcal{O})$, then $v_T = 0$. Indeed, let v_T satisfy (2.7).

- It follows from (2.6) and (2.7) that

$$\int_0^T \int_{\mathcal{O}} g \mathcal{N}_s v \, dx \, dt = 0,$$

for any $g \in L^2((0, T) \times \mathcal{O})$. By the fundamental lemma of the calculus of variations, we have that

$$\mathcal{N}_s v = 0 \quad \text{in} \quad (0, T) \times \mathcal{O}.$$

Proof Cont.

- It follows from the unique continuation property for solutions of the adjoint equation that $v = 0$ in $Q = (0, T) \times \Omega$.
- Since the solution of the adjoint system (2.3) is unique, we have that $v_T = 0$ on Ω.

The proof of the theorem is finished. □

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THANKS!