Control Theory of Fractional PDEs

Mahamadi Warma *
Department of Mathematical Sciences
George Mason University, Fairfax, VA 22030

1 General aspects of control theory

1.1 Introduction

Control Theory is certainly, at present, one of the most interdisciplinary areas of research. Control Theory arises in most modern applications. The same can be said about the very first technological discoveries of the industrial revolution. On the other hand, Control Theory has been a discipline where many mathematical ideas and methods have melt to produce a new body of important Mathematics. Accordingly, it is nowadays a rich crossing point of Engineering and Mathematics.

The word **control** has a double meaning. First controlling a system can be understood simply as testing or checking that its behavior is satisfactory. In a deeper sense, to control is also to act, to put things together in order to guarantee that the system behaves as desired.

Let us indicate briefly how control problems are stated nowadays in mathematical terms. To fix ideas, assume we want to get a good behavior of a physical system governed by the **state equation**

$$A(y) = f(v) \tag{1.1}$$

Here, y is the **state**, the unknown of the system that we are willing to control. It belongs to a vector space Y. On the other hand, v is the **control**. It belongs to the set of **admissible controls** $\mathcal{U}_{ad}$. This is the variable that we can choose freely in $\mathcal{U}_{ad}$ to act on the system.

Let us assume that $A : D(A) \subset Y \mapsto Y$ and $f : \mathcal{U}_{ad} \mapsto Y$ are two given (linear or nonlinear) mappings. The operator A determines the equation that

*Partially supported by Air Force Office of Scientific Research (AFOSR) and Army Research Office (ARO). Email: mwarma@gmu.edu

must be satisfied by the state variable y, according to the laws of Physics. The function f indicates the way the control v acts on the system governing the state.

Let us assume that, for each $v \in \mathcal{U}_{ad}$, the state equation (1.1) has exactly one solution $y = y(v)$ on Y. Then, roughly speaking, to control (1.1) is to find $v \in \mathcal{U}_{ad}$ such that the solution of (1.1) gets close to the prescribed state. The "best" among all the existing controls achieving the desired goal is frequently referred as the **optimal control**.

Another important underlying notion in Control Theory is **Optimization**. This can be regarded as a branch of Mathematics whose goal is to improve a variable in order to maximize a benefit (or minimize a cost). This is applicable to a lot of practical situations (the variable can be a temperature, a velocity field, a measure of information, etc.). Optimization Theory and its related techniques are such a broad subject that it would be impossible to make a unified presentation.

In order to understand why Optimization techniques and Control Theory are closely related, let us come back to (1.1). Assume that the set of admissible controls $\mathcal{U}_{ad}$ is a subset of a Banach space $\mathcal{U}$ and the state space Y is another Banach space. Also assume that $y_d \in Y$ is the preferred state and is chosen as a target for the state of the system. Then, the control problem consists in finding controls $v \in \mathcal{U}_{ad}$ such that the associated solution coincides or gets close to y_d.

It is then reasonable to think that a fruitful way to choose a good control v is by minimizing a **cost function** of the form:

$$J(v) = \frac{1}{2}\|y(v) - y_d\|_Y^2 \quad \forall\, v \in \mathcal{U}_{ad} \tag{1.2}$$

or, more generally

$$J(v) = \frac{1}{2}\|y(v) - y_d\|_Y^2 + \frac{\mu}{2}\|v\|_{\mathcal{U}}^2 \quad \forall\, v \in \mathcal{U}_{ad} \tag{1.3}$$

where $\mu \geq 0$.

These are (constrained) problems whose analysis corresponds to Optimization Theory.

It is interesting to analyze the two terms arising in the functional J in (1.3) when $\mu > 0$ separately, since they play complementary role. When minimizing the functional in (1.3), we are minimizing the balance between these two terms. The first one requires to get close to the target y_d while the second one penalizes using too much costly control. Thus, roughly speaking, when minimizing J we are trying to drive the system to a state close to the target y_d without too much effort.

More information and applications of Control Theory and Optimization can be found in the monograph [1] and the references therein.

1.2 Controllability versus optimization

As already mentioned, for systems of the form (1.1), the main goal of Control Theory is to find controls v leading the associated states $y(v)$, i.e. the solutions of the corresponding controlled systems, to a desired situation. There are however (at least) two ways of specifying a "desired prescribed situation":

1. To fix a desired state y_d and require

$$y(v) = y_d \tag{1.4}$$

   or, at least

$$y(v) \simeq y_d \tag{1.5}$$

   in some sense. This is the **controllability viewpoint**.

   The main question is then the existence of an admissible control v so that the corresponding state $y(v)$ satisfies (1.4) or (1.5). Once the existence of such a control v is established, it is meaningful to look for an optimal control, for instance, a control of **minimal size**. Other important questions arise in this context too. As we shall see in the lectures, this problem may be difficult (or even very difficult) to solve. In recent years, an important body of beautiful Mathematics has been developed in connection with these questions.

2. To fix a cost function $J = J(v)$ like for instance (1.2) or (1.3) and to look for a **minimizer** u of J. This is the **optimization or optimal control viewpoint**.

   As in (1.2) and (1.3), J is typically related to the "distance" to a prescribed state. Both approaches have the same ultimate goal, to bring the state close to the desired target but, in some sense, the second one is more realistic and easier to implement.

   The optimization viewpoint is, at least apparently, humble in comparison with the controllability approach. But it is many times much more realistic. In practice, it provides satisfactory results in many situations and, at the same time, it requires simpler mathematical tools.

1.3 History and applications

Our intention here is simply to recall some classical and well known results that have to some extent influenced the development of this discipline, pointing out several facts that, in our opinion, have been relevant for the recent achievements of Control Theory. Let us go back to the origins of Control Engineering and Control Theory and let us describe the role this discipline has played in History. Going backwards in time, we will easily conclude that Romans did use some elements of Control Theory in their aqueducts. Indeed, ingenious systems of regulating valves were used in these constructions in order to keep the water level constant.

Some people claim that, in the ancient Mesopotamia, more than 2000 years B.C., the control of the irrigation systems was also a well known art.

On the other hand, in the ancient Egypt the "harpenodaptai" (string stretchers), were specialized in stretching very long strings leading to long straight segments to help in large constructions. Somehow, this is an evidence of the fact that in the ancient Egypt the following two assertions were already well understood:

- The shortest distance between two points is the straight line (which can be considered to be the most classical assertion in Optimization and Calculus of Variations).
- This is equivalent to the following dual property: among all the paths of a given length the one that produces the longest distance between its extremes is the straight line as well.

The task of the "harpenodaptai" was precisely to build these "optimal curves". The work by Ch. Huygens and R. Hooke at the end of the XVII Century on the oscillations of the pendulum is a more modern example of development in Control Theory. Their goal was to achieve a precise measurement of time and location, so precious in navigation.

