What are the classical boundary conditions for the fractional Laplace operator?

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The Laplace operator

- Let $\Omega \subset \mathbb{R}^N$ be an open set and $u : \Omega \to \mathbb{R}$ a smooth function. The Laplacian Δu of u is defined by

$$\Delta u = \sum_{j=1}^{N} \frac{\partial^2 u(x)}{\partial x_j^2}.$$

- Δ is the typical local operator, that is, for every u

$$\text{supp}[\Delta u] \subset \text{supp}[u].$$

- To define boundary conditions for Δ one needs to introduce the Sobolev spaces.

Classical first order Sobolev spaces

Let $\Omega \subset \mathbb{R}^N$ be a bounded open set with boundary $\partial\Omega$. We let

$$W^{1,2}(\Omega) = \left\{ u \in L^2(\Omega),\ \int_\Omega |\nabla u|^2\, dx < \infty \right\}$$

and $W_0^{1,2}(\Omega) = \overline{D(\Omega)}^{W^{1,2}(\Omega)}$.

- By definition, $W_0^{1,2}(\Omega) \subseteq W^{1,2}(\Omega)$.
- If Ω is bounded, then $W_0^{1,2}(\Omega) \subsetneq W^{1,2}(\Omega)$.
- Notice that functions in $W_0^{1,2}(\Omega)$ are zeros on $\partial\Omega$ (in some sense).

Integration by parts formula for Δ

Let $\Omega \subset \mathbb{R}^N$ be bounded, smooth with boundary $\partial\Omega$. Let $u \in W^{1,2}(\Omega)$ be such that $\Delta u \in L^2(\Omega)$ and $\partial_\nu u := \nabla u \cdot \nu$ exists in $L^2(\partial\Omega)$.

- Then for every $v \in W^{1,2}(\Omega)$, we have

$$-\int_\Omega v\Delta u\, dx = \int_\Omega \nabla u \cdot \nabla v\, dx - \int_{\partial\Omega} v\partial_\nu u\, d\sigma. \tag{1.1}$$

- If $v \in W_0^{1,2}(\Omega)$, then (1.1) becomes

$$-\int_\Omega v\Delta u\, dx = \int_\Omega \nabla u \cdot \nabla v\, dx.$$

The Dirichlet BC for Δ

- If Ω is smooth, then Δ_D is the operator defined by

$$D(\Delta_D) = W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega),\ \Delta_D u = -\Delta u.$$

- For every Ω, Δ_D is the operator associated with the form

$$\mathcal{E}_D(u,v) = \int_\Omega \nabla u \cdot \nabla v\ dx,\ u,v \in W_0^{1,2}(\Omega)$$

in the sense that

$$D(\Delta_D) = \Big\{ u \in W_0^{1,2}(\Omega),\ \exists\, f \in L^2(\Omega), \mathcal{E}_D(u,v) = (f,v)_{L^2(\Omega)} \\ \forall v \in W_0^{1,2}(\Omega) \Big\},\ \Delta_D u = f.$$

We have: $D(\Delta_D) = \{u \in W_0^{1,2}(\Omega) :\ \Delta u \in L^2(\Omega)\}, \Delta_D u = -\Delta u.$

The Neumann BC for Δ

- If Ω is smooth, Δ_N is the operator defined by

$$D(\Delta_N) = \Big\{u \in W^{2,2}(\Omega):\ \partial_\nu u = 0 \ \text{ on } \partial\Omega\Big\},\ \Delta_N u = -\Delta u.$$

- For every Ω, Δ_N is the operator associated with the form

$$\mathcal{E}_N(u,v) = \int_\Omega \nabla u \cdot \nabla v \, dx,\ u, v \in W^{1,2}(\Omega)$$

in the sense that

$$D(\Delta_N) = \Big\{u \in W^{1,2}(\Omega),\ \exists\, f \in L^2(\Omega), \mathcal{E}_N(u,v) = (f,v)_{L^2(\Omega)} \\ \forall v \in W^{1,2}(\Omega)\Big\},\ \Delta_N u = f.$$

Assume that Ω has a Lipschitz boundary. Then
$D(\Delta_N) = \Big\{u \in W^{1,2}(\Omega):\ \Delta u \in L^2(\Omega),\ \partial_\nu u = 0 \ \text{ on } \ \partial\Omega\Big\}$.

Spectrum of Δ_D and Δ_N

Let $\Omega \subset \mathbb{R}^N$ be any bounded domain.

- Δ_D has a discrete spectrum formed of eigenvalues satisfying

$$0 < \lambda_1^D \le \lambda_2^D \le \cdots \le \lambda_n^D \le \cdots, \quad \lim_{n\to\infty} \lambda_n^D = \infty.$$

- If Ω is Lipschitz, then Δ_N has a discrete spectrum formed of eigenvalues satisfying

$$0 = \lambda_1^N \le \lambda_2^N \le \cdots \le \lambda_n^N \le \cdots, \quad \lim_{n\to\infty} \lambda_n^N = \infty.$$

The heat equation

Let $A = -\Delta_D$ or $A = -\Delta_N$.

- For every $u_0 \in L^2(\Omega)$, the Cauchy problem (or heat equation)

$$\partial_t u = Au \quad \text{in} \quad \Omega \times (0, \infty), \quad u(\cdot, 0) = u_0 \quad \text{in} \quad \Omega \tag{1.2}$$

  is well posed.

- The solution u of (1.2) is given by

$$u(t, x) = e^{tA} u_0(x),$$

  where the family of operators $T(t) := e^{tA} : L^2(\Omega) \to L^2(\Omega)$ is the so called semigroup generated by the operator A. That is, $T(t + s) = T(t)T(s)$, $\forall\ t, s \geq 0$ and $T(0) = I$.

The Dirichlet-to-Neumann operator

Let $\Omega \subset \mathbb{R}^N$ be bounded, Lipschitz with boundary $\partial\Omega$. Given $g \in L^2(\partial\Omega)$ and $\lambda \in \mathbb{R} \setminus \sigma(\Delta_D)$ (where $\sigma(\Delta_D)$=Spectrum of Δ_D), let $u \in W^{1,2}(\Omega)$ be the unique solution of the Dirichlet problem

$$-\Delta u = \lambda u \quad \text{in} \quad \Omega, \quad u = g \quad \text{on} \quad \partial\Omega. \tag{1.3}$$

The operator $\mathbb{D}_{1,\lambda}$ defined on $L^2(\partial\Omega)$ by

$$\begin{cases} D(\mathbb{D}_{1,\lambda}) = \Big\{g \in L^2(\partial\Omega),\ \exists\ u \in W^{1,2}(\Omega) \ \text{ solution of } \ (1.3), \\ \qquad\qquad\qquad\qquad\qquad\qquad \partial_\nu u \ \text{ exists in } \ L^2(\partial\Omega)\Big\}, \\ \mathbb{D}_{1,\lambda} g = \partial_\nu u \end{cases} \tag{1.4}$$

is called the Dirichlet-to-Neumann operator.

Remark

- Some properties of $\mathbb{D}_{1,\lambda}$ have been used to give another proof of

$$\lambda^N_{n+1} \leq \lambda^D_n \quad \text{for all} \quad n \in \mathbb{N}.$$

- The operator $\mathbb{D}_{1,0}$ has been defined on very rough domains by Arendt & ter Elst: JDE (2011).
- $\mathbb{D}_{1,0}$ has been defined on exterior domains by Arendt & ter Elst: PA (2015).
- For every $u_0 \in L^2(\partial\Omega)$, the Cauchy problem

$$\partial_t u + \mathbb{D}_{1,\lambda} u = 0 \quad \text{on} \quad \partial\Omega \times (0,\infty), \quad u(x,0) = u_0 \quad \text{on} \quad \partial\Omega,$$

is well-posed. The solution is also given by $u(x,t) = e^{-t\mathbb{D}_{1,\lambda}} u_0(x)$ and the family of operators $(e^{-t\mathbb{D}_{1,\lambda}})_{t\geq 0}$ satisfies the semigroup properties.

Derivation of singular integrals: Long jump random walks

Let $\mathcal{K}: \mathbb{R}^N \to [0,\infty)$ be an even function such that

$$\sum_{k\in\mathbb{Z}^N} \mathcal{K}(k) = 1. \tag{2.1}$$

Given a small $h > 0$, we consider a random walk on the lattice $h\mathbb{Z}^N$.

- We suppose that at any unit time τ (which may depend on h) a particle jumps from any point of $h\mathbb{Z}^N$ to any other point.
- The probability for which a particle jumps from a point $hk \in h\mathbb{Z}^N$ to the point $h\tilde{k}$ is taken to be $\mathcal{K}(k-\tilde{k}) = \mathcal{K}(\tilde{k}-k)$. Note that, differently from the standard random walk, in this process the particle may experience arbitrarily long jumps, though with small probability.

Long jump random walks: Continue

- Let $u(x,t)$ be the probability that our particle lies at $x \in h\mathbb{Z}^N$ at time $t \in \tau\mathbb{Z}$.
- Then $u(x,t+\tau)$ is the sum of all the probabilities of the possible positions $x+hk$ at time t weighted by the probability of jumping from $x+hk$ to x. That is,

$$u(x,t+\tau) = \sum_{k\in\mathbb{Z}^N} \mathcal{K}(k)u(x+hk,t).$$

- Using (2.1) we have the evolution law:

$$u(x,t+\tau) - u(x,t) = \sum_{k\in\mathbb{Z}^N} \mathcal{K}(k)\left[u(x+hk,t) - u(x,t)\right]. \quad (2.2)$$

Long jump random walks: Continue

- In particular, in the case when $\tau = h^{2s}$ and $\mathcal{K}$ is homogeneous (i.e., $\mathcal{K}(y) = |y|^{-(N+2s)}$ for $y \neq 0$, $\mathcal{K}(0) = 0$, and $0 < s < 1$), (2.1) holds and $\mathcal{K}(k)/\tau = h^N \mathcal{K}(hk)$.
- Therefore, we can rewrite (2.2) as follows:

$$\frac{u(x, t+\tau) - u(x,t)}{\tau} = h^N \sum_{k \in \mathbb{Z}^N} \mathcal{K}(hk) \left[u(x+hk, t) - u(x,t)\right]. \tag{2.3}$$