These works were later adapted to regulate the velocity of windmills. The main mechanism was based on a system of balls rotating around an axis, with a velocity proportional to the velocity of the windmill. When the rotational velocity increases, the balls get farther from the axis, acting on the wings of the mill through appropriate mechanisms.

J. Watt adapted these ideas when he invented the steam engine and this constituted a magnificent step in the industrial revolution. In this mechanism, when the velocity of the balls increases, one or several valves open to let the vapor escape. This makes the pressure diminish. When this happens, i.e. when the pressure inside the boiler becomes weaker, the velocity begins to go down. The goal of introducing and using this mechanism is of course to keep the velocity as close as possible to a constant.

The British astronomer G. Airy was the first scientist to analyze mathematically the regulating system invented by Watt. But the first definitive mathematical description was given only in the works by J.C. Maxwell, in 1868, where some of the erratic behaviors encountered in the steam engine were described and some control mechanisms were proposed.

The central ideas of Control Theory gained soon a remarkable impact and, in the twenties, engineers were already preferring the continuous processing and using semi-automatic or automatic control techniques. In this way, Control Engineering germinated and got the recognition of a distinguished discipline.

In the thirties important progresses were made on automatic control and design and analysis techniques. The number of applications increased covering amplifiers in telephone systems, distribution systems in electrical plants, stabilization of aeroplanes, electrical mechanisms in paper production, Chemistry, petroleum and steel Industry, etc.

By the end of that decade, two emerging and clearly different methods or approaches were available: a first method based on the use of differential equations and a second one, of frequential nature, based on the analysis of amplitudes and phases of ”inputs” and ”outputs”.

After 1960, the methods and ideas mentioned above began to be considered as part of ”classical” Control Theory. The war made clear that the models considered up to that moment were not accurate enough to describe the complexity of the real word. Indeed, by that time it was clear that true systems are often nonlinear and nondeterministic, since they are affected by ”noise”. This generated important new efforts in this field.

The contributions of the U.S. scientist R. Bellman in the context of dynamic programming, R. Kalman in filtering techniques and the algebraic approach to linear systems and the Russian L. Pontryagin with the maximum principle for nonlinear optimal control problems established the foundations of modern Control Theory.

2 The lectures

Here we give a summary of the material that will be discussed in details during the lectures.

2.1 Introduction

The main focus of the series of three lectures is to examine the above mentioned Optimal Control and Controllability problems, when the underlying operator A given in (1.1) is a fractional (nonlocal) operator. We will focus on the cases where $A = (-\Delta)^s$ is the **Fractional Laplace Operator** or $A = \partial_t + (-\Delta)^s$ $(0 < s < 1)$.

Before we state our control problems, let us first introduce the fractional Laplace operator $(-\Delta)^s$ and the needed fractional order Sobolev spaces.

- **Fractional order Sobolev spaces**: Let $\Omega \subset \mathbb{R}^N$ $(N \geq 1)$ be a bounded open set and $0 < s < 1$. We let

$$H^s(\Omega) := \left\{ u \in L^2(\Omega) \ : \ \int_\Omega \int_\Omega \frac{|u(x) - u(y)|^2}{|x-y|^{N+2s}} \, dxdy < \infty \right\},$$

  and we endow it with the norm defined by

$$\|u\|_{H^s(\Omega)} := \left(\int_\Omega |u|^2 \, dx + \int_\Omega \int_\Omega \frac{|u(x) - u(y)|^2}{|x-y|^{N+2s}} \, dxdy \right)^{\frac{1}{2}}.$$

  We also need to define

$$H_0^s(\Omega) := \left\{ u \in H^s(\mathbb{R}^N) \ : \ u = 0 \text{ in } \mathbb{R}^N \setminus \Omega \right\}.$$

  Then

$$\|u\|_{H_0^s(\Omega)} := \left(\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^2}{|x-y|^{N+2s}} \, dxdy \right)^{\frac{1}{2}}$$

  defines an equivalent norm on $H_0^s(\Omega)$.

  We shall use $H^{-s}(\mathbb{R}^N)$ and $H^{-s}(\Omega)$ to denote the dual spaces of $H^s(\mathbb{R}^N)$ and $H_0^s(\Omega)$, respectively, and $\langle \cdot, \cdot \rangle$ to denote their duality pairing whenever it is clear from the context.

- **The fractional Laplace operator**: To introduce the fractional Laplace operator, we set

$$\mathbb{L}_s^1(\mathbb{R}^N) := \left\{ u : \mathbb{R}^N \to \mathbb{R} \ \text{ measurable} : \int_{\mathbb{R}^N} \frac{|u(x)|}{(1+|x|)^{N+2s}} \, dx < \infty \right\}.$$

For $u \in \mathbb{L}^1_s(\mathbb{R}^N)$ and $\varepsilon > 0$, we let

$$(-\Delta)^s_\varepsilon u(x) = C_{N,s} \int_{\{y\in\mathbb{R}^N, |y-x|>\varepsilon\}} \frac{u(x)-u(y)}{|x-y|^{N+2s}} dy, \quad x \in \mathbb{R}^N,$$

where the normalized constant $C_{N,s}$ is given by

$$C_{N,s} := \frac{s2^{2s}\Gamma\left(\frac{2s+N}{2}\right)}{\pi^{\frac{N}{2}}\Gamma(1-s)} = \left(\int_{\mathbb{R}^N} \frac{1-\cos(\xi_1)}{|\xi|^{N+2s}}\, d\xi\right)^{-1} \tag{2.1}$$

and Γ is the usual Euler Gamma function. The **fractional Laplacian** $(-\Delta)^s$ is defined for $u \in \mathbb{L}^1_s(\mathbb{R}^N)$ by the formula

$$(-\Delta)^s u(x) = C_{N,s}\text{P.V.} \int_{\mathbb{R}^N} \frac{u(x)-u(y)}{|x-y|^{N+2s}} dy = \lim_{\varepsilon\downarrow 0} (-\Delta)^s_\varepsilon u(x), \tag{2.2}$$

provided that the limit exists for a.e. $x \in \mathbb{R}^N$.

Other equivalent definitions of $(-\Delta)^s$ will be given during the lectures.

It is known that for $u \in \mathcal{D}(\Omega)$, we have

$$\lim_{s\uparrow 1^-} \int_{\mathbb{R}^N} u(-\Delta)^s u\, dx = \int_{\mathbb{R}^N} |\nabla u|^2 dx = -\int_{\mathbb{R}^N} u\Delta u\, dx = -\int_\Omega u\Delta u\, dx,$$

that is where the constant $C_{N,s}$ plays a crucial role.

- **The nonlocal normal derivative**: Next, for $u \in H^s(\mathbb{R}^N)$ we define the nonlocal normal derivative $\mathcal{N}_s u$ of u as:

$$\mathcal{N}_s u(x) := C_{N,s} \int_\Omega \frac{u(x)-u(y)}{|x-y|^{N+2s}}\, dy, \quad x \in \mathbb{R}^N \setminus \overline{\Omega}. \tag{2.3}$$

We shall also call $\mathcal{N}_s$ the *interaction operator*. Clearly $\mathcal{N}_s$ is a nonlocal operator and it is well defined on $H^s(\mathbb{R}^N)$.

Lemma 2.1 *The interaction operator $\mathcal{N}_s$ maps continuously $H^s(\mathbb{R}^N)$ into $H^s_{\text{loc}}(\mathbb{R}^N \setminus \Omega)$.*

Despite the fact that $\mathcal{N}_s$ is defined on $\mathbb{R}^N \setminus \overline{\Omega}$, it is still known as the “normal" derivative. This is due to its similarity with the classical normal derivative as we discuss next.

Proposition 2.2 *Let $\Omega \subset \mathbb{R}^N$ be a bounded open set with a Lipschitz continuous boundary. Then the following assertions hold.*

(a) **The divergence theorem for $(-\Delta)^s$.** *Let $u \in C_0^2(\mathbb{R}^N)$, i.e., C^2-functions on $\mathbb{R}^N$ that vanish at $\pm\infty$. Then*

$$\int_\Omega (-\Delta)^s u \, dx = -\int_{\mathbb{R}^N \setminus \Omega} \mathcal{N}_s u \, dx.$$

(b) **The integration by parts formula for $(-\Delta)^s$.** *Let $u \in H^s(\mathbb{R}^N)$ be such that $(-\Delta)^s u \in L^2(\Omega)$ and $\mathcal{N}_s u \in L^2(\mathbb{R}^N \setminus \Omega)$. Then for every $v \in H^s(\mathbb{R}^N)$ we have that*

$$\begin{aligned} &\int_\Omega v(-\Delta)^s u \, dx \\ =&\frac{C_{N,s}}{2} \int\int_{\mathbb{R}^{2N} \setminus (\mathbb{R}^N \setminus \Omega)^2} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{N+2s}} \, dxdy \\ &- \int_{\mathbb{R}^N \setminus \Omega} v\mathcal{N}_s u \, dx, \end{aligned} \tag{2.4}$$

where $\mathbb{R}^{2N} \setminus (\mathbb{R}^N \setminus \Omega)^2 = (\Omega \times \Omega) \cup (\Omega \times (\mathbb{R}^N \setminus \Omega)) \cup ((\mathbb{R}^N \setminus \Omega) \times \Omega)$.

(c) **The limit as $s \uparrow 1^-$.** *Let $u, v \in C_0^2(\mathbb{R}^N)$. Then*

$$\lim_{s \uparrow 1^-} \int_{\mathbb{R}^N \setminus \Omega} v\mathcal{N}_s u \, dx = \int_{\partial\Omega} v \frac{\partial u}{\partial \nu} \, d\sigma,$$

where $\partial u / \partial \nu := \nabla u \cdot \vec{\nu}$ denotes the normal derivative of the function u

Remark 2.3 Comparing the properties (a)-(c) in Proposition 2.2 with the classical properties of the standard Laplacian Δ we can immediately infer that $\mathcal{N}_s$ plays the same role for $(-\Delta)^s$ that the classical normal derivative does for Δ. For this reason, we call $\mathcal{N}_s$ the nonlocal normal derivative.

- **The Dirichlet problem for** $(-\Delta)^s$:

  (a) Let $g \in C(\partial\Omega)$. The classical Dirichlet problem for Δ is given by

  $$\Delta u = 0 \quad \text{in} \ \ \Omega, \quad u = g \ \text{on} \ \ \partial\Omega.$$

  It is nowadays well-known that the classical Dirichlet problem is well-posed if and only if Ω is regular in the sense of Wiener.

(b) Let $g \in C(\partial\Omega)$. Then the Dirichlet problem

$$(-\Delta)^s u = 0 \quad \text{in} \quad \Omega, \quad u = g \text{ on } \partial\Omega, \tag{2.5}$$

is not well-posed. This follows from the fact that

$$(-\Delta)^s u(x) = C_{N,s} \left(\int_\Omega \frac{u(x) - u(y)}{|x-y|^{N+2s}}\, dy + \int_{\mathbb{R}^N \setminus \Omega} \frac{u(x) - u(y)}{|x-y|^{N+2s}}\, dy \right).$$

(c) Let $g \in C_0(\mathbb{R}^N \setminus \Omega)$. The well-posed Dirichlet problem is given by

$$(-\Delta)^s u = 0 \quad \text{in} \quad \Omega, \quad u = g \quad \text{in} \quad \mathbb{R}^N \setminus \Omega.$$

- The zero Dirichlet exterior condition (EC) for $(-\Delta)^s$: The above items suggest that conditions for $(-\Delta)^s$ with be given in $\mathbb{R}^N \setminus \Omega$.

(a) The zero Dirichlet boundary boundary condition for Δ is given by $u = 0$ on $\partial\Omega$.

(b) Let $(-\Delta)^s_D$ be the operator on $L^2(\Omega)$ given by

$$\begin{cases} D((-\Delta)^s_D) = \Big\{ u \in H^s_0(\Omega) : \ (-\Delta)^s u \in L^2(\Omega) \Big\}, \\ (-\Delta)^s_D u = (-\Delta)^s u. \end{cases}$$

Then $(-\Delta)^s_D$ is the realization in $L^2(\Omega)$ of $(-\Delta)^s$ with the zero Dirichlet exterior condition.

(c) The zero Dirichlet exterior condition for $(-\Delta)^s$ is characterized by $u = 0$ in $\mathbb{R}^N \setminus \Omega$.

(d) Do not confuse $(-\Delta)^s_D$ with $(-\Delta_D)^s$ (the spectral fractional Laplacian). The two operators are different. They eigenvalues and eigenfunctions are different.

2.2 An important theorem to know

Here we state an important theorem that will be frequently used throughout the lectures. The result is due to J. L. Lions and is a generalized version of the classical Lax-Milgram Lemma and it can be applied to several PDEs problems to prove existence of various notions of solutions.

Theorem 2.4 (Lions' Existence Theorem) *Let $\big(F, \|\cdot\|_F\big)$ be a Hilbert space. Let Φ be a subspace of F endowed with a pre-Hilbert scalar product $(((\cdot,\cdot)))$ and associated norm $|||\cdot|||$. Moreover, let $E : F \times \Phi \to \mathbb{C}$ be a sesquilinear form. Assume that the following hold:*

(a) The embedding $\Phi \hookrightarrow F$ is continuous, that is, there is a constant $C_1 > 0$ such that

$$\|\varphi\|_F \le C_1 |||\varphi||| \quad \forall\, \varphi \text{ in } \Phi. \tag{2.6}$$

(b) For all $\varphi \in \Phi$, the mapping $u \mapsto E(u,\varphi)$ is continuous on F.