- Notice that the term on the right-hand side of (2.3) is just the approximating Riemann sum of

$$\int_{\mathbb{R}^N} \mathcal{K}(y) \left[u(x+y, t) - u(x,t)\right] dy.$$

Long jump random walks: Continue

- Thus letting $\tau = h^{2s} \to 0^+$ in (2.3), we obtain

$$\partial_t u(x,t) = \int_{\mathbb{R}^N} \frac{u(x+y,t) - u(x,t)}{|y|^{N+2s}} dy. \tag{2.4}$$

- The integral in (2.4) has a singularity at $y = 0$. However when $0 < s < 1$ and u is smooth, we have

$$\begin{aligned} &\lim_{\varepsilon \downarrow 0} \int_{\mathbb{R}^N \setminus B(0,\varepsilon)} \frac{u(x+y,t) - u(x,t)}{|y|^{N+2s}} dy \\ &= \lim_{\varepsilon \downarrow 0} \int_{\mathbb{R}^N \setminus B(x,\varepsilon)} \frac{u(z,t) - u(x,t)}{|z-x|^{N+2s}} dz \\ &= -\left(C_{N,s}\right)^{-1} (-\Delta)^s u(x,t), \end{aligned} \tag{2.5}$$

for a proper normalizing constant $C_{N,s} > 0$.

Long jump random walks: Continue

This shows that a simple random walk with possibly long jumps produces, at the limit a singular integral with a homogeneous kernel.

The fractional Laplace operator: Using Fourier Analysis

Let $0 < s < 1$. Using Fourier analysis, we have that the fractional Laplace operator $(-\Delta)^s$ can be defined as the pseudo-differential operator with symbol $|\xi|^{2s}$. That is,

$$(-\Delta)^s u = C_{N,s}\mathcal{F}^{-1}\left(|\xi|^{2s}\mathcal{F}(u)\right),$$

where $\mathcal{F}$ and $\mathcal{F}^{-1}$ denotes the Fourier transform and the inverse Fourier transform, respectively, and $C(N,s)$ is an appropriate constant.

The fractional Laplace operator: Using Singular Integrals

Let $0 < s < 1$ and

$$\mathcal{L}^1_s(\mathbb{R}^N) := \left\{ u : \mathbb{R}^N \to \mathbb{R} \text{ measurable },\ \int_{\mathbb{R}^N} \frac{|u(x)|}{(1+|x|)^{N+2s}}\,dx < \infty \right\}.$$

For $u \in \mathcal{L}^1_s(\mathbb{R}^N)$ and $\varepsilon > 0$ we let

$$(-\Delta)^s_\varepsilon u(x) = C_{N,s} \int_{\{y\in\mathbb{R}^N:\ |x-y|>\varepsilon\}} \frac{u(x)-u(y)}{|x-y|^{N+2s}}\,dy, \quad x \in \mathbb{R}^N.$$

The fractional Laplacian $(-\Delta)^s u$ of u is defined for $x \in \mathbb{R}^N$ by,

$$(-\Delta)^s u(x) = C_{N,s}\text{P.V.} \int_{\mathbb{R}^N} \frac{u(x)-u(y)}{|x-y|^{N+2s}}\,dy = \lim_{\varepsilon\downarrow 0} (-\Delta)^s_\varepsilon u(x)$$

provided that the limit exists, where $C_{N,s} := \dfrac{s2^{2s}\Gamma\left(\frac{N+2s}{2}\right)}{\pi^{\frac{N}{2}}\Gamma(1-s)}$.

The fractional Laplace operator: Caffarelli-Silvestre extension

Let $0 < s < 1$. For $u : \mathbb{R}^N \to \mathbb{R}$, consider the extension $w : \mathbb{R}^N \times [0, \infty) \to \mathbb{R}$ that satisfies the Dirichlet problem

$$\begin{cases} \Delta_x w + \frac{1-2s}{y} w_y + w_{yy} = 0 \text{ in } \mathbb{R}^N \times (0, \infty), \\ w(x, 0) = u(x). \end{cases}$$

Then the fractional Laplace operator can be defined as

$$(-\Delta)^s u(x) = -d_s \lim_{y \to 0+} y^{1-2s} w_y(x, y),$$

where the constant d_s is given by

$$d_s := 2^{2s-1} \frac{\Gamma(s)}{\Gamma(1-s)}.$$

All the definitions coincide

- Let $0 < s < 1$. Then

$$\begin{aligned}(-\Delta)^s u(x) =& C_{N,s}\mathcal{F}^{-1}\left(|\xi|^{2s}\mathcal{F}(u)\right)\\ =& C_{N,s}\text{P.V.}\int_{\mathbb{R}^N}\frac{u(x)-u(y)}{|x-y|^{N+2s}}\,dy\\ =& -d_s\lim_{y\to 0+} y^{1-2s}w_y(x,y),\end{aligned}$$

  where $w : \mathbb{R}^N \times [0,\infty) \to \mathbb{R}$ is a solution of the Dirichlet problem

$$\begin{cases}\Delta_x w + \frac{1-2s}{y}w_y + w_{yy} = 0 \text{ in } \mathbb{R}^N \times (0,\infty),\\ w(x,0) = u(x).\end{cases}$$

- $(-\Delta)^s$ is the typical nonlocal operator. That is, $\text{supp}[(-\Delta)^s u] \not\subseteq \text{supp}[u]$.

The regional fractional Laplace operator

Let $\Omega \subset \mathbb{R}^N$ be an open set. For $0 < s < 1$, $u \in \mathcal{L}^1_s(\Omega)$ and $\varepsilon > 0$ we let

$$(-\Delta)^s_{\Omega,\varepsilon} u(x) = C_{N,s} \int_{\{y\in\Omega\ |x-y|>\varepsilon\}} \frac{u(x)-u(y)}{|x-y|^{N+2s}}\, dy, \quad x \in \Omega.$$

The regional fractional Laplacian $(-\Delta)^s_\Omega u$ of u is defined for $x \in \Omega$ by,

$$(-\Delta)^s_\Omega u(x) = C_{N,s}\text{P.V.} \int_\Omega \frac{u(x)-u(y)}{|x-y|^{N+2s}}\, dy = \lim_{\varepsilon\downarrow 0} (-\Delta)^s_{\Omega,\varepsilon} u(x)$$

provided that the limit exists. Note that $(-\Delta)^s_\Omega$ depends on Ω.

- $(-\Delta)^s_\Omega$ is a nonlocal operator.

The operators $(-\Delta)^s$ and $(-\Delta)^s_\Omega$ are different

- For every $u \in \mathcal{D}(\Omega)$, we have

$$(-\Delta)^s u(x) = C_{N,s}\text{P.V.} \int_{\mathbb{R}^N} \frac{u(x)-u(y)}{|x-y|^{N+2s}}\, dy$$
$$= C_{N,s}\text{P.V.} \int_{\Omega} \frac{u(x)-u(y)}{|x-y|^{N+2s}}\, dy + u(x)C_{N,s}\int_{\mathbb{R}^N\setminus\Omega} \frac{dy}{|x-y|^{N+2s}}\, dy$$

- That is for $u \in \mathcal{D}(\Omega)$, we have, $(-\Delta)^s u = (-\Delta)^s_\Omega u + V_\Omega(x)u$, where the potential V_Ω is given by

$$v(x) := C_{N,s}\int_{\mathbb{R}^N\setminus\Omega} \frac{dy}{|x-y|^{N+2s}}\, dy.$$

- The potential $V_\Omega(x)$ is difficult to manipulate.

The limit as $s \uparrow 1$

Let $\Omega \subset \mathbb{R}^N$ a bounded open set. Then $\forall\ u, v \in \mathcal{D}(\Omega)$,

$$\lim_{s\uparrow 1}\int_\Omega v(-\Delta)^s_\Omega u dx = \lim_{s\uparrow 1}\int_{\mathbb{R}^N} v(-\Delta)^s u dx = -\int_\Omega v\Delta u dx = \int_\Omega \nabla u\cdot\nabla v dx.$$

Proof

First, let $u \in \mathcal{D}(\Omega)$, since $\lim_{s\uparrow 1}(1-s)\Gamma(1-s) = 1$, we get that

$$\begin{aligned}
&\lim_{s\uparrow 1}\int_\Omega u(-\Delta)^s_\Omega u dx\\
=&\lim_{s\uparrow 1}\frac{s2^{2s-1}\Gamma\left(\frac{N+2s}{2}\right)}{\pi^{\frac{N}{2}}(1-s)\Gamma(1-s)}(1-s)\int_\Omega\int_\Omega\frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}dxdy\\
=&\int_\Omega|\nabla u|^2dx = -\int_\Omega u\Delta u dx.
\end{aligned}$$

Proof Cont.

- Proceeding similarly, we also have that for $u \in \mathcal{D}(\Omega)$,

$$\lim_{s\uparrow 1}\int_{\mathbb{R}^N} u(-\Delta)^s u dx = \int_{\mathbb{R}^N} |\nabla u|^2 dx = -\int_{\mathbb{R}^N} u\Delta u dx = -\int_\Omega u\Delta u dx.$$

- Replacing u by $u+v$ for $u, v \in \mathcal{D}(\Omega)$, we get the equality for every $u, v \in \mathcal{D}(\Omega)$.

Objectives in the rest of the talk

- Find a right formulation for the Dirichlet problems associated with the operators $(-\Delta)^s$ and $(-\Delta)^s_\Omega$.
- Find the right definition of Dirichlet and Neumann boundary conditions for the operators $(-\Delta)^s$ and $(-\Delta)^s_\Omega$.
- Find a right definition of a fractional Dirichlet-to-Neumann type operator associated with $(-\Delta)^s$ or/and $(-\Delta)^s_\Omega$.

Fractional order Sobolev Spaces

Let $\Omega \subset \mathbb{R}^N$ be an arbitrary open set and $s \in (0,1)$.

- We denote

$$W^{s,2}(\Omega) := \left\{ u \in L^2(\Omega) : \int_\Omega \int_\Omega \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\, dx\, dy < \infty \right\}.$$

  We let

$$W_0^{s,2}(\Omega) = \overline{\mathcal{D}(\Omega)}^{W^{s,2}(\Omega)}.$$

- We define

$$W_0^{s,2}(\overline{\Omega}) = \left\{ u \in W^{s,2}(\mathbb{R}^N) : \; u = 0 \; \text{ a.e. on } \; \mathbb{R}^N \setminus \Omega \right\}.$$

- There is no obvious inclusion between $W_0^{s,2}(\Omega)$ and $W_0^{s,2}(\overline{\Omega})$.