(c) There is a constant $C_2 > 0$ such that

$$|E(\varphi,\varphi)| \ge C_2 |||\varphi|||^2 \quad \text{for all} \quad \varphi \in \Phi. \tag{2.7}$$

If $\varphi \mapsto L(\varphi)$ is a continuous linear functional on Φ, then there exists a function $u \in F$ verifying

$$E(u,\varphi) = L(\varphi) \quad \text{for all} \quad \varphi \in \Phi.$$

2.3 Lecture 1: Optimal control with fractional elliptic PDEs

This lecture is based on the material contained in [2] and the reference therein.

Let

$$Z_D := L^2(\mathbb{R}^N \setminus \Omega), \quad U_D := L^2(\Omega).$$

Given $\xi \ge 0$ a constant penalty parameter, we consider the minimization problem:

$$\min_{(u,z)\in(U_D,Z_D)} J(u) + \frac{\xi}{2}\|z\|^2_{Z_D}, \tag{2.8a}$$

subject to the fractional Dirichlet exterior value problem: Find $u \in U_D$ solving

$$\begin{cases} (-\Delta)^s u &= 0 \quad \text{in } \Omega, \\ u &= z \quad \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{2.8b}$$

and the control constraints

$$z \in Z_{ad,D}, \tag{2.8c}$$

with $Z_{ad,D} \subset Z_D$ being a closed and convex subset. The precise conditions on the functional J depend on the result we would like to obtain. For this reason they will be given in the statements of our results.

Remark 2.5 Boundary Optimal Control replaced with Exterior Control!

(a) For elliptic problems associated with the fractional Laplacian, the notion of boundary control does not make sense. As we have already mentioned in Section 2, this follows from the fact that the following stationary problem is not well-posed:

$$(-\Delta)^s u = f \text{ in } \Omega, \quad u = g \text{ on } \partial\Omega.$$

(b) The exterior control plays the role for the fractional Laplacian that the boundary control does for the Laplace operator, as the following stationary problem is well-posed:

$$(-\Delta)^s u = f \text{ in } \Omega, \quad u = g \text{ on } \mathbb{R}^N \setminus \Omega.$$

Next, we begin by rewriting (2.8b) in a more general form. That is,

$$\begin{cases} (-\Delta)^s u = f & \text{in } \Omega, \\ u = z & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases} \tag{2.9}$$

Here is our notion of weak solutions.

Definition 2.6 (Weak solution) *Let $f \in H^{-s}(\Omega)$, $z \in H^s(\mathbb{R}^N \setminus \Omega)$, and $\widetilde{z} \in H^s(\mathbb{R}^N)$ be such that $\widetilde{z}|_{\mathbb{R}^N \setminus \Omega} = z$. A function $u \in H^s(\mathbb{R}^N)$ is said to be a weak solution to (2.9) if $u - \widetilde{z} \in H_0^s(\Omega)$ and*

$$\mathcal{E}(u,v) := \frac{C_{N,s}}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+2s}} \, dxdy = \langle f, v \rangle,$$

for every $v \in H_0^s(\Omega)$.

Firstly, we notice that since Ω is assumed to have a Lipschitz continuous boundary, we have that, for $z \in H^s(\mathbb{R}^N \setminus \Omega)$, there exists $\widetilde{z} \in H^s(\mathbb{R}^N)$ such that $\widetilde{z}|_{\mathbb{R}^N \setminus \Omega} = z$. Secondly, the existence and uniqueness of a weak solution u to (2.9) and the continuous dependence of u on the data f and z can be easily proved. More precisely we have the following result.

Proposition 2.7 *Let $f \in H^{-s}(\Omega)$ and $z \in H^s(\mathbb{R}^N \setminus \Omega)$. Then there exists a unique weak solution u to (2.9) in the sense of Definition 2.6. In addition there is a constant $C > 0$ such that*

$$\|u\|_{H^s(\mathbb{R}^N)} \le C \left(\|f\|_{H^{-s}(\Omega)} + \|z\|_{H^s(\mathbb{R}^N \setminus \Omega)} \right). \tag{2.10}$$

Even though such a result is typically sufficient in most situations, nevertheless it is not directly useful in the current context of the optimal control problem (2.8) since we are interested in taking the space $Z_D = L^2(\mathbb{R}^N \setminus \Omega)$. Thus we need existence of solutions (in some sense) to the fractional Dirichlet problem (2.9) when $z \in L^2(\mathbb{R}^N \setminus \Omega)$. In order to tackle this situation we introduce our notion of very-weak solution for (2.9).

Definition 2.8 (Very-weak solution) *Let $z \in L^2(\mathbb{R}^N \setminus \Omega)$ and $f \in H^{-s}(\Omega)$. A function $u \in L^2(\mathbb{R}^N)$ is said to be a very-weak solution to* (2.9) *(or a solution by transposition) if the identity*

$$\int_\Omega u(-\Delta)^s v \, dx = \langle f, v \rangle - \int_{\mathbb{R}^N \setminus \Omega} z \mathcal{N}_s v \, dx, \tag{2.11}$$

holds for every $v \in V := \{v \in H_0^s(\Omega) \; : \; (-\Delta)^s v \in L^2(\Omega)\}$.

Next we state the existence and uniqueness of a very-weak solution to (2.9).

Theorem 2.9 *Let $f \in H^{-s}(\Omega)$ and $z \in L^2(\mathbb{R}^N \setminus \Omega)$. Then there exists a unique very-weak solution u to* (2.9) *according to Definition 2.6 that fulfills*

$$\|u\|_{L^2(\Omega)} \leq C \left(\|f\|_{H^{-s}(\Omega)} + \|z\|_{L^2(\mathbb{R}^N \setminus \Omega)} \right), \tag{2.12}$$

for a constant $C > 0$. In addition, if $z \in H^s(\mathbb{R}^N \setminus \Omega)$, then the following assertions hold.

(a) Every weak solution of (2.9) *is also a very-weak solution.*

(b) Every very-weak solution of (2.9) *that belongs to $H^s(\mathbb{R}^N)$ is also a weak solution.*

We then have the following well-posedness result of the optimal control problem (2.8).