Theorem: Grisvard (book 1985) & W. (Potential Analysis 2015)

Let $\Omega \subset \mathbb{R}^N$ be a bounded open set with boundary $\partial\Omega$.

- If Ω is Lipschitz then

$$W^{s,2}(\Omega) = W_0^{s,2}(\Omega) \Longleftrightarrow 0 < s \leq \frac{1}{2}.$$

- Let $C \subset [0,1]$ be the Cantor set and let $\Omega := (0,1) \setminus C$. Let $\dim_H(\partial\Omega)$ be the Hausdorff dimension of $\partial\Omega$. Note that

$$0 < \dim_H(\partial\Omega) = d := \frac{\ln(2)}{\ln(3)} < 1.$$

  Then

$$W^{s,2}(\Omega) = W_0^{s,2}(\Omega) \Longleftrightarrow 0 < s \leq \frac{1}{2}(1-d).$$

Sobolev embedding

Let $\Omega \subset \mathbb{R}^N$ be an arbitrary bounded open set and $0 < s < 1$. Let

$$q := \frac{2N}{N-2s} \text{ if } N > 2s \text{ and } 1 \leq q < \infty \text{ if } N = 2s.$$

Then the following assertions hold.

- If $N \geq 2s$, then $W_0^{s,2}(\Omega) \hookrightarrow L^q(\Omega)$.
- If $N < 2s$, then $W_0^{s,2}(\Omega) \hookrightarrow C^{0,s-\frac{N}{2}}(\mathbb{R}^N)$.
- If Ω is Lipschitz and $N \geq 2s$, then $W^{s,2}(\Omega) \hookrightarrow L^q(\Omega)$.
- If Ω is Lipschitz and $N < 2s$, then $W^{s,2}(\Omega) \hookrightarrow C^{0,s-\frac{N}{2}}(\overline{\Omega})$.

The Dirichlet problem for $(-\Delta)^s$

Let $\Omega \subset \mathbb{R}^N$ be smooth with boundary $\partial\Omega$.

- If $g \in C(\mathbb{R}^N)$ then the Dirichlet problem

$$(-\Delta)^s u = 0 \quad \text{in} \quad \Omega, \quad u = g \text{ on } \quad \partial\Omega, \tag{3.1}$$

is not well-posed. The well-posed Dirichlet problem is given by

$$(-\Delta)^s u = 0 \quad \text{in} \quad \Omega, \quad u = g \quad \text{on} \quad \mathbb{R}^N \setminus \Omega. \tag{3.2}$$

This follows from the fact that

$$(-\Delta)^s u(x) = C_{N,s} \int_\Omega \frac{u(x) - u(y)}{|x-y|^{N+2s}}\, dy + C_{N,,s} \int_{\mathbb{R}^N\setminus\Omega} \frac{u(x) - u(y)}{|x-y|^{N+2s}}\, dy.$$

- If $g \in W^{s,2}(\mathbb{R}^N) \setminus \Omega$ then the Dirichlet problem (3.2) is well-posed.

The Dirichlet problem for $(-\Delta)^s_\Omega$

Let $\Omega \subset \mathbb{R}^N$ be bounded and Lipschitz with boundary $\partial\Omega$.

- If $\frac{1}{2} < s < 1$ and $g \in C(\partial\Omega)$, then the Dirichlet problem

$$(-\Delta)^s_\Omega u = 0 \quad \text{in} \quad \Omega, \quad u = g \text{ on } \quad \partial\Omega, \tag{3.3}$$

  is well-posed.
- If $\frac{1}{2} < s < 1$ and $g \in W^{s-\frac{1}{2},2}(\partial\Omega)$, then the Dirichlet problem (3.3) is well-posed.
- We will see later why the restriction $\frac{1}{2} < s < 1$?

The Dirichlet B.C. for $(-\Delta)^s$

Let $\Omega \subset \mathbb{R}^N$ open and $\mathcal{E}$ with $D(\mathcal{E}) = W_0^{s,2}(\overline{\Omega})$ be given by

$$\mathcal{E}(u,v) = \frac{C_{N,s}}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+2s}}\, dx\, dy.$$

Let $(-\Delta)^s_D$ be the operator on $L^2(\Omega)$ associated with $\mathcal{E}$. Then

$$D((-\Delta)^s_D) = \left\{u \in W_0^{s,2}(\overline{\Omega}) :\ (-\Delta)^s u \in L^2(\Omega)\right\},\ (-\Delta)^s_D u = (-\Delta)^s u.$$

- Here the Dirichlet B.C. is characterized by

$$u = 0 \quad \text{on} \quad \mathbb{R}^N \setminus \Omega.$$

The Dirichlet B.C. for $(-\Delta)^s_\Omega$

Let $\Omega \subset \mathbb{R}^N$ open and $\mathcal{E}_D$ with $D(\mathcal{E}_D) = W_0^{s,2}(\Omega)$ be given by

$$\mathcal{E}_D(u,v) = \frac{C_{N,s}}{2} \int_\Omega \int_\Omega \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+2s}}\, dx\, dy.$$