Theorem 2.10 *Let $Z_{ad,D}$ be a closed and convex subset of Z_D. Let either $\xi > 0$ or $Z_{ad,D}$ be bounded and let $J : U_D \to \mathbb{R}$ be weakly lower-semicontinuous. Then there exists a solution $\bar{z}$ to* (2.8)*. If either J is convex and $\xi > 0$ or J is strictly convex and $\xi \geq 0$, then $\bar{z}$ is unique.*

In view of Theorem 2.9 the following (solution-map) control-to-state map

$$S : Z_D \to U_D, \quad z \mapsto Sz = u,$$

is well-defined, linear, and continuous. For $z \in Z_D$, we have that $u := Sz \in L^2(\mathbb{R}^N)$.

We next derive the first order necessary optimality conditions for (2.8). We begin by identifying the structure of the adjoint operator S^*.

Lemma 2.11 *For the state equation* (2.8b) *the adjoint operator* $S^* : U_D \to Z_D$ *is given by*

$$S^* w = -\mathcal{N}_s p \in Z_D,$$

where $w \in U_D$ *and* $p \in H_0^s(\Omega)$ *is the weak solution to the dual problem*

$$\begin{cases} (-\Delta)^s p &= w \quad \text{in } \Omega, \\ p &= 0 \quad \text{in } \mathbb{R}^N \setminus \Omega. \end{cases} \tag{2.13}$$

Theorem 2.12 *Assume that* $\xi > 0$. *Let the assumptions of Theorem 2.10 hold. Let* $\mathcal{Z}$ *be an open set in* Z_D *such that* $Z_{ad,D} \subset \mathcal{Z}$. *Let* $u \mapsto J(u) : U_D \to \mathbb{R}$ *be continuously Fréchet differentiable with* $J'(u) \in U_D$. *If* $\bar{z}$ *is a minimizer of* (2.8) *over* $Z_{ad,D}$, *then the first order necessary optimality conditions are given by*

$$\int_{\mathbb{R}^N \setminus \Omega} \left(-\mathcal{N}_s \bar{p} + \xi \bar{z}\right)(z - \bar{z}) \, dx \geq 0, \quad \forall z \in Z_{ad,D}, \tag{2.14}$$

where $\bar{p} \in H_0^s(\Omega)$ *solves the dual equation*

$$\begin{cases} (-\Delta)^s \bar{p} = J'(\bar{u}) & \text{in } \Omega, \\ \bar{p} = 0 & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases} \tag{2.15}$$

Equivalently we can write (2.14) *as*

$$\bar{z} = \mathcal{P}_{Z_{ad,D}} \left(\frac{1}{\xi} \mathcal{N}_s \bar{p} \right), \tag{2.16}$$

where $\mathcal{P}_{Z_{ad,D}}$ *is the projection onto the set* $Z_{ad,D}$. *If* J *is convex, then* (2.14) *is also a sufficient condition.*

Remark 2.13 (Regularity for the optimization variable) We recall a rather surprising result for the adjoint equation (2.13). The standard maximal elliptic regularity that is known to hold for the classical Laplacian on smooth open sets does not hold in the case of the fractional Laplacian i.e., p does not always belong to $H^{2s}(\Omega)$. Notice that $w \in L^2(\Omega)$ and $p = [(-\Delta)^s_D]^{-1} w$. More precisely assume that Ω is a smooth bounded open set. Then we have the following situations.

(a) If $0 < s < \frac{1}{2}$, then $D((-\Delta)^s_D) = H^{2s}_0(\Omega)$ and hence, $p \in H^{2s}(\Omega)$ in that case.

(b) But if $\frac{1}{2} \leq s < 1$, then $D((-\Delta)^s_D) \not\subset H^{2s}(\Omega)$, thus in that case p does not always belong to $H^{2s}(\Omega)$.

2.4 Lecture 2: Optimal control with fractional parabolic PDEs

This lecture is based on the material contained in [3] and the reference therein.

Given $\xi \geq 0$ a constant penalty parameter we consider the following minimization problem:

$$\min_{(u,z)\in(\mathcal{U}_D,\mathcal{Z}_D)} \left(J(u) + \frac{\xi}{2}\|z\|^2_{Z_D} \right), \tag{2.17a}$$

subject to the fractional parabolic Dirichlet exterior value problem: Find $u \in \mathcal{U}_D$ solving

$$\begin{cases} \partial_t u + (-\Delta)^s u = 0 & \text{in } Q := (0,T)\times\Omega, \\ u = z & \text{in } \Sigma := [0,T)\times(\mathbb{R}^N\setminus\Omega), \\ u(0,\cdot) = 0 & \text{in } \Omega, \end{cases} \tag{2.17b}$$

and the control constraints

$$z \in \mathcal{Z}_{ad,D}, \tag{2.17c}$$

with $\mathcal{Z}_{ad,D} \subset \mathcal{Z}_D$ being a closed and convex subset. Here,

$$\mathcal{Z}_D := L^2((0,T)\times(\mathbb{R}^N\setminus\Omega)), \;\; \mathcal{U}_D := L^2((0,T)\times\Omega))$$

and the functional J is assumed to be weakly lower-semicontinuous and satisfies suitable conditions.

Remark 2.14 (Boundary Control replaced with Exterior Control!) As we have already mentioned in the elliptic case in Section 2.3, here also, the exterior control plays the role for the fractional Laplacian that the boundary control does for the Laplace operator. Since the elliptic equation

$$(-\Delta)^s u = f \;\text{ in }\; \Omega, \;\; u = g \;\text{ on }\; \mathbb{R}^N\setminus\Omega.$$

is not well-posed, the associated evolution equation can be well-posed.

Throughout the following, for a Banach space $\mathbb{X}$, we shall let

$$H^1_{0,0}((0,T);\mathbb{X}) := \{u \in H^1((0,T);\mathbb{X}) :\ u(\cdot,0) = 0\}$$

and

$$H^1_{0,T}((0,T);\mathbb{X}) := \{u \in H^1((0,T);\mathbb{X}) :\ u(\cdot,T) = 0\}.$$

We next introduce our notion of weak solutions to the nonhomogeneous parabolic problem (2.17b). Notice the higher regularity requirement on the datum z.

Definition 2.15 (Weak solution) *Let $z \in H^1_{0,0}((0,T);H^s(\mathbb{R}^N \setminus \Omega))$ and $\tilde{z} \in H^1_{0,0}((0,T);H^s(\mathbb{R}^N))$ be such that $\tilde{z}|_\Sigma = z$. Then a function $u \in \mathbb{U} := L^2((0,T);H^s(\mathbb{R}^N)) \cap H^1_{0,0}((0,T);H^{-s}(\Omega))$ is said to be a weak solution to* (2.17b) *if $u - \tilde{z} \in \mathbb{U}_0$ and*

$$\langle \partial_t u(t,\cdot), v\rangle + \frac{C_{N,s}}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{(u(t,x)-u(t,y))(v(x)-v(y))}{|x-y|^{N+2s}}\,dxdy = 0,$$

for every $v \in H^s_0(\Omega)$ and almost every $t \in (0,T)$.