Let $(-\Delta)^s_{\Omega,D}$ be the operator on $L^2(\Omega)$ associated with $\mathcal{E}_D$. Then

$$D((-\Delta)^s_{\Omega,D}) = \left\{ u \in W_0^{s,2}(\Omega) : \ (-\Delta)^s_\Omega u \in L^2(\Omega) \right\},$$
$$(-\Delta)^s_{\Omega,D} u = (-\Delta)^s_\Omega u.$$

- Here the Dirichlet B.C. is characterized by

$$u = 0 \quad \text{on} \quad \partial\Omega.$$

The operators $(-\Delta)^s_D$ and $(-\Delta)^s_{\Omega,D}$ are different

Assume $\Omega \subset \mathbb{R}^N$ is bounded. Then the following hold.

- $(-\Delta)^s_D$ has a compact resolvent with eigenvalues satisfying

$$0 < \lambda^D_{s,1} \leq \lambda^D_{s,2} \leq \cdots \leq \lambda^D_{s,n} \leq \cdots$$

- $(-\Delta)^s_{\Omega,D}$ has a compact resolvent with eigenvalues satisfying

$$0 < \lambda^{\Omega,D}_{s,1} \leq \lambda^{\Omega,D}_{s,2} \leq \cdots \leq \lambda^{\Omega,D}_{s,n} \leq \cdots$$

- $(-\Delta)^s_D$ and $(-\Delta)^s_{\Omega,D}$ are different in the sense that they have different eigenvalues and eigenfunctions. In particular

$$0 < \lambda^{\Omega,D}_{s,1} < \lambda^D_{s,1}.$$

What is needed to define Neumann B.C?

- One needs a notion of fractional normal derivative.
- One needs an integration by parts formula, that is, a Green type formula for the fractional Laplace operator $(-\Delta)^s$ and/or the regional fractional Laplace operator $(-\Delta)^s_\Omega$.

The nonlocal fractional normal derivative (Gunzburger et al)

Let $\Omega \subset \mathbb{R}^N$ be bounded and Lipschitz. For $0 < s < 1$ let

$$W^{s,2}_\Omega := \Big\{ u : \mathbb{R}^N \to \mathbb{R} \ \text{ measurable, } \ \|u\|_{W^{s,2}_\Omega} < \infty \Big\},$$

where

$$\|u\|^2_{W^{s,2}_\Omega} := \|u\|^2_{L^2(\Omega)} + \int\int_{\mathbb{R}^{2N}\setminus(\mathbb{R}^N\setminus\Omega)^2} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}} dxdy.$$

Notice that $W^{s,2}_\Omega \hookrightarrow W^{s,2}(\Omega)$.
For $u \in W^{s,2}_\Omega$ we define the nonlocal fractional normal derivative as

$$\mathcal{N}_s u(x) = C_{N,s} \int_\Omega \frac{u(x)-u(y)}{|x-y|^{N+2s}}\, dy, \quad x \in \mathbb{R}^N \setminus \overline{\Omega}. \tag{3.4}$$

Integration by part formula for $(-\Delta)^s$ (Dipierro, Ros-Oton and Valdinoci)

Let $\Omega \subset \mathbb{R}^N$ be bounded and Lipschitz. For $0 < s < 1$ and $u, v \in C^2(\mathbb{R}^N)$,

$$\int_\Omega v(-\Delta)^s u \, dx = \frac{C_{N,s}}{2} \int\int_{\mathbb{R}^{2N}\setminus(\mathbb{R}^N\setminus\Omega)^2} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+2s}} dxdy - \int_{\mathbb{R}^N\setminus\Omega} v\mathcal{N}_s u \, dx. \tag{3.5}$$

The Neumann B.C. for $(-\Delta)^s$

Let $\mathcal{E}_{s,\mathcal{N}}$ with $D(\mathcal{E}_{s,\mathcal{N}}) := W^{s,2}_\Omega$ be given by

$$\mathcal{E}_{s,\mathcal{N}}(u,v) := \frac{C_{N,s}}{2}\int\int_{\mathbb{R}^{2N}\setminus(\mathbb{R}^N\setminus\Omega)^2} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+2s}}dxdy.$$

Let $(-\Delta)^s_\mathcal{N}$ be the self-adjoint operator on $L^2(\Omega)$ associated with $\mathcal{E}_{s,\mathcal{N}}$. Using (3.5) we have that

$$\begin{cases} D((-\Delta)^s_\mathcal{N}) = \Big\{u \in W^{s,2}_\Omega,\ (-\Delta)^s u \in L^2(\Omega),\ \ \mathcal{N}_s u = 0 \ \text{ on } \ \mathbb{R}^N \setminus \overline{\Omega}\Big\}, \\ (-\Delta)^s_\mathcal{N} u = (-\Delta)^s u. \end{cases}$$

- The Neumann B.C. is characterized by $\mathcal{N}_s u = 0$ on $\mathbb{R}^N \setminus \Omega$. The operator $\mathcal{N}_s$ is nonlocal too.

The limit as $s \uparrow 1$

Let $\Omega \subset \mathbb{R}^N$ be bounded, Lipschitz. Then for all $u, v \in C_0^2(\mathbb{R}^N)$,

$$\lim_{s\uparrow 1} \frac{C_{N,s}}{2} \int\int_{\mathbb{R}^{2N}\setminus(\mathbb{R}^N\setminus\Omega)^2} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+2s}} dxdy$$

$$- \lim_{s\uparrow 1} \int_{\mathbb{R}^N\setminus\Omega} v\mathcal{N}_s u \, dx$$

$$= \int_\Omega \nabla u \cdot \nabla v \, dx - \int_{\partial\Omega} \frac{\partial u}{\partial \nu} v \, d\sigma.$$

A local fractional normal derivative (Q.Y. Guan & Z.M. Ma: 2006)

Let $\Omega \subset \mathbb{R}^N$ be a bounded open set of class $C^{1,1}$. Let $\frac{1}{2} < s \leq 1$ and

$$C^2_{2s}(\overline{\Omega}) := \Big\{u : \ u(x) = f(x)\rho(x)^{2s-1} + g(x),\ \forall\, x \in \Omega,$$

$$\text{for some } \ f, g \in C^2(\overline{\Omega})\Big\},$$

where $\rho(x) := \mathrm{dist}(x, \partial\Omega)$, $x \in \Omega$. For $u \in C^2_{2s}(\overline{\Omega})$ and $z \in \partial\Omega$, we define the (local) fractional normal derivative $\mathcal{N}^{2-2s}u$ of u by

$$\mathcal{N}^{2-2s}u(z) = \lim_{t\downarrow 0} \frac{du(z+\nu(z)t)}{dt} t^{2-2s} \qquad (3.6)$$

$$= \lim_{t\downarrow 0} \frac{u(z+\nu(z)t) - u(z)}{t^{2s-1}},$$

where $\nu(z)$ denotes the outer normal vector to Ω at the point z.

Some properties of the fractional normal derivative

Let $\frac{1}{2} < s \leq 1$ and $\Omega \subset \mathbb{R}^N$ be a bounded open set of class $C^{1,1}$.

- If $\frac{1}{2} < s < 1$ and $u \in C^1(\overline{\Omega})$, then $\mathcal{N}^{2-2s}u(z) = 0 \; \forall \, z \in \partial\Omega$.
- If $s = 1$ and $u \in C^1(\overline{\Omega})$, then $\mathcal{N}^0 u(z) = \partial_\nu u(z)$.
- If $\frac{1}{2} < s < 1$ and $u \in C^2_{2s}(\overline{\Omega})$, then

$$\mathcal{N}^{2-2s}u(z) = \lim_{\Omega \ni x \to z} \frac{u(x) - u(z)}{\rho^{2s-1}(x)}, \; \forall \, z \in \partial\Omega.$$

Integration by parts formula (Q.Y. Guan & Z.M. Ma: 2006) +(W. 2015))

Let $\frac{1}{2} < s < 1$ and $\Omega \subset \mathbb{R}^N$ of class $C^{1,1}$. Then for every $u \in C^2_{2s}(\overline{\Omega})$ and $v \in W^{s,2}(\Omega)$, one has $(-\Delta)^s_\Omega u \in L^2(\Omega)$, $\mathcal{N}^{2-2s}u \in L^2(\partial\Omega)$ and

$$\int_\Omega v(-\Delta)^s_\Omega u\, dx = \frac{C_{N,s}}{2}\int_\Omega\int_\Omega \frac{(v(x)-v(y))(u(x)-u(y))}{|x-y|^{N+2s}}\, dxdy - B_s\int_{\partial\Omega} v\mathcal{N}^{2-2s}u\, d\sigma, \tag{3.7}$$

where B_s is a constant depending only on s.

- If $u \in C^1(\overline{\Omega})$, then in (3.7) there is no boundary term. This is surprising! But there is an explanation due to the nonlocality.
- Formula (3.7) is not true if one replaces $(-\Delta)^s_\Omega u$ by $(-\Delta)^s u$.

The Neumann B.C. for $(-\Delta)^s_\Omega$

Let $\frac{1}{2} < s < 1$ and $\mathcal{E}_{\Omega,N}$ with $D(\mathcal{E}_{\Omega,N}) = W^{s,2}(\Omega)$ be given by

$$\mathcal{E}_{\Omega,N}(u,v) = \frac{C_{N,s}}{2}\int_\Omega\int_\Omega \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+2s}}\,dx\,dy.$$

Let $(-\Delta)^s_{\Omega,N}$ be the operator on $L^2(\Omega)$ associated with $\mathcal{E}_{\Omega,N}$. Then

$$D((-\Delta)^s_{\Omega,N}) = \left\{u \in W^{s,2}(\Omega) :\ (-\Delta)^s_\Omega u \in L^2(\Omega), \mathcal{N}^{2-2s}u = 0 \text{ on } \partial\Omega\right\}$$

$$(-\Delta)^s_{\Omega,N}u = (-\Delta)^s_\Omega u.$$

- The Neumann B.C. is characterized by $\mathcal{N}^{2-2s}u = 0$ on $\partial\Omega$. The operator $\mathcal{N}^{2-s}$ is local.