Throughout the following we shall let

$$\mathbb{U}_0 := L^2((0,T);H^s_0(\Omega)) \cap H^1_{0,0}((0,T);H^{-s}(\Omega))$$

and

$$\mathbb{U} := L^2((0,T);H^s(\mathbb{R}^N)) \cap H^1_{0,0}((0,T);H^{-s}(\Omega)).$$

Next, we show the well-posedness of (2.17b).

Theorem 2.16 (Existence of weak solution) *Let the function $z \in H^1_{0,0}((0,T);H^{s,2}(\mathbb{R}^N \setminus \Omega))$. Then there exists a unique weak solution $u \in \mathbb{U}$ to* (2.17b)*. In addition, there is a constant $C > 0$ such that*

$$\|u\|_{\mathbb{U}} \le C\|z\|_{H^1((0,T);H^s(\mathbb{R}^N\setminus\Omega))}. \tag{2.18}$$

Remark 2.17 Let $(\varphi_n)_{n\in\mathbb{N}}$ be the orthonormal basis of eigenfunctions of $(-\Delta)^s_D$ associated with the eigenvalues $(\lambda_n)_{n\in\mathbb{N}}$. In Theorem 2.16 the unique weak solution u of (2.17b) is given by

$$u(t,x) = -\sum_{n=1}^{\infty}\left(\int_0^t \left(z(t-\tau,\cdot), \mathcal{N}_s\varphi_n\right)_{L^2(\mathbb{R}^N\setminus\Omega)} e^{-\lambda_n\tau}\,d\tau\right)\varphi_n(x).$$

Our next goal is to reduce the regularity requirements on the datum z in both space and time. We shall call the resulting solution u a very-weak solution.

Definition 2.18 (Very-weak solution) *Let $z \in L^2((0,T) \times (\mathbb{R}^N \setminus \Omega))$. A function $u \in L^2((0,T) \times \mathbb{R}^N)$ is said to be a very-weak solution to* (2.17b) *(or a solution by transposition) if the identity*

$$\int_Q u \left(-\partial_t v + (-\Delta)^s v\right)\, dxdt = -\int_\Sigma z \mathcal{N}_s v \, dxdt, \tag{2.19}$$

holds for every $v \in L^2((0,T); V) \cap H^1_{0,T}((0,T); L^2(\Omega))$, where we recall that $V := \{v \in H^s_0(\Omega) : \ (-\Delta)^s v \in L^2(\Omega)\}$.

The following result shows the existence and uniqueness of a very-weak solution to (2.17b) in the sense of Definition 2.18.

Theorem 2.19 (Existence of very-weak solution) *Let $z \in L^2((0,T) \times (\mathbb{R}^N \setminus \Omega))$. Then there exists a unique very-weak solution u to* (2.17b) *according to Definition 2.18 that fulfills*

$$\|u\|_{L^2((0,T);L^2(\Omega))} \le C \|z\|_{L^2((0,T);L^2(\mathbb{R}^N \setminus \Omega))}, \tag{2.20}$$

for a constant $C > 0$. In addition, if $z \in H^1_{0,0}((0,T); H^s(\mathbb{R}^N \setminus \Omega))$, then the following assertions hold.

- *(a) Every weak solution of* (2.17b) *is also a very-weak solution.*
- *(b) Every very-weak solution of* (2.17b) *that belongs to $\mathbb{U}$ is also a weak solution.*

We recall the function spaces $\mathcal{Z}_D$ and $\mathcal{U}_D$ given by

$$\mathcal{Z}_D := L^2((0,T); L^2(\mathbb{R}^N \setminus \Omega)), \quad \mathcal{U}_D := L^2((0,T); L^2(\Omega)).$$

Due to Theorem 2.19, the control-to-state (solution) map

$$S : \mathcal{Z}_D \to \mathcal{U}_D, \quad z \mapsto Sz =: u,$$

is well-defined, linear and continuous. Furthermore, for $z \in \mathcal{Z}_D$, we have $u := Sz \in L^2((0,T); L^2(\mathbb{R}^N))$. Let $J : \mathcal{U}_D \to \mathbb{R}$ and consider the reduced functional

$$\mathcal{J} : \mathcal{Z}_D \to \mathbb{R}, \ \ z \mapsto \mathcal{J}(z) := \Big(J(Sz) + \frac{\xi}{2} \|z\|^2_{Z_D} \Big).$$

Then we can write the reduced Dirichlet exterior parabolic optimal control problem as follows:

$$\min_{z \in \mathcal{Z}_{ad,D}} \mathcal{J}(z). \tag{2.21}$$

Next, we state the well-posedness result for (2.17) and equivalently for (2.21).

Theorem 2.20 *Let $\mathcal{Z}_{ad,D}$ be a closed and convex subset of $\mathcal{Z}_D$. Let either $\xi > 0$ with $J \geq 0$ or $\mathcal{Z}_{ad,D}$ be bounded and $J : \mathcal{U}_D \to \mathbb{R}$ be weakly lower-semicontinuous. Then there exists a solution $\bar{z}$ to* (2.21) *and equivalently to* (2.17)*. If either J is convex and $\xi > 0$ or J is strictly convex and $\xi \geq 0$, then $\bar{z}$ is unique.*

In order to derive the first order necessary optimality conditions, we need an expression of the adjoint operator S^*.

Lemma 2.21 *The adjoint operator $S^* : \mathcal{U}_D \to \mathcal{Z}_D$ for the state equation* (2.17b) *is given by*

$$S^* w = -\mathcal{N}_s p \in \mathcal{Z}_D,$$

where $w \in \mathcal{U}_D$ and $p \in L^2((0,T); H_0^s(\Omega)) \cap H^1_{0,T}((0,T); H^{-s}(\Omega))$ is the weak solution to the following adjoint problem:

$$\begin{cases} -\partial_t p + (-\Delta)^s p &= w \quad \text{in } Q, \\ p &= 0 \quad \text{in } \Sigma, \\ p(T,\cdot) &= 0 \quad \text{in } \Omega. \end{cases} \tag{2.22}$$

Theorem 2.22 *Assume that $\xi > 0$. Let $\mathcal{Z} \subset \mathcal{Z}_D$ be open such that $\mathcal{Z}_{ad,D} \subset \mathcal{Z}$ and let the assumptions of Theorem 2.20 hold. Moreover, let $u \mapsto J(u) : \mathcal{U}_D \to \mathbb{R}$ be continuously Fréchet differentiable with $J'(u) \in \mathcal{U}_D$. If $\bar{z}$ is a minimizer of* (2.21) *over $\mathcal{Z}_{ad,D}$, then the first order necessary optimality conditions are given by*

$$\int_{\mathbb{R}^N \setminus \Omega} \left(-\mathcal{N}_s \bar{p} + \xi \bar{z}\right)(z - \bar{z}) \; dx \geq 0, \quad \forall z \in \mathcal{Z}_{ad,D}, \tag{2.23}$$

where $\bar{p} \in L^2((0,T); H_0^s(\Omega)) \cap H^1_{0,T}((0,T); H^{-s}(\Omega))$ solves the adjoint equation

$$\begin{cases} -\partial_t \bar{p} + (-\Delta)^s \bar{p} = J'(\bar{u}) & \text{in } Q, \\ \bar{p} = 0 & \text{in } \Sigma, \\ \bar{p}(T,\cdot) = 0 & \text{in } \Omega, \end{cases} \tag{2.24}$$

with $\bar{u} := S\bar{z}$. *Finally,* (2.23) *is equivalent to*

$$\bar{z} = \mathcal{P}_{\mathcal{Z}_{ad},D}\left(\xi^{-1}\mathcal{N}_s\bar{p}\right), \tag{2.25}$$

where $\mathcal{P}_{\mathcal{Z}_{ad},D}$ *is the projection onto the set* $\mathcal{Z}_{ad,D}$. *Moreover, if* J *is convex, then* (2.23) *is a sufficient condition.*

2.5 Lecture 3: Controllability of fractional parabolic PDEs

This lecture is based on the material contained in [4, 5, 7, 6] and the references therein.

Here we discuss the controllability properties of fractional heat equations. We will consider interior and exterior controls. As we have already mentioned in Section 1, these problems may be difficult (very difficult) to solve. For every result, we shall give an equivalent characterization. Several topics in this section are still open problems and are excellent problems of research for graduate students and/or post-doctoral fellows who are interested in this field of mathematics and its applications.

2.5.1 Interior controllability properties

The present lecture is concerned with the interior controllability properties of a fractional heat equation involving the fractional Laplace operator on a bounded smooth domain $\Omega \subset \mathbb{R}^N$ $(N \geq 1)$. That is, the following control problem:

$$\begin{cases} y_t + (-\Delta)^s y = f\chi_\omega & \text{in } Q := \Omega \times (0,T), \\ y = 0 & \text{in } \Sigma := (\mathbb{R}^N \setminus \Omega) \times (0,T), \\ y(\cdot,0) = y_0, & \text{in } \Omega, \end{cases} \tag{2.26}$$

where $(-\Delta)^s$ for $0 < s < 1$ is the fractional Laplace operator. In (2.26), y is the state to be controlled, f is the control function which is localized in a nonempty open set $\omega \subset \Omega$, and χ_ω stands for the characteristic function of the set ω.

Definition 2.23 *By a finite energy solution of* (2.26) *we mean a function* $y \in C([0,T]; L^2(\Omega)) \cap L^2((0,T); H_0^s(\Omega)) \cap H^1((0,T); H^{-1}(\Omega))$ *such that* $y(0,\cdot) = y_0$ *and the equality*

$$\langle y_t, \phi\rangle_{H^{-s}(\Omega),H_0^s(\Omega)} + \mathcal{E}(y,\phi) = \langle f, \phi\rangle_{H^{-s}(\Omega),H_0^s(\Omega)} \tag{2.27}$$

holds for every $\phi \in H_0^s(\Omega)$ *and a.e.* $t \in (0,T)$*, where for* $u, v \in H_0^s(\Omega)$*, we recall that*

$$\mathcal{E}(u,v) := \frac{C_{N,s}}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+2s}}\, dxdy.$$

Theorem 2.24 *Let* $y_0 \in L^2(\Omega)$ *and* $f \in L^2((0,T); H^{-s}(\Omega))$*. Then* (2.26) *has a unique finite energy solution* y*. In addition, if* $y_0 = 0$ *and* $f \in L^2((0,T) \times \omega))$*, then* $y \in L^\infty((0,T); H_0^s(\Omega)) \cap H^1((0,T); L^2(\Omega))$*.*

Remark 2.25 The set of reachable states is given by

$$\mathcal{R}(y_0, T) := \{y(\cdot, T) :\ f \in L^2((0,T) \times \omega))\}$$

Definition 2.26 *We have the following three notions of controllability.*

(a) The system (2.26) *is said to be null controllable in time* $T > 0$ *if*

$$0 \in \mathcal{R}(y_0, T).$$

This is equivalent that there is a control function $f \in L^2((0,T) \times \omega))$ *such that the finite energy solution* y *satisfies*

$$y(T, \cdot) = 0 \ \textit{a.e. in } \Omega.$$

(b) The system (2.26) *is said to be exactly controllable in time* $T > 0$ *if*

$$\mathcal{R}(y_0, T) = L^2(\Omega).$$

This is equivalent to for every given target $y_d \in L^2(\Omega)$ *there is a control function* $f \in L^2((0,T) \times \omega))$ *such that the finite energy solution* y *satisfies*

$$y(T, \cdot) = y_d \ \textit{a.e. in } \Omega.$$

(c) The system (2.26) *is said to be approximately controllable in time* $T > 0$ *if*

$$\mathcal{R}(y_0, T) \textit{ is dense in } L^2(\Omega).$$

This is equivalent to for every $y_0, y_1 \in L^2(\Omega)$ *and* $\varepsilon > 0$*, there is a control function* $f \in L^2((0,T) \times \omega))$ *such that the finite energy solution* y *satisfies*

$$\|y(T, \cdot) - y_1\|_{L^2(\Omega)} < \varepsilon.$$

The following result shows that for our system, null and exact controllabilities are the same notions.

Proposition 2.27 *The following assertions are equivalent.*

(a) The system (2.26) *is null controllable in time* $T > 0$.

(b) The system (2.26) *is exactly controllable in time* $T > 0$.

We have the following characterization of null/exact controllability.

Theorem 2.28 *The following assertions are equivalent.*

(a) The system (2.26) *is null or exactly controllable in* $T > 0$.

(b) The following observability inequality holds:

$$\|v(0,\cdot)\|^2_{L^2(\Omega)} \leq C \int_0^T \int_\omega |v|^2 \, dxdt, \tag{2.28}$$

where v *is the unique solution of the associated dual system:*

$$\begin{cases} -v_t(t,x) + (-\Delta)^s v(t,x) = 0 & in \;\; Q, \\ v = 0 & on \;\; \Sigma, \\ v(T,\cdot) = v_T & in \;\; \Omega. \end{cases} \tag{2.29}$$

Remark 2.29 *For the approximate controllability of the system* (2.26) *it is enough to consider the case where* $y_0 = 0$.

Now we can characterize the approximate controllability property.

Theorem 2.30 *The following assertions are equivalent.*

(a) The system (2.26) *is approximately controllable in time* $T > 0$.