The limit as $s \uparrow 1$

Let $\Omega \subset \mathbb{R}^N$ bounded Lipschitz. $\forall\, u \in C^2(\overline{\Omega}), v \in W^{1,2}(\Omega)$ we have

$$\lim_{s\uparrow 1} \int_\Omega v(-\Delta)^s_\Omega u dx = \int_\Omega \nabla u \nabla v dx = -\int_\Omega v \Delta u dx + \int_{\partial\Omega} \frac{\partial u}{\partial \nu} v d\sigma.$$

Proof

We have $W^{1,2}(\Omega) \hookrightarrow W^{s,2}(\Omega)$. Let $u \in C^2(\overline{\Omega})$. Then $\mathcal{N}^{2-2s} u = 0$.

$$\begin{aligned}
\lim_{s\uparrow 1} \int_\Omega u(-\Delta)^s_\Omega u dx =& \frac{1}{2} \lim_{s\uparrow 1} C_{N,s} \int_\Omega \int_\Omega \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}} dxdy \\
=& \frac{1}{2} \lim_{s\uparrow 1} \frac{s 2^{2s} \Gamma\left(\frac{N+2s}{2}\right)(1-s)}{\pi^{\frac{N}{2}}(1-s)\Gamma(1-s)} \int_\Omega \int_\Omega \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}} dxdy \\
=& \int_\Omega |\nabla u|^2 dx = -\int_\Omega u \Delta u dx + \int_{\partial\Omega} \frac{\partial u}{\partial \nu} u d\sigma.
\end{aligned}$$

Proof Cont.

It follows that for every $u, v \in C^2(\overline{\Omega})$, we have

$$\lim_{s\uparrow 1} \int_\Omega v(-\Delta)^s_\Omega u dx = -\int_\Omega v\Delta u dx + \int_{\partial\Omega} \frac{\partial u}{\partial \nu} v d\sigma.$$

Now we obtain the identity for $u \in C^2(\overline{\Omega})$ and $v \in W^{1,2}(\Omega)$ by density.

Why $\frac{1}{2} < s < 1$?

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with Lipschitz boundary $\partial\Omega$.

- If $0 < s \leq \frac{1}{2}$, then Dirichlet and Neumann B.C. for $(-\Delta)^s_\Omega$ coincide. That is,

$$D((-\Delta)^s_{\Omega,D}) = D((-\Delta)^s_{\Omega,N}) \text{ and } (-\Delta)^s_{\Omega,D}u = (-\Delta)^s_{\Omega,N}u.$$

- This follows from the fact that $W^{s,2}(\Omega) = W^{s,2}_0(\Omega) \Leftrightarrow 0 < s \leq \frac{1}{2}$.
- For these reasons we assume without any restriction that $\frac{1}{2} < s < 1$.

A fractional D-to-N operator for $(-\Delta)^s_\Omega$: Definition

Let $\sigma((-\Delta)^s_{\Omega,D})$ denote the spectrum (which is discrete) of $(-\Delta)^s_{\Omega,D}$.
Let $\frac{1}{2} < s < 1$ and $\lambda \in \mathbb{R} \setminus \sigma((-\Delta)^s_{\Omega,D})$. Then, given $g \in L^2(\partial\Omega)$, there exists $u \in W^{s,2}(\Omega)$ solution of the Dirichlet problem

$$(-\Delta)^s_\Omega u = \lambda u \quad \text{in} \quad \Omega, \quad u = g \text{ on } \quad \partial\Omega. \tag{4.1}$$

The fractional D-to-N operator $\mathbb{D}_{s,\lambda}$ is defined on $L^2(\partial\Omega)$ by

$$\begin{cases} D(\mathbb{D}_{s,\lambda}) = \Big\{ g \in L^2(\partial\Omega),\ \exists\, u \in W^{s,2}(\Omega) \ \text{ solution of } \ (4.1), \\ \qquad\qquad\qquad\qquad\qquad\qquad \mathcal{N}^{2-2s} u \ \text{ exists in } \ L^2(\partial\Omega) \Big\}, \\ \mathbb{D}_{s,\lambda} u = C_s \mathcal{N}^{2-2s} u. \end{cases}$$

Sign of the first eigenvalue of $\mathbb{D}_{s,\lambda}$

Let $\lambda \in \mathbb{R} \setminus \sigma((-\Delta)^s_{\Omega,D})$ and let $\eta_{1,s}(\lambda)$ be the first eigenvalue of $\mathbb{D}_{s,\lambda}$. Then the following assertions hold.

- If $\lambda < 0$ then $\eta_{1,s}(\lambda) > 0$.
- if $\lambda > 0$ then $\eta_{1,s}(\lambda) < 0$.
- $\eta_{1,s}(0) = 0$.

Theorem (W. CPAA, 2015)

Let $\frac{1}{2} < s < 1$, $n \in \mathbb{N}$, $\lambda_{n,s}^{\Omega,D}$ the n-th eigenvalue of $(-\Delta)^s_{\Omega,D}$ and $\lambda_{n,s}^{\Omega,N}$ the n-th eigenvalue of $(-\Delta)^s_{\Omega,N}$. Then

$$0 \leq \lambda_{n+1,s}^{\Omega,N} \leq \lambda_{n,s}^{\Omega,D}.$$

THANKS

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