(b) The dual system (2.29) *has the unique continuation property: That is, let* $\omega \subset \Omega$ *be an arbitrary nonempty open set and* v *the unique solution of* (2.29). *If* $v = 0$ *in* $(0,T) \times \omega$, *then* $v = 0$ *in* $(0,T) \times \Omega$.

Proposition 2.31 *Consider the mapping*

$$F : L^2((0,T) \times \omega) \to L^2(\Omega) : \; f \mapsto y(\cdot, T)$$

where y *is the unique solution of* (2.26) *with* $y_0 = 0$. *Then the following assertions are equivalent.*

(a) *The system* (2.26) *is approximately controllable in time* $T > 0$.

(b) *The range of* F, *that is, Ran*(F) *is dense in* $L^2(\Omega)$.

(c) *Ker*($F^\star$) $= 0$, *where* $F^\star$ *is the adjoint of* F.

We can prove a positive result of approximate controllability.

Theorem 2.32 *Let* $\omega \subset \Omega$ *be an arbitrary nonempty open set. Then the system* (2.26) *is always approximately controllable for any* $T > 0$ *and* $f \in L^2((0,T) \times \omega)$.

2.5.2 Exterior controllability properties

Here we consider the exterior control problem:

$$\begin{cases} u_t(t,x) + (-\Delta)^s u(t,x) = 0 & \text{in } (0,T) \times \Omega, \\ u = g\chi_{\mathcal{O}} & \text{in } (0,T) \times (\mathbb{R}^N \setminus \Omega), \\ u(0,\cdot) = u_0 & \text{in } \Omega. \end{cases} \tag{2.30}$$

Here, u is the state to be controlled and g is the control function which is localized in a non-empty open set $\mathcal{O} \subset (\mathbb{R}^N \setminus \Omega)$.

According to Lecture 2, we have the following existence result of the system (2.30).

Theorem 2.33 *For every* $g \in L^2((0,T) \times \mathcal{O})$, *the system* (2.30) *has a unique very-weak solution (or solution by transposition)* $u \in L^2(\mathbb{R}^N)$.

Definition 2.34 *The three notions of controllability are defined similarly, where here the set of reachable states is given by*

$$\mathcal{R}(u_0, T) = \{u(\cdot, T) : \; g \in L^2((0,T) \times \mathcal{O})\}.$$

As in the case of the interior control, we also have the following result.

Lemma 2.35 *The following assertions are equivalent.*

(a) *The system* (2.30) *is null controllable in time* $T > 0$.

(b) *The system* (2.30) *is exactly controllable in time* $T > 0$.

Theorem 2.36 *The following assertions are equivalent.*

(a) *The system* (2.30) *is null or exactly controllable in* $T > 0$.

(b) *The following observability inequality holds:*

$$\|w(0,\cdot)\|^2_{L^2(\Omega)} \le C \int_0^T \int_{\mathcal{O}} |\mathcal{N}_s w|^2 \, dxdt, \tag{2.31}$$

where w *is the unique solution of the associated dual system:*

$$\begin{cases} -w_t(t,x) + (-\Delta)^s w(t,x) = 0 & \text{in } (0,T)\times\Omega, \\ w = 0 & \text{on } (0,T)\times(\mathbb{R}^N\setminus\Omega), \\ w(T,\cdot) = w_T & \text{in } \Omega, \end{cases} \tag{2.32}$$

and we recall that $\mathcal{N}_s w$ *is the nonlocal normal derivative of* w *given by:*

$$\mathcal{N}_s w(x) = C_{N,s} \int_\Omega \frac{w(x)-w(y)}{|x-y|^{N+2s}} \, dy, \quad x \in (\mathbb{R}^N \setminus \overline{\Omega}).$$

As for the case of the interior control, we have the following results.

Remark 2.37 *For the approximate controllability of the system* (2.30) *it is also enough to consider the case where* $u_0 = 0$.

Theorem 2.38 *The following assertions are equivalent.*

(a) *The system* (2.30) *is approximately controllable in time* $T > 0$.

(b) *The dual system* (2.31) *has the unique continuation property: That is, let* $\mathcal{O} \subset (\mathbb{R}^N \setminus \Omega)$ *be an arbitrary nonempty open set and* w *the unique solution of* (2.32). *If* $\mathcal{N}_s w = 0$ *in* $(0,T)\times\mathcal{O}$, *then* $w = 0$ *in* $(0,T)\times\Omega$.

Proposition 2.39 *Consider the mapping*

$$G : L^2((0,T)\times\mathcal{O}) \to L^2(\Omega) : \; g \mapsto u(\cdot,T)$$

where u *is the unique very weak solution of* (2.30) *with* $u_0 = 0$. *Then the following assertions are equivalent.*

(a) *The system* (2.30) *is approximately controllable in time* $T > 0$.

(b) *The range of* G, *that is, Ran*(G) *is dense in* $L^2(\Omega)$.

(c) *Ker*$(G^\star) = 0$, *where* $G^\star$ *is the adjoint of* G.

We have the following positive approximate controllability result.

Theorem 2.40 *Let* $\mathcal{O} \subset (\mathbb{R}^N \setminus \Omega)$ *be an arbitrary nonempty open set. Then the system* (2.30) *is always approximately controllable for any* $T > 0$ *and* $g \in L^2((0,T)\times\mathcal{O})$.

References

[1] F. Alabau-Boussouira, R. Brockett, O. Glass, J. Le Rousseau, and E. Zuazua. Control of partial differential equations. Lecture Notes in Mathematics, 2048. Springer, Heidelberg; 2012.

[2] H. Antil, R. Khatri, and M. Warma. External optimal control of nonlocal PDEs. Inverse Problems 35 (2019), no. 8, 084003, 35 pp.

[3] H. Antil, D. Verma, and M. Warma. External optimal control of fractional parabolic PDEs. ESAIM Control Optim. Calc. Var. 26 (2020), Paper No. 20, 33 pp.

[4] U. Biccari and Hernández-Santamaria. Controllability of a one-dimensional fractional heat equation: theoretical and numerical aspects. IMA J. Math. Control Inform. 36 (2019), no. 4, 1199–1235.

[5] U. Biccari, M. Warma, and E. Zuazua. Controllability of the one-dimensional fractional heat equation under positivity constraints. Commun. Pure Appl. Anal. 19 (2020), no. 4, 1949–1978.

[6] M. Warma and S. Zamorano. Null controllability from the exterior of a one-dimensional nonlocal heat equation. Control Cybernet. 48 (2019), no. 3, 417–438.

[7] M. Warma. Approximate controllability from the exterior of space-time fractional diffusive equations. SIAM J. Control Optim. 57 (2019), no. 3, 2037–2063.